~~The Existence of Solutions for Second Order Hamiltonian Systems with Periodic Potentials~~

Abstract This paper is dedicated to investigating the solutions of second order Hamiltonian systems with periodic potentials. We generalized some results to the operator equation $Ax - \nabla\Phi(x) = 0$ by virtue of the critical point theory and the index theory of operator equations. And then we discuss the Sturm-Liouville boundary value problem and the generalized periodic boundary value problem with periodic potential, the existence of solutions is obtained.

Key words Second order Hamiltonian system; Periodic potential; Solutions; Critical point; Operator equation.

1 Introduction

In last decades, variational methods and critical point theorem have been used successfully in studying the existence and multiplicity solutions for second Hamiltonian systems by many mathematicians. Jean Mawhin and Michel Willen in [10] (see also [23] for reference) investigated the existence of solutions to the second Hamiltonian system

$$\ddot{x}(t) + V'(t,x) = 0, \tag{1.1}$$

$$x(0) - x(1) = \dot{x}(0) - \dot{x}(1) = 0, \tag{1.2}$$

where $V \in C([0,1] \times \mathbf{R}^n, \mathbf{R})$ and $V'(t,x)$ denotes the gradient of $V(t,x)$ with respect to x. Under the periodic conditions: $V(t, x + T_i e_i) = V(t,x)(1 \le i \le n)$ for all $(t,x) \in [0,1] \times \mathbf{R}^n$, some positive real numbers $T_1, T_2, \cdots, T_n$ and the canonical basis $\{e_i\}_{i=1}^n$ of $\mathbf{R}^n$, they have proved that (1.1)-(1.2) has one solution. These above conditions have also been used by many mathematicians, see [2, 10, 7, 23] for examples. J. Pipan, M. Schechter and L. Lin have made outstanding contributions to solve the second order Hamiltonian systems [11, 12, 13, 14, 15, 17]. They obtained some brilliant results for non-autonomous second order Hamiltonian systems by linking methods. Y. Dong considered different boundary values such as $x(1) = Mx(0), \dot{x}(1) = N\dot{x}(0), x(0)\cos\alpha - \dot{x}(0)\sin\alpha = 0, x(1)\cos\beta - \dot{x}(1)\sin\beta = 0$ and obtained some excellent results by index theory[4, 16]. Very recently, C. Li studied the solutions with minimal period for non-autonomous second-order Hamiltonian systems with $x(0) = x(pT), \dot{x}(0) = \dot{x}(pT)$ by critical point [18, 19]. Z.Wang and his collaborators unified many previous known results of the non-autonomous second order in [22].

In this paper, we will generalize the result to operator equations. Let X be a real infinite-dimensional Hilbert space with norm $||\cdot||_X$ and inner product $(\cdot,\cdot)_X$. Let $A : D(A) \subset X \to X$ be an unbounded self-adjoint operator satisfying $\sigma(A) = \sigma_d(A)$ and is bounded below by 0 and assume

(Φ_0) $\Phi : Z \equiv D(|A|^{\frac{1}{2}}) \to \mathbf{R}$ is differentiable and for any $x \in Z$ there exists $M > 0$ such that $|\Phi'(x)y| \leq M||y||_X, \forall y \in Z$.

From the Riesz Representation Theorem, it is easy to see that (Φ_0) implies that for any $x \in Z$ there exists a unique element in X denoted by $\nabla\Phi(x)$ such that $\Phi'(x)y = (\nabla\Phi(x), y)_X$ for all $y \in Z$.

We consider the following operator equations:

$$Ax - \nabla\Phi(x) = 0. \tag{1.3}$$

If we define $X = L^2([0,1], \mathbf{R}^n)$ and $(Ax)(t) = -\ddot{x}(t)$ with $D(A) = \{x \in H^2([0,1], \mathbf{R}^n) | x$ satisfies (1.2)$\}$, $\Phi(x) = \int_0^1 V(t,x)dt$ defined on $Z = D(|A|^{\frac{1}{2}}) = \{x \in H^1([0,1], \mathbf{R}^n) | x(0) - x(1) = 0\}$ is continuously differentiable via [4, Prop. 7.3.2] and

$$\Phi'(x)y = \int_0^1 (V'(t, x(t)), y(t))dt, \forall y \in Z.$$

Therefore, (1.1)-(1.2) is a special case of (1.3).

