# Application of a new approach to the Adomian method to the solution of fractional-order integro-differential equations

# Abstract

In this paper we solve fractional order integro-differential equations of Fredholm type and Volterra type. For the solution we use a new Adomian decompositional method [5]. In the first part we give the basic notions on fractional operators, essential to our work. The second part is devoted to the description and convergence of the method. In the third part, the method has been used to solve fractional order integro-differential equations of Fredholm type and Volterra type. The last part is devoted to the conclusion and some bibliographical references.


# Introduction

Many physical phenomena can be modelled by fractional-order differential equations. In recent years, several researchers have studied fractional-order differential equations. An analytical and numerical study has been carried out on a class of fractional-order delay differential equations [17].
An integral-differential equation is an equation which involves both the derivatives of a function and its integrals. It is used in a number of fields including physics, astrophysics, electricity and economics etc.
Several numerical methods can be used to solve these equations, including the Adomian method [16] ,[11] [3],[14] [5] the SBA method [3] [2], [4] [6] [7], [8], [9] [10] and the Mellin-SBA method[11]. The retrograde finite difference and Nyström methods can also be used successfully to solve fractional order intro-differential equations of the Fredholm type [18].
In this paper we present a new approach to solving fractional-order integrodifferential equations.

**Key words :** partial differential equation, fractional integral, fractional derivative, fractional integro-differential equations.
**2020 Mathematics Subject Classification :** 35-XX 65-XX, 45-XX.

# 0.1 Preliminary notions

In this section, we give the basic definitions of fractional analysis [11], [9] [7] [12], [13].

## 0.1.1 special funtions

### 0.1.1.1 Gamma function

Where $x$ is a strictly positive real number (or a complex number with a positive real part), the Gamma function is the function defined on $]0, +\infty[$ by :

$$\Gamma(x) = \int_0^{+\infty} t^{x-1} e^{-t} dt \tag{1}$$

$$\Gamma(\frac{1}{2}) = \int_0^{+\infty} t^{\frac{-1}{2}} e^{-t} dt$$

$\Gamma(\frac{1}{2}) = \sqrt{\pi}$
$\Gamma(1,5) = 0.5\Gamma(0.5) = 0.5\sqrt{\pi}$

The Gamma function can be seen as a generalization of the factorial function.
we have :

$$\Gamma(x+1) = x\Gamma(x) \tag{2}$$

in particular :

$$\Gamma(n+1) = n!$$

### 0.1.2 Beta function

Let $x$ and $y$ be two strictly positive real numbers, and the Beta function is defined by :

$$B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}dt \tag{3}$$

For all strictly positive real numbers $x$ and $y$, we have :

$$B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} \tag{4}$$

$$B(x,y) = B(y,x) \tag{5}$$

### 0.1.3 Mittag-Leffler function

The Mittag-Leffler function, known as $E_{alpha,\beta}(z)$, is a special function that applies in the complex plane and depends on two real parameters $\alpha$ and $\beta$. It is defined by :

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{+\infty} \frac{z^k}{\Gamma(\alpha k+\beta)} \tag{6}$$

we note $E_\alpha(z)$ if $\beta = 1$.

$$E_\alpha(z) = \sum_{k=0}^{+\infty} \frac{z^k}{\Gamma(\alpha k+1)}. \tag{7}$$

The Mittag-Leffler function is a convergent serie. we have :

$$E_{1,1}(x) = e^x$$

$$E_{1,2}(x) = \sum_{k=0}^{+\infty} \frac{x^k}{(k+1)!} = \frac{e^x-1}{x}$$

$$E_{2,1}(x^2) = cosh(x)$$

### 0.1.4 Fractional integration and derivation

Fractional derivation [15] [1] is a concept that uses derivatives of non-integer order.

#### 0.1.4.1 Fractional integral in the sense of Riemann Liouville

If $a$ is a real number and $\alpha$ a strictly positive real number, we denote by $f$ a locally integrable function defined on $[a;+\infty[$.
The fractional integral of order $\alpha$ of lower bound $a$ is :

$$({}_a^{RL}I_t^\alpha f)(t) = \frac{1}{\Gamma(\alpha)}\int_a^t (t-\tau)^{\alpha-1}f(\tau)d\tau \tag{8}$$

If there is no ambiguity, we simply note $I_t^\alpha f(t)$ or $I^\alpha f(t)$.

#### 0.1.4.2 Fractional derivative in the sense of Riemann Liouville

Designate by $a$ a real number, by $\alpha$ a strictly positive réel number and by $n$ a non-zero natural number such that : $n-1<\alpha\leq n$.
The fractional derivative in the sense of Riemman Liouvielle is defined by :

$$ {}_a^{RL}D_t^{\alpha}f(t)=\frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dt^n}\int_a^t(t-\tau)^{n-\alpha-1}f(\tau)d\tau \tag{9}$$

**properties :**
Let $\alpha$ and $\beta$ be two strictly positive real numbers, denote by $a$ a real number, $f$ a continuous and locally integrable function définite on [a,+$\infty$[ and by $n$ a natural number such that : $n-1<\alpha\leq n$. We have :

$$ {}_a^{RL}D_t^{\alpha}f(t)=\frac{d^n}{dt^n}[{}_aI_t^{n-\alpha}f(t)] \tag{10}$$

$$ {}_a^{RL}D_t^{\alpha}[{}_a^{RL}I_t^{\beta}f(t)]={}_a^{RL}D_t^{\alpha-\beta}f(t) \tag{11}$$

with $\alpha\geq\beta\geq0$.

$$ {}_a^{RL}D_t^{\alpha}[{}_a^{RL}I_t^{\alpha}f(t)]=f(t) \tag{12}$$

$$ {}_aI_t^{\alpha}[{}_a^{RL}D_t^{\alpha}f(t)]=f(t)-\sum_{j=1}^{n}[{}_aD_t^{\alpha-j}f(t)]_{t=a}\frac{(t-a)^{\alpha-j}}{\Gamma(\alpha-j+1)} \tag{13}$$

Let $m$ be a non-zero natural number, and we have :

$$ \frac{d^m}{dt^m}[{}_a^{RL}D_t^{\alpha}f(t)]={}_a^{RL}D_t^{m+\alpha}f(t)] \tag{14}$$

and

$$ {}_a^{RL}D_t^{\alpha}f^{(m)}(t)]={}_a^{RL}D_t^{m+\alpha}f(t)-\sum_{j=1}^{m}f^{(j)}(a)\frac{(t-a)^{j-m-\alpha}}{\Gamma(j+1-m-\alpha)} \tag{15}$$

