*Original Research Article*

CONDITIONS FOR CONVEX OPTIMIZATION IN -SPACES

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ABSTRACT

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| This paper establishes the optimality conditions for convex optimization in -spaces. We prove that if a lower semi-continuous function is Lipschitz continuous in an -space , then it is weakly lower semi-continuous and must attain a unique global minimizer for the convex optimization problem on a sequentially bounded convex constraint set . We also provide further necessary conditions for optimality using the concepts of compactness and coercivity of semi-continuous functions on sequentially bounded domains. Additionally, we prove the existence of minimizers using the concepts of Gateux and Frchet differentiability |

*Keywords: -space, local minimizer, global minimizer, Gateux-differentiable function and Frchet-differentiable function.*

1. **INTRODUCTION**

A lot of studies involving convex optimization have been conducted over a long period of time with interesting conditions for optimality obtained. Alexandrian [1] conducted a study on convex optimization in Hilbert spaces, focusing on determining minimizers for convex programs in such spaces. The study used optimization tools involving lower semi-continuous functions and convex functionals, and applied the generalized Weierstrass Theorem to present conditions required for minimizers to be attained for Hilbert space convex problems. Since this assertion holds for a reflexive space such as which is a Hilbert space, it would be interesting to see whether it holds for general -spaces, where 1 ≤ p < ∞. This study examined the applicability of Alexandrian's findings to general -spaces and sought to discuss properties of convex optimization in -spaces.

Houska and Chachuat [8] dealt with non-convex optimization problems and used a complete-search algorithm to identify feasible solutions. Bay, Grammont, and Maatouk [3] formulated interpolation problems as convex programs governed by linear constraints in Hilbert spaces. They developed an algorithm that approached a constrained interpolating function through the convergence of approximate solutions. However, their study was limited to inner product norms, whereas in the current research, the norms were defined as -norms. This change in norm structure provided a new perspective on the behavior of the functions representing the optimization problems and offered insights into determining solutions for these problems.

Okelo [14] conducted a study on optimization in Hilbert spaces, showing that if a function is weakly sequentially lsc, then the function attains a minimizer on the convex set . The study also established that if is closed, then the optimization problem admits at least one global minimizer. Offia [13] minimized COPs operating on infinite-dimensional Hilbert spaces using functions. However, none of these studies dealt with convex optimization in -spaces.

Peypouquet [15] studied convex optimization in normed spaces, particularly Banach spaces, characterizing properties such as topological duals and linear functionals in these spaces. Devore and Temlyakov [5] examined the application of convex optimization in Banach spaces using interior point methods and investigated recent advances in structural optimization. However, neither of these studies considered convex optimization in -spaces. Unser [16] worked on COPs expressing the solutions as component sums in Banach spaces, regularizing the COPs through the penalization of norms of the minima. In [11] Marraffa and Satco investigated convergence theorems for varying measures using convexity. A wide class of continuous Stietjes differential equations were generalized, with the convexity condition replacing the monotonicity condition to come up with conditions for convergence of measurable functions.

In the recent times, some researchers have delved into the field of online convex optimization. Lin *et al* [10] used coordinate descent algorithms to carry out online convex optimization formulating the problem as a varying objective function. The optimal value was attained after a finite number of step-wise iterations. Zhang *et al* [17] applied online convex optimization to event-triggered communication problems where the constraints were set as time varying inequalities. The feasible set included a network of neighbor-collaborative agents. The distributed event-triggered online primal dual algorithm was used in this optimization. These studies have not focused on convex optimization in -spaces.

Therefore, the current paper aimed to investigate convex optimization in -spaces and address the gap left by Peypouquet and other researchers who established optimality conditions in complete normed spaces. We have explored the conditions for convex optimization in -spaces, taking into consideration the underlying -norm structures and the range of p, i.e., 1 ≤ p < ∞.

1. **PRELIMINARIES**

In this section, useful preliminary results used in later discussion are stated and key concepts are defined. We start by defining a special type of Banach space referred to as the −space.

**Definition 2.1**: Let be a measure space. For a number an Lp space which consists of measurable functions is defined as:

The −norm of is defined by

Now we proceed to define a global minimizer and a local minimizer.

**Definition 2.2 [14]**

A point is termed as a global minimizer of the program for, if for all and .

**Definition 2.3 [14]**

A point is termed as a local minimizer of the program , for , if there exists such that for all whenever satisfies.