#### 0.1.4.3 Fractional derivative in Caputo's sense

The fractional derivative of order $\alpha$ of $f$ of lower bound $a$ in Caputo's sense is defined by :

$$ {}_a^{C}D_t^{\alpha}f(t)=\frac{1}{\Gamma(n-\alpha)}\int_a^t(t-\tau)^{n-\alpha-1}f^{(n)}(\tau)d\tau \tag{16}$$

$f^{(n)}$ denotes the derivative of order $n$ of $f$ and $n$ is a natural number such that
$n-1<\alpha<n$.
we also note ${}^cD_t^{\alpha}$ if $a=0$.
**Properties**

$$ {}_a^{C}D_t^{\alpha}f(t)={}_aI_t^{n-\alpha}[\frac{d^n}{dt^n}f(t)] \tag{17}$$

$$ {}_a^{C}D_t^{\alpha}[{}_aI_t^{\beta}f(t)]={}_a^{C}D_t^{\alpha-\beta}f(t) \tag{18}$$

in particular

$$ {}_aI_t^{\alpha}[{}_a^{C}D_t^{\alpha}f(t)]=f(t)-\sum_{k=0}^{n-1}\frac{t^k}{k!}f^{(k)}(0^+) \tag{19}$$

**Examples :**
${}^cD_t^{\alpha}(t-a)^{\beta}=\frac{\Gamma(\beta+1)}{\Gamma(-\alpha+\beta+1)}(t-a)^{\beta-\alpha}$.

${}^{c}D_t^{\alpha}C = 0.$
${}_0I_t^{\alpha}t^{\beta} = \frac{\Gamma(\beta+1)}{\Gamma(\alpha+\beta+1)}t^{\beta+\alpha}$
${}_0I_t^{\alpha}\sqrt{t} = 0.5\sqrt{\pi}t$
${}_aI_t^{\alpha}C = \frac{C}{\Gamma(1+\alpha)}(t-a)^{\alpha}$

### 0.1.5 Fredholm-type integro-differential equations :

The standard form of a Fredholm-type fractional-order integro-differential equation is given by :

$$ {}^{c}Du(x) = f(x) + \frac{1}{\Gamma(\alpha)}\int_a^b (x-t)^{\alpha-1}K(x,t)u(t)dt \tag{20} $$

$K(x,t)$ is the kernel of the equation.

#### 0.1.5.1 Volterra-type integro-differential equation

The standard form of a Volterra-type fractional-order integro-differential equation is given by :

$$ {}^{c}Du(x) = f(x) + \frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1}K(x,t)u(t)dt \tag{21} $$

$K(x,t)$ is the kernel of the equation.

## 0.2 Resolution method

The method we are going to describe is an improved version of Adomian's method [5]. The method consists of introducing a new function $\phi(x)$ which we determine. We then derive the unknown function $\varphi(x)$[5].

### 0.2.1 Description of the method on a Fredholm fractional-order integro-differential equation

Consider the following problem :

$$ \begin{cases} \frac{\partial^{\alpha}\varphi(x)}{\partial^{\alpha}x} = f(x) + \lambda\int_a^b K(x,t)\varphi(t)dt \\ \varphi(0) = A \end{cases} \tag{22} $$

with $0 \leq a \leq b$.

#### 0.2.1.1 Description

Applying the fractional integral to (22), we obtain :

$$ \varphi(x) = A + I_x^{\alpha}(f(x)) + \lambda I_x^{\alpha}(\int_a^b K(x,t)\varphi(t)dt) \tag{23} $$

We introduce the function $\phi(x)$ defined by :

$$ \phi(x) = I_x^{\alpha}(\int_a^b K(x,t)\varphi(t)dt) \tag{24} $$

we get

$$\varphi(x) = A + I_x^{\alpha}(f(x)) + \lambda\phi(x) \tag{25}$$

Replacing $\varphi(t)$ by its expression in ( 24), we obtain :

$$\phi(x) = \lambda I_x^{\alpha} \int_a^b K(x,t)[A + I_t^{\alpha}(f(t)) + \lambda\phi(t)]dt. \tag{26}$$

We get

$$\begin{aligned}\phi(x) = I_x^{\alpha} \int_a^b K(x,t)Adt + I_x^{\alpha}(\int_a^b K(x,t)I_t^{\alpha}(f(t)))dt \\ + \lambda I_x^{\alpha}(\int_a^b K(x,t)\phi(t))dt\end{aligned} \tag{27}$$

Posing
$F(x) = I_x^{\alpha} \int_a^b K(x,t)Adt + I_x^{\alpha} \int_a^b K(x,t)I_t^{\alpha}(f(t))dt$,
we get

$$\phi(x) = F(x) + \lambda I_x^{\alpha}(\int_a^b K(x,t)\phi(t))dt \tag{28}$$

The solution $\phi(x)$ is found in the form

$$\phi(x) = \sum_{n=0}^{+\infty} \phi_n(x) \tag{29}$$

We derive the Adomian algorithm below :

$$\begin{cases} \phi_0(x) = F(x) \\ \phi_n(x) = \lambda I_x^{\alpha}(\int_a^b K(x,t)\phi_{n-1}(t))dt \;\; n \geq 1 \end{cases} \tag{30}$$

If the algorithm (30) converges, then we obtain $\phi(x)$ in the form :

$$\phi(x) = \sum_{n=0}^{+\infty} \phi_n(x) \tag{31}$$

The solution of (22) is deduced :

$$\varphi(x) = A + I_x^{\alpha}(f(x)) + \lambda\phi(x) \tag{32}$$

**Note**
In some cases, for fast convergence, we can use **the modified Adomian algorithm** [5], which consists in decomposing $F(x)$ into the form

$$F(x) = F_1(x) + F_2(x) \tag{33}$$

We get the **modified Adomian algorithm** :

$$\begin{cases} \phi_0(x) = F_1(x) \\ \phi_1(x) = F_2(x) + \lambda I_x^{\alpha} \int_a^b K(x,t)\phi_0(t)dt \;\; n \geq 1 \\ \phi_n(x) = \lambda I_x^{\alpha}(\int_a^b K(x,t)\phi_{n-1}(t))dt \;\; n \geq 2 \end{cases} \tag{34}$$

#### 0.2.1.2 Method convergence

Let's go back to the algorithm (30) :

$$\begin{cases}\phi_0(x) = F(x) \\ \phi_n(x) = \lambda I_x^\alpha(\int_a^b K(x,t)\phi_{n-1}(t))dt \;\; n \geq 1\end{cases} \tag{35}$$

with

$$F(x) = I_x^\alpha \int_a^b K(x,t)Adt + I_x^\alpha \int_a^b K(x,t) I_t^\alpha(f(t))dt$$

#### 0.2.1.3 Proposition

Under the assumptions, $f \in C([0,T]$, $K \in C([a,T]^2)$, $t \in [0,T]$ and $|\frac{M(b-a)T^\alpha}{\Gamma(\alpha+1)}| < 1$ algorithm (30) converges.

**Proof** :
$f \in C([0,T]$ and $K \in C([0,T]^2)$ so there are two real numbers $m$ and $M$ such that $|f(x)| \leq m$ and $|K(x,t)| \leq M$

$$\begin{aligned}
|\phi_0(x)| = |F(x)| &= |I_x^\alpha \int_a^b K(x,t)Adt + I_x^\alpha \int_a^b K(x,t)I_t^\alpha(f(t))dt| \\
&= |I_x^\alpha[\int_a^b K(x,t)(A + I_t^\alpha(f(t))dt]| \\
&\leq |I_x^\alpha[\int_a^b K(x,t)(A + \frac{mt^\alpha}{\Gamma(\alpha+1)})dt]| \\
&\leq I_x^\alpha[\int_a^b M|A + \frac{mT^\alpha}{\Gamma(\alpha+1)}|dt]
\end{aligned}$$

let : $q = |A + \frac{mT^\alpha}{\Gamma(\alpha+1)}|$.
we get :

$$\begin{aligned}
|\Phi_0(x)| &\leq MI_x^\alpha \int_a^b qdt \\
&\leq Mq(b-a)\frac{x^\alpha}{\Gamma(\alpha+1)} \\
&\leq Mq(b-a)\frac{T^\alpha}{\Gamma(\alpha+1)} \\
&\leq Mq\frac{T^\alpha}{\Gamma(\alpha+1)}(b-a)
\end{aligned}$$

we obtain

$$\begin{aligned}
|\phi_n(x)| &= |\lambda I_x^\alpha \int_a^b K(x,t)\phi_{n-1}(t)dt| \\
&\leq \lambda|I_x^\alpha \int_a^b M\phi_{n-1}(t)dt|.
\end{aligned}$$

We get :

$$\begin{aligned}|\phi_1(x)| &\leq \lambda |I_x^\alpha \int_a^b M\phi_0(t)dt| \\ &\leq \lambda |I_x^\alpha \int_a^b M^2 q \frac{T^\alpha}{\Gamma(\alpha+1)}(b-a)dy \\ &\leq q\lambda |\frac{M(b-a)T^\alpha}{\Gamma(\alpha+1)}|^2\end{aligned}$$

Similarly, we have :

$$\begin{aligned}|\phi_2(x)| &= \lambda |I_x^\alpha \int_a^b K(x,t)\phi_1(t)dt| \\ &\leq \lambda |I_x^\alpha \int_a^b Mq\lambda |\frac{M(b-a)T^\alpha}{\Gamma(\alpha+1)}|^2 dt \\ &\leq q\lambda^2 |\frac{M(b-a)T^\alpha}{\Gamma(\alpha+1)}|^3\end{aligned}$$

Recurrently :

$$|\phi_{n-1}(x)| \leq \frac{q}{\lambda} |\frac{\lambda M(b-a)T^\alpha}{\Gamma(\alpha+1)}|^n \tag{36}$$

we obtain

$$\sum_{i=0}^{n-1} |\phi_i(x)| \leq \frac{q}{\lambda} \frac{1-(\frac{M(b-a)T^\alpha}{\Gamma(\alpha+1)})^n}{1-\frac{M(b-a)T^\alpha}{\Gamma(\alpha+1)}} \tag{37}$$

Under the conditions $|\frac{M(b-a)T^\alpha}{\Gamma(\alpha+1)}| < 1$, we get

$$\sum_{n=0}^{+\infty} |\phi_n(x)| \leq \frac{q}{\lambda} \frac{1}{1-\frac{M(b-a)T^\alpha}{\Gamma(\alpha+1)}} \tag{38}$$

The series

$$\sum_{n=0}^{+\infty} \phi_n(x) \tag{39}$$

is therefore absolutely convergent.
We deduce that the algorithm (30) is convergent.

### 0.2.2 Description of the method on a Volterra fractional-order integro-differential equation

Consider the following problem :

$$\begin{cases}\frac{\partial^\alpha \varphi(x)}{\partial^\alpha x} = f(x) + \lambda \int_0^x K(x,t)\varphi(t)dt \\ \varphi(0) = a\end{cases} \tag{40}$$

#### 0.2.2.1 Description

Let's apply the fractional integral to (40) we obtain :

$$\varphi(x) = a + I_x^\alpha(f(x)) + \lambda I_x^\alpha(\int_0^x K(x,t)\varphi(t)dt) \tag{41}$$

We introduce the function $\phi(x)$ defined by :

$$\phi(x) = I_x^\alpha(\int_0^x K(x,t)\varphi(t)dt) \tag{42}$$

We get

$$\varphi(x) = a + I_x^\alpha(f(x)) + \lambda\phi(x) \tag{43}$$

If we replace $\varphi(t)$ by its expression in ( 42), we obtain :

$$\phi(x) = I_x^\alpha[\int_0^x K(x,t)(a + I_t^\alpha(f(t)) + \lambda\phi(t))]dt. \tag{44}$$

We get

$$\begin{aligned}\phi(x) = I_x^\alpha \int_0^x K(x,t)adt + I_x^\alpha(\int_0^x K(x,t)I_t^\alpha(f(t)))dt \\ + \lambda I_x^\alpha(\int_0^x K(x,t)\phi(t))dt.\end{aligned} \tag{45}$$