**Theorem 2.4 [2]**

Let be convex on the convex set . Given that the local minimum for over is , then is also the global minimum of over .

**Proof:**

If we set a local minimum for to be at it means that throughout the neighborhood of q ∈ 𝓠. Suppose a positive number p satisfies and there exists that satisfies for all where Now, . Therefore, Thus, by Jensen's inequality, we have . Hence, implying that the minimum is global.

We now define Gateux differentiability and Fréchet differentiability.

**Definition 2.5 [4]**

Let be an -space. Suppose is open. Then is Gâteaux differentiable at if for all for all . Here, is termed as Gateux -variation of with respect to .

**Definition 2.6 [4]**

A function from a subset of an Lp -space is Fréchet differentiable at if a bounded operator exists that satisfies. We call ) Fréchet derivative of with respect to .

1. **Main results**

Now we give the main results in which we have presented the requirements necessary for convex optimization in -spaces. We begin by showing that if a function in a strongly sequentially bounded convex sub-space of a convex -space taking a convex closed set to the extended real line is Lipschitz continuous and , then it must attain minimizers in its domain.

**Proposition 3.1** Let the sub-space of a convex -space be strongly sequentially bounded. If a function where is a convex closed set, is Lipschitz continuous in , then is lower semi-continuous and attains a minimizer on .

*Proof.* Let be a sequence converging strongly to . Since is bounded (from hypothesis), a subsequence of exists, converging strongly to . The closure of implies that Now, since is Lipschitz continuous and converges to , we have Clearly, is in . We proceed to show that a minimizer exists in Given that the sequence is convergent, we have for each . This shows that is minimized on by . Since is strongly sequentially bounded and closed, there exists a subsequence of converging strongly to . Furthermore, if is a minimizer on , since is lsc in , we obtain Therefore, is the required minimizer on .

In the following lemma we characterize the solvability property of a convex optimization problem in an -space with a weakly lower semi-continuous objective function.

**Lemma 3.2** *Let be an* -*space and be a convex set. Assume a function in satisfying is weak lower semi-continuous (w-lsc) and coercive. If is a convex optimization problem, then attains a solution .*

*Proof.* Suppose and assume minimizes a convergent sequence in G, with as Since is coercive and, then is a bounded sequence. Therefore, there exists strongly. Furthermore, since is weakly lower semi-continuous, Hence, .

The next result proves that if a convex function is coercive and weakly lower semi-continuous then it attains a unique global minimizer.

**Theorem 3.3** *Suppose a finite function is convex and coercive in an* -*space . Assume that is weak lower semi-continuous (w-lsc). If is a convex optimization problem, then attains a solution . Furthermore, strict convexity in θ guarantees a unique solution .*

*Proof.* Let be a convex optimization problem. Since is *lsc*, convex and coercive (from hypothesis), then as for all (by Lemma 3.2). Thus, attains an optimal solution To prove the uniqueness of this solution, assume are two optimal solutions for the unconstrained convex optimization problem Then we have . This is a contradiction. Thus, .

**Theorem 3.4** *Let a function in an* -*space be Gateaux-differentiable over a convex set . If the Gateaux-derivative is given by for , then for all is necessary for to minimize .*

*Proof.* Let minimize , then for all , . Hence, Therefore, . Now, since is convex, we have:

So,

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The findings in the following result show the conditions for existence of minimizers in sequentially bounded and compact regions.

**Proposition 3.5** Let G be a sequentially bounded and compact set. Let be an lsc function in an -space . If the convex function is compact, then there is a local minimizer of

*Proof.* Suppose Since is bounded, there exists As becomes sufficiently large, we have , implying that is in a compact set. Since the bounded sequence is compact, there exists tending to for some . Since is , we have showing that the global minimizer of is . Hence, is also the local minimizer because is convex.

In the next theorem we have proved that a closed and convex lower semi-continuous function on a a sequentially bounded compact set attains minimizers for a convex optimization problem.

**Theorem 3.6***Let G be a sequentially bounded and compact set. Let*   *be an lsc function in an* -space *. If satisfies the compactness and convexity conditions, then the set of all local minimizers of is compact.*

*Proof.* Given that the set of constraints is sequentially bounded and compact, then by Proposition 3.5 and convexity of , local minimizers of exist and they lie in the level set This shows pre-compactness. It now suffices to prove the closedness property. Since is then for all in the closure of , we obtain:

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The next proposition shows a condition for optimality in convex optimization using the notion of Frchet-differentiability.