Let
$F(x) = I_x^\alpha(\int_0^x K(x,t)a)dt + I_x^\alpha \int_0^x K(x,t)I_t^\alpha(f(t))dt$,
we obtain

$$\phi(x) = F(x) + \lambda I_x^\alpha(\int_0^x K(x,t)\phi(t))dt. \tag{46}$$

We are looking for the solution $\phi(x)$ in the form

$$\phi(x) = \sum_{n=0}^{+\infty} \phi_n(x) \tag{47}$$

The algorithm below can be deduced from this :

$$\begin{cases}\phi_0(x) = F(x) \\ \phi_n(x) = \lambda I_x^\alpha(\int_0^x K(x,t)\phi_{n-1}(t))dt \;\; n \geq 1\end{cases} \tag{48}$$

If the series

$$\sum_0^{+\infty} \phi_n(x)$$

converge the we obtain $\phi_(x)$ such that :

$$\phi(x) = \sum_0^{+\infty} \phi_n(x) \tag{49}$$

This leads to the solution of (40) :

$$\varphi(x) = a + I_x^\alpha(f(x)) + \lambda\phi(x) \tag{50}$$

#### 0.2.2.2 Convergence study

Let's go back to the algorithm (48) :

$$\begin{cases}\phi_0(x) = F(x)\\ \phi_n(x) = \lambda I_x^\alpha(\int_0^x K(x,t)\phi_{n-1}(t))dt \;\; n \geq 1\end{cases}$$

with $F(x) = I_x^\alpha \int_0^x K(x,t)a)dt + I_x^\alpha \int_0^x K(x,t) I_t^\alpha(f(t))dt$

#### 0.2.2.3 Proposition

It is assumed that $f \in C([0,T]$ and $K \in C([0,T]^2)$. The algorithm (48) Converges.
**Proof** :
$f \in C([0,T]$ and $K \in C([0,T]^2)$ so there are two real numbers $m$ and $M$ such that $|f(x)| \leq m$ and $|K(x,t)| \leq M$

$$\begin{aligned}|\phi_0(x)| &= |I_x^\alpha \int_0^x K(x,t)adt + I_x^\alpha \int_0^x K(x,t)I_t^\alpha(f(t))dt|\\ &= |I_x^\alpha[\int_0^x K(x,t)(a + I_t^\alpha(f(t))dt]|\\ &\leq |I_x^\alpha[\int_0^x K(x,t)(a + \frac{mt^\alpha}{\Gamma(\alpha+1)})dt]|\\ &\leq I_x^\alpha[\int_0^x K(x,t)|a + \frac{mT^\alpha}{\Gamma(\alpha+1)}|dt]\end{aligned}$$

There is a real $L$ such that : $|a + \frac{mT^\alpha}{\Gamma(\alpha+1)}| < L$. We obtain

$$\begin{aligned}|\Phi_0(x) &\leq I_x^\alpha[\int_0^x K(x,t)Ldt]\\ &\leq I_x^\alpha[\int_0^x MLdt]\\ &\leq I_x^\alpha xMLdt]\\ &\leq \frac{MLx^{\alpha+1}}{\Gamma(\alpha+2)}\end{aligned}$$

We get

$$\begin{aligned}|\phi_1(x)| &= |\lambda I_x^\alpha \int_0^x K(x,t)\phi_0(t)dt|\\ &\leq |\lambda I_x^\alpha(\frac{M^2Lx^{\alpha+2}}{\Gamma(\alpha+3)})| = |\lambda(\frac{M^2Lx^{2\alpha+2}}{\Gamma(2\alpha+3)})|\end{aligned}$$

Similarly, we have

$$\begin{aligned}
|\phi_2(x)| &= |\lambda I_x^\alpha \int_0^x K(x,t)\phi_1(t)dt| \\
&\leq |\lambda^2 I_x^\alpha (\frac{M^3 L t^{2\alpha+3}}{\Gamma(2\alpha+4)})dt| \\
&\leq |\lambda^2 (\frac{M^3 L x^{3\alpha+3}}{\Gamma(3\alpha+4)})|
\end{aligned}$$

Recurrently :

$$|\phi_{n-1}(x)| \leq |\frac{L}{\lambda}(\frac{\lambda^n M^n x^{n\alpha+n}}{\Gamma(n\alpha+n+1)})| \tag{51}$$

and

$$|\phi_{n-1}(x)| \leq \frac{L}{\lambda}|\frac{(\lambda M x^{(\alpha+1)})^n}{\Gamma(n(\alpha+1)+1)}| \tag{52}$$

indeed

$$\begin{aligned}
|\phi_n(x)| &= |\lambda I_x^\alpha (\int_0^x K(x,t)\phi_{n-1}(t))dt| \\
&\leq \lambda |I_x^\alpha (\int_0^x K(x,t)(\frac{L}{\lambda}\frac{(\lambda M t^{(\alpha+1)})^n}{\Gamma(n(\alpha+1)+1)})|dt \\
|\phi_n(x)| &\leq I_x^\alpha (\frac{L\lambda^n M^{n+1} x^{n\alpha+n+1}}{\Gamma(n\alpha+n+2)}) \\
|\phi_n(x)| &\leq \frac{L}{\lambda}(\frac{\lambda^{n+1} M^{n+1} x^{n\alpha+\alpha+n+1}}{\Gamma(n\alpha+\alpha+n+2)})
\end{aligned}$$

From this we deduce

$$|\phi_n(x)| \leq \frac{L}{\lambda}\frac{(\lambda M x^{\alpha+1})^{n+1}}{\Gamma((n+1)(\alpha+1)+1)} \tag{53}$$

We obtain :

$$\begin{aligned}
\sum_{n=0}^{+\infty} |\phi_n(x)| &\leq \sum_{n=1}^{+\infty} \frac{L}{\lambda}|\frac{(\lambda M t^{(\alpha+1)})^n}{\Gamma(n(\alpha+1)+1)}| \\
\sum_{n=0}^{+\infty} |\phi_n(x)| &\leq \frac{L}{\lambda} E_{\alpha+1}(\lambda M t^{\alpha+1}) - \frac{L}{\lambda}
\end{aligned}$$

The series

$$\sum_{n=0}^{+\infty} \phi_n(x) \tag{54}$$

is therefore absolutely convergent.
We deduce that the algorithm (48) is convergent.

# 0.3 Applications

## 0.3.1 Example 1 :Application to a fractional order integro-differential equation of Volterra type in dimension 1

We consider the following problem :

$$ {}^{c}Du(x) = \frac{6}{\Gamma(4-\alpha)}x^{3-\alpha} - \beta(\alpha;5)x^{\alpha+5} + \int_0^x (x-t)^{\alpha-1}xtu(t)dt \tag{55} $$

Let us apply the Riemann fractional integral. We obtain :

$$ u(x) = x^3 - \frac{\beta(\alpha;5)\Gamma(\alpha+6)}{\Gamma(6+2\alpha)}x^{2\alpha+6} +_x I^\alpha(\int_0^x (x-t)^{\alpha-1}xtu(t)dt) \tag{56} $$

Let

$$ \Phi(x) = I_x^\alpha(\int_0^x (x-t)^{\alpha-1}xtu(t)dt). \tag{57} $$

The equation (55) deviates :

$$ u(x) = x^3 - \frac{\beta(\alpha;5)\Gamma(\alpha+6)}{\Gamma(6+2\alpha)}x^{2\alpha+6} + \Phi(x) \tag{58} $$

Let us replace in (57) the expression of $u(x)$ obtained in (58). We obtain :

$$ \begin{aligned} \Phi(x) &= I_x^\alpha(\int_0^x (x-t)^{\alpha-1}xtu(t)dt) \\ &= I_x^\alpha[(\int_0^x (x-t)^{\alpha-1}xt(t^3 - \frac{\beta(\alpha;5)\Gamma(\alpha+6)}{\Gamma(6+2\alpha)}t^{2\alpha+6} + \Phi(t))dt)] \\ &= I_x^\alpha[(\int_0^x (x-t)^{\alpha-1}xt^4dt - \frac{\beta(\alpha;5)\Gamma(\alpha+6)}{\Gamma(6+2\alpha)}(\int_0^x (x-t)^{\alpha-1}xt^{2\alpha+7}dt] + I_x^\alpha(\int_0^x (x-t)^{\alpha-1}xt\Phi(t)dt) \\ &= I_x^\alpha[\beta(\alpha,5)x^{\alpha+5}) - \frac{\beta(\alpha;5)\Gamma(\alpha+6)}{\Gamma(6+2\alpha)}\int_0^x xt^{7+2\alpha}(x-t)^{\alpha-1}dt] + I_x^\alpha[\int_0^x (x-t)^{\alpha-1}xt\Phi(t)dt] \\ &= \frac{\beta(\alpha+5)\Gamma(\alpha+6)}{\Gamma(2\alpha+6)}x^{2\alpha+6} - \frac{\beta(\alpha+5)\Gamma(\alpha+6)\beta(\alpha;8+2\alpha)}{\Gamma(2\alpha+6)}I_x^\alpha(x^{3\alpha+8}) + I_x^\alpha[\int_0^x (x-t)^{\alpha-1}xt\Phi(t)dt)] \end{aligned} $$

We get :
$\Phi(x) = \frac{\beta(\alpha+5)\Gamma(\alpha+6)}{\Gamma(2\alpha+6)}x^{2\alpha+6} - \frac{\beta(\alpha+5)\Gamma(\alpha+6)\beta(\alpha;8+2\alpha)\Gamma(3\alpha+9)}{\Gamma(2\alpha+6)\Gamma(4\alpha+9)}x^{4\alpha+8} + I_x^\alpha \int_0^x (x-t)^{\alpha-1}xt\Phi(t)dt$

**Adomian Algorithm**

$$ \begin{cases} \Phi_0(x) &= \frac{\beta(\alpha+5)\Gamma(\alpha+6)}{\Gamma(2\alpha+6)}x^{2\alpha+6} \\ \Phi_1(x) &= -\frac{\beta(\alpha+5)\Gamma(\alpha+6)\beta(\alpha;8+2\alpha)\Gamma(3\alpha+9)}{\Gamma(2\alpha+6)\Gamma(4\alpha+9)}x^{4\alpha+8} + I_x^\alpha[\int_0^x (x-t)^{\alpha-1}xt\Phi_0(t)dt] \\ \Phi_n(x) &= I_x^\alpha[\int_0^x (x-t)^{\alpha-1}xt\Phi_{n-1}(t)dt] \end{cases} \tag{59} $$

**Calculation of $\Phi_1(x)$**

$$\begin{aligned}
\Phi_1(x) &= -\frac{\beta(\alpha+5)\Gamma(\alpha+6)\beta(\alpha;8+2\alpha)\Gamma(3\alpha+9)}{\Gamma(2\alpha+6)\Gamma(4\alpha+9)}x^{4\alpha+8} + I_x^\alpha[\int_0^x (x-t)^{\alpha-1}xt\Phi_0(t)dt] \\
&= -\frac{\beta(\alpha+5)\Gamma(\alpha+6)\beta(\alpha;8+2\alpha)\Gamma(3\alpha+9)}{\Gamma(2\alpha+6)\Gamma(4\alpha+9)}x^{4\alpha+8} \\
&+ I_x^\alpha[\int_0^x (x-t)^{\alpha-1}xt(\frac{\beta(\alpha+5)\Gamma(\alpha+6)}{\Gamma(2\alpha+6)}t^{2\alpha+6})dt] \\
&= -\frac{\beta(\alpha+5)\Gamma(\alpha+6)\beta(\alpha;8+2\alpha)\Gamma(3\alpha+9)}{\Gamma(2\alpha+6)\Gamma(4\alpha+9)}x^{4\alpha+8} \\
&+ (\frac{\beta(\alpha+5)\Gamma(\alpha+6)}{\Gamma(2\alpha+6)}I_x^\alpha[\int_0^x (x-t)^{\alpha-1}xt^{2\alpha+7}]dt \\
&= -\frac{\beta(\alpha+5)\Gamma(\alpha+6)\beta(\alpha;8+2\alpha)\Gamma(3\alpha+9)}{\Gamma(2\alpha+6)\Gamma(4\alpha+9)}x^{4\alpha+8} \\
&+ \frac{\beta(\alpha+5)\Gamma(\alpha+6)\beta(\alpha;8+2\alpha)\Gamma(3\alpha+9)}{\Gamma(2\alpha+6)\Gamma(4\alpha+9)}x^{4\alpha+8} \\
&= 0
\end{aligned}$$

$$\begin{cases}
\Phi_0(x) &= \frac{\beta(\alpha+5)\Gamma(\alpha+6)}{\Gamma(2\alpha+6)}x^{2\alpha+6} \\
\Phi_1(x) &= -\frac{\beta(\alpha+5)\Gamma(\alpha+6)\beta(\alpha;8+2\alpha)\Gamma(3\alpha+9)}{\Gamma(2\alpha+6)\Gamma(4\alpha+9)}x^{4\alpha+8} + I_x^\alpha[\int_0^x (x-t)^{\alpha-1}xt\Phi_0(t)dt] = 0 \\
\vdots \\
\Phi_n(x) &= 0; n = 2; 3...
\end{cases} \tag{60}$$