**Proposition 3.7** *Let be an* -*space and be a convex function from a convex set to the extended real line If is Fréchet-differentiable, then it satisfies*

*Proof.* If ), define linear functionals in such that and :

Thus, as . This shows that in , hence

The following theorem proves the classical result that for convex objective functions every local minimizer is a global minimizer holds in -spaces.

**Theorem 3.8** *Let the set be a convex constraint set for the convex optimization problem If a function in an -space is convex, then every single local minimum forms a global minimum, and moreover, forms a minimizer for if and only if*

***Proof.***Assume that minimizes ) locally and not globally. So, there exists such that Convexity of yields

 for all .

Also, claiming convexity of , we deduce ) for as . This contradicts the assumption that minimizes strictly locally. Hence, is a global minimizer.

Now, suppose Then, :

showing that is a global minimizer.

Conversely, assume minimizes globally and let. Then ), implying that is not a minimizer. This is a contradiction. Hence,

1. **Conclusion**

We have discussed the conditions for convex optimization in -spaces. We have proved that if a lower semi-continuous function is Lipschitz continuous in an -space then it is weakly lower semi-continuous and must attain a unique global minimizer for the convex optimization problemon a sequentially bounded convex constraint set We have further proved that if a lower semi-continuous function in an -space which is convex and coercive is Gateaux-differentiable or Fréchet-differentiable then it attains a minimizer on a convex set. The open question is: Can these results hold in Sobolev spaces on manifolds?

Disclaimer (Artificial Intelligence)

The author declares that no generative AI technologies such as large language models (chatgpt, copilot, etc.) and text-to-image generators have been used during the writing or editing of this manuscript.

Competing Interests

Author has declared that no competing interests exist.

References

[1] Alexanderian A., *Optimization in Infinite Dimensional Hilbert Spaces*, University of Texas, Austin, 2013.

[2] Barbu V. and Precupanu T., *Convexity and Optimization in Banach Spaces*, Springer, 2012.

[3] Bay X., Grammont I., and Maatouk H., "A new Method for Interpolating in a Convex Subset of a Hilbert space", *Springer, Verlag*, 68, 1 (2017) 95-120.

[4] Behmardi D. and Nayeri E., "Introduction of Fréchet and Gâteaux Derivative", *Applied Mathematical Sciences*, 2, 20 (2008) 975-980.

[5] Devore R. and Temlyakov V., *Convex optimization on Banach Spaces*, arXiv:1401.0334v1, Stat. ML, 2014.

[6] Dhara A., and Dutta J., *Optimality Conditions in Convex Optimization, A Finite Dimensional View*, Taylor and Francis Group, CRC Press, New York, 2012.

[7] Ekeland I. and Temam R., *Convex Analysis and Variational problems*, North Holland, Amsterdam, 1976.

[8] Houska B. and Chachuat B., "Global Optimization in Hilbert Space", *Math. Program. Ser. A, Springer*, 173, (2019), 221-249.

[9] Kumlin P., "A Note on Lp Spaces", *Functional Analysis Lecture Notes*, Chalmers G, U, 2003.

[10] Lin Y., Shames I. and Nesic D., Online Convex Optimization Using Coordinate Descent Algorithms, *J. Automatica,* 165 (2024) 1-13.

[11] Marraffa V. and Satco B., Convergence Theorems for Varying Measures Under Convexity Conditions and Applications, *Mediterr. J. Math,* 19, 274 (2022), 01-18.

[12] Mitter S., "Convex Optimization in Infinite Dimensional Spaces", *Springer-Verlag Berlin, Heidelberg*, (2008) 161-179.

[13] Offia A., "On Convex Optimization in Hilbert Spaces", *IJMSI*, 8, 4, (2020) 07-09.

[14] Okelo N.B., "On Certain Conditions for Convex Optimization in Hilbert Spaces", *arxiv:1903.10177, Math FA*, (2019).

[15] Peypouquet J., *Convex Optimization in Normed spaces: Theory, Methods and Examples*, Springer, New York, 2015.

[16] Unser M. and Aziznejad S., "Convex Optimization in Sums of Banach Spaces", *arXiv:2104.13127, Math OC*, 2, 2021.

[17] Zhang K., Yi X., Cao M., Chai T. and Yang T., Distributed Online Constrained Convex Optimization With Event-triggered Communication, *European Journal of Control* 80 (2024) 101042, 1-8.