We obtain :

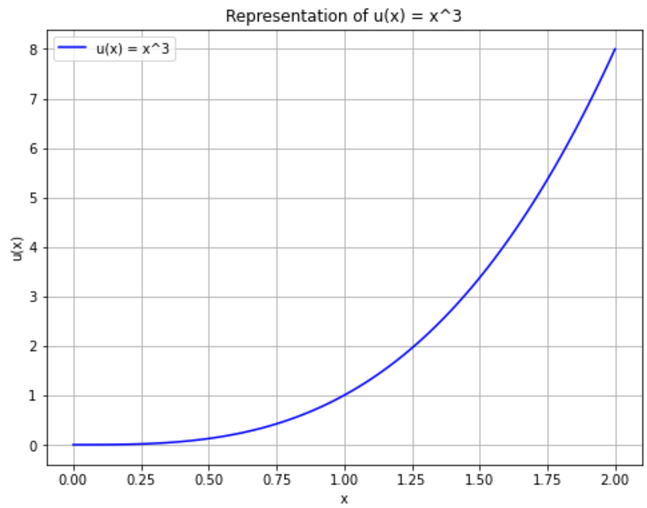
$$\Phi(x) = \sum_{n=0}^{+\infty}\Phi_n(x) = \frac{\beta(\alpha+5)\Gamma(\alpha+6)}{\Gamma(2\alpha+6)}x^{2\alpha+6} \tag{61}$$

According to (58), we have :

$$\begin{aligned}
u(x) &= x^3 - \frac{\beta(\alpha;5)\Gamma(\alpha+6)}{\Gamma(6+2\alpha)}x^{2\alpha+6} + \Phi(x) \\
&= x^3 - \frac{\beta(\alpha;5)\Gamma(\alpha+6)}{\Gamma(6+2\alpha)}x^{2\alpha+6} + \frac{\beta(\alpha+5)\Gamma(\alpha+6)}{\Gamma(2\alpha+6)}x^{2\alpha+6} \\
&= x^3.
\end{aligned}$$

We deduce the general solution to the problem :

$$u(x) = x^3 \tag{62}$$




### 0.3.2 Example 2 : Application to a fractional order integro-differential equation of volterra type in dimension 2

We consider the following problem :

$$\begin{cases} {}^cDu(x,t) = \frac{1}{\Gamma(2-\alpha)} t^{1-\alpha} e^x - \frac{x^{2+\alpha} e^x}{\Gamma(\alpha)} \beta(\alpha, 2) + \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} x u(x,t) dt \\ u(x,0) = 0 \end{cases} \tag{63}$$

Let's apply $I_t^\alpha$.

$$u(x,t) = \frac{e^x \Gamma(2-\alpha)}{\Gamma(2)\Gamma(2-\alpha)} t - \frac{x^{2+\alpha} e^x}{\Gamma(\alpha)\Gamma(1+\alpha)} \beta(\alpha, 2) t^\alpha + I_t^\alpha [\frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} x u(x,t) dt]. \tag{64}$$

We obtain :

$$u(x,t) = t - \frac{x^{2+\alpha} e^x \beta(\alpha, 2)}{\Gamma(\alpha)\Gamma(1+\alpha)} t^\alpha + I_t^\alpha [\frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} x u(x,t) dt]. \tag{65}$$

Let

$$\Phi(x,t) = I_t^\alpha [\frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} x u(x,t) dt]. \tag{66}$$

The equation (65) becomes :

$$u(x,t) = e^x t - \frac{x^{2+\alpha} e^x \beta(\alpha, 2)}{\Gamma(\alpha)\Gamma(1+\alpha)} t^\alpha + \Phi(x,t) \tag{67}$$

Let us replace $u(x,t)$ with its expression in (66).
We obtain :

$$\begin{aligned}
\Phi(x,t) &= I_t^\alpha[\frac{1}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1}x[te^x - \frac{x^{2+\alpha}e^x\beta(\alpha,2)}{\Gamma(\alpha)\Gamma(1+\alpha)}t^\alpha + \Phi(x,t)]]dt \\
&= I_t^\alpha[\frac{1}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1}xe^x tdt] - I_t^\alpha[\frac{1}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1}\frac{x^{3+\alpha}e^x\beta(\alpha,2)}{\Gamma(\alpha)\Gamma(1+\alpha)}t^\alpha \\
&+ I_t^\alpha[\frac{1}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1}x\Phi(x,t)]dt \\
&= I_t^\alpha[\frac{x^{\alpha+2}e^x}{\Gamma(\alpha)}\beta(\alpha,2)] - I_t^\alpha[\frac{x^{3+\alpha}e^x\beta(\alpha,2)}{\Gamma^2(\alpha)\Gamma(1+\alpha)}\int_0^x (x-t)^{\alpha-1}t^\alpha dt] \\
&+ I_t^\alpha[\frac{1}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1}x\Phi(x,t)dt] \\
&= I_t^\alpha[\frac{x^{\alpha+2}e^x}{\Gamma(\alpha)}\beta(\alpha,2)] - I_t^\alpha[\frac{x^{3+3\alpha}e^x\beta(\alpha,2)}{\Gamma^2(\alpha)\Gamma(1+\alpha)}\beta(\alpha;\alpha+1) + I_t^\alpha[\frac{1}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1}x\Phi(x,t)dt] \\
&= \frac{x^{\alpha+2}\beta(\alpha,2)e^x}{\Gamma(\alpha)\Gamma(\alpha+1)}t^\alpha - \frac{x^{3+3\alpha}e^x\beta(\alpha,2)\beta(\alpha;\alpha+1)}{\Gamma^2(\alpha)\Gamma^2(1+\alpha)}t^\alpha + I_t^\alpha[\frac{1}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1}x\Phi(x,t)dt]
\end{aligned}$$

We obtain the canonical form of Adomian :

$$\Phi(x,t) = \frac{x^{\alpha+2}\beta(\alpha,2)e^x}{\Gamma(\alpha)\Gamma(\alpha+1)}t^\alpha - \frac{x^{3+3\alpha}e^x\beta(\alpha,2)\beta(\alpha;\alpha+1)}{\Gamma^2(\alpha)\Gamma^2(1+\alpha)}t^\alpha + I_t^\alpha[\frac{1}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1}x\Phi(x,t)dt \tag{68}$$

We get :

$$\begin{cases}
\Phi_0(x,t) &= \frac{x^{\alpha+2}\beta(\alpha,2)e^x}{\Gamma(\alpha)\Gamma(\alpha+1)}t^\alpha \\
\Phi_1(x,t) &= -\frac{x^{3+3\alpha}e^x\beta(\alpha,2)\beta(\alpha;\alpha+1)}{\Gamma^2(\alpha)\Gamma^2(1+\alpha)}t^\alpha + I_t^\alpha[\frac{1}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1}x\Phi_0(x,t)dt \\
\Phi_n(x,t) &= I_t^\alpha[\frac{1}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1}x\Phi_{n-1}(x,t)dt
\end{cases} \tag{69}$$

Calculation of $\Phi_1(x,t)$ :

$$\begin{aligned}
\Phi_1(x,t) &= -\frac{x^{3+3\alpha}e^x\beta(\alpha,2)\beta(\alpha;\alpha+1)}{\Gamma^2(\alpha)\Gamma^2(1+\alpha)}t^\alpha +_t I^\alpha[\frac{1}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1}x\Phi_0(x,t)dt \\
\Phi_1(x,t) &= -[\frac{x^{3+3\alpha}e^x\beta(\alpha,2)\beta(\alpha;\alpha+1)}{\Gamma^2(\alpha)\Gamma^2(1+\alpha)}t^\alpha +_t I^\alpha[\frac{1}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1}x(\frac{x^{\alpha+2}\beta(\alpha,2)e^x}{\Gamma(\alpha)\Gamma(\alpha+1)}t^\alpha)dt] \\
&= -[\frac{x^{3+3\alpha}e^x\beta(\alpha,2)\beta(\alpha;\alpha+1)}{\Gamma^2(\alpha)\Gamma^2(1+\alpha)}t^\alpha +_t I^\alpha[\frac{x^{\alpha+2}\beta(\alpha,2)e^x}{\Gamma^2(\alpha)\Gamma(\alpha+1)}\int_0^x (x-t)^{\alpha-1}xt^\alpha dt] \\
&= -[\frac{x^{3+3\alpha}e^x\beta(\alpha,2)\beta(\alpha;\alpha+1)}{\Gamma^2(\alpha)\Gamma^2(1+\alpha)}t^\alpha +_t I^\alpha[\frac{x^{3\alpha+3}\beta(\alpha,2)e^x}{\Gamma^2(\alpha)\Gamma(\alpha+1)}\beta(\alpha;\alpha+1) \\
&= -[\frac{x^{3+3\alpha}e^x\beta(\alpha,2)\beta(\alpha;\alpha+1)}{\Gamma^2(\alpha)\Gamma^2(1+\alpha)}t^\alpha + \frac{x^{3+3\alpha}e^x\beta(\alpha,2)\beta(\alpha;\alpha+1)}{\Gamma^2(\alpha)\Gamma^2(1+\alpha)}t^\alpha \\
&= 0
\end{aligned}$$

We obtain

$$\begin{cases} \Phi_0(x,t) & = \frac{x^{\alpha+2}\beta(\alpha,2)e^x}{\Gamma(\alpha)\Gamma(\alpha+1)}t^\alpha \\ \Phi_1(x,t) & = -\frac{x^{3+3\alpha}e^x\beta(\alpha,2)\beta(\alpha;\alpha+1)}{\Gamma^2(\alpha)\Gamma^2(1+\alpha)}t^\alpha + I_t^\alpha[\frac{1}{\Gamma(\alpha)}\int_0^x(x-t)^{\alpha-1}x\Phi_0(x,t)dt = 0 \\ \vdots \\ \Phi_n(x,t) & = I_t^\alpha[\frac{1}{\Gamma(\alpha)}\int_0^x(x-t)^{\alpha-1}x\Phi_{n-1}(x,t)dt = 0 \end{cases} \tag{70}$$

Hence

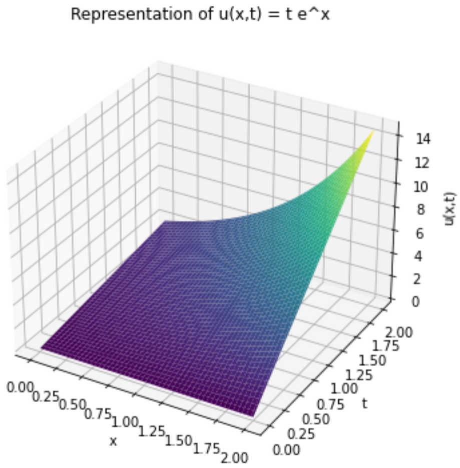
$$\Phi(x,t) = \sum_0^{+\infty}\Phi_n(x,t) = \frac{x^{\alpha+2}\beta(\alpha,2)e^x}{\Gamma(\alpha)\Gamma(\alpha+1)}t^\alpha \tag{71}$$

We deduce from this

$$\begin{aligned} u(x,t) &= e^x t - \frac{x^{2+\alpha}e^x\beta(\alpha,2)}{\Gamma(\alpha)\Gamma(1+\alpha)}t^\alpha + \Phi(x,t) \\ &= e^x t \end{aligned}$$

We obtain the general solution to the problem :

$$u(x,t) = te^x \tag{72}$$




### 0.3.3 Example 3 : Application to a fractional integro-differential equation of Fredholm type in dimension 2

Consider the following problem :

$$\begin{cases} D_t^\alpha u(x,t) = \frac{t^{1-\alpha}}{\Gamma(2-\alpha)}sinx - \frac{1}{3}\lambda xsinx + \lambda\int_0^1 xtu(x,t)dt \\ u(x,0) = 0 \end{cases} \tag{73}$$

Let's apply $I_t^\alpha$. we get :

$$u(x,t) = sinxt - \frac{\lambda xsinx}{3}\frac{1}{\Gamma(\alpha+1)}t^\alpha + \frac{\lambda t^\alpha}{\Gamma(\alpha+1)}\int_0^1 xtu(x,t)dt \tag{74}$$

Let's ask

$$\phi(x,t) = \frac{t^\alpha}{\Gamma(\alpha+1)}\int_0^1 xtu(x,t)dt. \tag{75}$$

We obtain

$$u(x,t) = tsinx - \frac{\lambda xsinx}{3}\frac{1}{\Gamma(\alpha+1)}t^{\alpha} + \lambda\phi(x). \tag{76}$$

We get :

$$\begin{aligned}
\phi(x,t) &= \frac{t^{\alpha}}{\Gamma(\alpha+1)}\int_0^1 xt(tsinx - \frac{\lambda xsinx}{3}\frac{1}{\Gamma(\alpha+1)}t^{\alpha} + \lambda\phi(x))dt \\
&= \frac{t^{\alpha}}{\Gamma(\alpha+1)}\int_0^1 xt^2 sinxdt - \frac{\lambda xsinx}{3}\frac{1}{\Gamma(\alpha+1)}\frac{t^{\alpha}}{\Gamma(\alpha+1)}\int_0^1 xt^{\alpha+1} + \lambda\frac{t^{\alpha}}{\Gamma(\alpha+1)}\int_0^1 xt\phi(x))dt \\
&= \frac{t^{\alpha}xsinx}{\Gamma(\alpha+1)}\int_0^1 t^2 dt - \frac{\lambda x^2 sinxt^{\alpha}}{3((\Gamma(\alpha+1))^2}\int_0^1 t^{\alpha+1}dt + \lambda\frac{xt^{\alpha}}{\Gamma(\alpha+1)}\int_0^1 t\phi(x,t))dt \\
&= \frac{t^{\alpha}xsinx}{3\Gamma(\alpha+1)} - \frac{\lambda x^2 sinxt^{\alpha}}{3(\alpha+2)(\Gamma(\alpha+1))^2} + \frac{\lambda xt^{\alpha}}{\Gamma(\alpha+1)}\int_0^1 t\phi(x,t))dt
\end{aligned}$$

We deduce from this

$$\phi(x,t) = \frac{t^{\alpha}xsinx}{3\Gamma(\alpha+1)} - \frac{\lambda x^2 sinxt^{\alpha}}{3(\alpha+2)(\Gamma(\alpha+1))^2} + \frac{\lambda xt^{\alpha}}{\Gamma(\alpha+1)}\int_0^1 t\phi(x,t))dt. \tag{77}$$

We seek the solution in the form :

$$\phi(x,t) = \sum_{n=0}^{+\infty}\phi_n(x,t) \tag{78}$$

By injecting (78) into (77) and proceeding with an identification, we obtain the following algorithm :

$$\begin{cases}
\phi_0(x,t) = \frac{t^{\alpha}xsinx}{3\Gamma(\alpha+1)} \\
\phi_1(x,t) = -\frac{\lambda x^2 sinxt^{\alpha}}{3(\alpha+2)(\Gamma(\alpha+1))^2} + \frac{\lambda xt^{\alpha}}{\Gamma(\alpha+1)}\int_0^1 t\phi_0(x,t))dt \\
\phi_n(x,t) = \frac{\lambda xt^{\alpha}}{\Gamma(\alpha+1)}\int_0^1 t\phi_{n-1}(x,t))dt;\ n \geq 1
\end{cases} \tag{79}$$

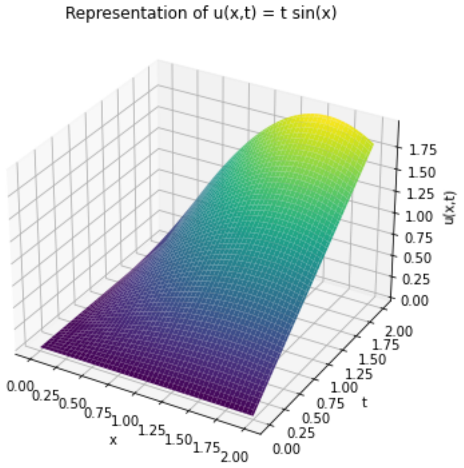
$$\begin{aligned}
\phi_1(x,t) &= -\frac{\lambda x^2 sinxt^{\alpha}}{3(\alpha+2)(\Gamma(\alpha+1))^2} + \frac{\lambda xt^{\alpha}}{\Gamma(\alpha+1)}\int_0^1 t\phi_0(x,t))dt \\
&= -\frac{\lambda x^2 sinxt^{\alpha}}{3(\alpha+2)(\Gamma(\alpha+1))^2} + \frac{\lambda xt^{\alpha}}{\Gamma(\alpha+1)}\int_0^1 t(\frac{t^{\alpha}xsinx}{3\Gamma(\alpha+1)})dt \\
&= -\frac{\lambda x^2 sinxt^{\alpha}}{3(\alpha+2)(\Gamma(\alpha+1))^2} + \frac{\lambda x^2 sinxt^{\alpha}}{3(\Gamma(\alpha+1))^2}\int_0^1 t^{\alpha+1}dt \\
&= -\frac{\lambda x^2 sinxt^{\alpha}}{3(\alpha+2)(\Gamma(\alpha+1))^2} + \frac{\lambda x^2 sinxt^{\alpha}}{3(\alpha+2)(\Gamma(\alpha+1))^2} = 0
\end{aligned}$$

We deduce :
$\phi_1(x,t) = \phi_2(x,t) = ... = \phi_n(x,t) = 0.$
We obtain

$$\phi(x,t) = \sum_{n=0}^{+\infty}\phi_n(x,t) = \frac{t^{\alpha}xsinx}{3\Gamma(\alpha+1)}. \tag{80}$$

The solution to the problem is :

$$\begin{aligned} u(x,t) &= tsinx - \frac{\lambda x sinx}{3}\frac{1}{\Gamma(\alpha+1)}t^{\alpha} + \lambda\phi(x) \\ &= tsinx - \frac{\lambda x sinx}{3\Gamma(\alpha+1)}t^{\alpha} + \lambda\frac{t^{\alpha} x sinx}{3\Gamma(\alpha+1)} \\ u(x,t) &= tsinx. \end{aligned}$$




# Conclusion

In this article we describe a new approach to solving linear fractional integro-differential equations. The advantage of this method in linear fractional integro-differential equations is that it converges more quickly to the solution if it exists. In the future, we plan to use the method on fractional-order integro-differential equations of the Fredholm-Volterra type.

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