The main results of this paper is the following theorem.

Theorem 1.1 Assume that Φ satisfies (Φ_0),

(Φ_1) $\Phi \in C^1(Z, \mathbf{R})$ and Φ is weak continuous i.e. $\Phi(x_j) \to \Phi(x_0)$ as $x_j \rightharpoonup x_0$,

(Φ_2) $\Phi(x) \leq C$ for all $x \in X$ and $C > 0$ is a constant, and

(Φ_3) if $\dim\ker(A) = m \geq 1$, there exist a basis $\{e_1, e_2, \cdots, e_m\}$ of $\ker(A)$ and some positive real numbers $T_1, T_2, \cdots, T_m$ such that $\Phi(x + T_j e_j) = \Phi(x)$ for all $x \in Z, 1 \leq j \leq m$.

Then (1.3) has one solution.

In section 2 we will prove Theorem 1.1, and in the last section as applications we will investigate second order Hamiltonian systems.

2 Proof

In this section, we prove Theorem 1.1. Recall that A is a self-adjoint operator. Let E be the spectral measure associated with A. Set $P^+ = E(0, +\infty), P^0 = E(\{0\})$ and $P^- = (-\infty, 0)$, and set $Z^+ = P^+, Z^0 = P^0 Z$ and $Z^- = P^- Z$. Since $\sigma(A) = \sigma_d(A)$ is bounded from below by 0, then $Z = Z^0 \bigoplus Z^+$. We define an equivalent norm on Z by

$$||x||^2 = |||A|^{\frac{1}{2}}||_X^2 + ||x^0||_X^2, \tag{2.4}$$

and define

$$I(x) = \frac{1}{2}||x^+||^2 - \Phi(x), \tag{2.5}$$

where $x \in Z, x^+ \in Z^+$ and $x^0 \in Z^0$. It is easy to see that

$$I'(x)y = (x^+, y^+) - \Phi'(x)y, \forall x, y \in Z. \tag{2.6}$$

In [3], we see that if assumption (Φ_0) holds, then any critical point of $I(x)$ is a solution of (1.3).

Lemma 2.1 ([10], Theorem 1.1) If I is w.l.s.c.(weakly lower semi-continuous) on a Hilbert space X and has a bounded minimizing sequence, then I has a minimum on X.

Proof of Theorem 1.1 Set $I_1(x) = \frac{1}{2}||x^+||^2$. Since $I_1(x)$ is convex and continuous, $I_1(x)$ is w.l.s.c.. By (Φ_1), $I(x)$ is w.l.s.c..

For any $x \in Z$,

$$\begin{aligned} I(x) &= \frac{1}{2}||x^+||^2 - \Phi(x) \\ &\geq \frac{1}{2}||x^+||^2 - C \end{aligned} \tag{2.7}$$

via (Φ_2). Let $\{x_j\}$ be a minimizing sequence for I such that $I(x_j) \to \inf I$. By (2.7), $||x_j^+||$ is bounded and $x_j = x_j^+ + x_j^0$. If we write $x_j^0 = \sum_{j=1}^{m} c_j e_j$, then $c_j = \hat{c_j} + k_j T_j (1 \leq j \leq m)$ for some $k_j \in Z$ and $\hat{c_j} \in [0, T_j)$. Set $\hat{x_j} = \hat{x}_j^+ + \hat{x_j}^0 = \hat{x}_j^+ + \sum_{j=1}^{m} \hat{c}_j e_j, \hat{x}_j^+ = x_j^+$ and $\hat{x_j}^0$ is bounded. From (Φ_3),

$$I(x_j) = \frac{1}{2}||x_j^+||^2 - \Phi(x_j)$$

$$= \frac{1}{2}||x_j^+||^2 - \Phi(x_j^+ + \hat{x}_j^0 + \sum_{j=1}^{m} k_j T_j e_j)$$

$$= \frac{1}{2}||x_j^+||^2 - \Phi(x_j^+ + \hat{x}_j^0)$$

$$= I(\hat{x}_j) \tag{2.8}$$

Hence $I(\hat{x}_j) = I(x_j) \to \inf_Z I$ as $j \to \infty$, that is $\{\hat{x}_j\}$ is a bounded minimizing sequence of I. By Lemma 2.1, $I(x)$ has a minimum on Z. The minimum point is a critical point, and is also a solution of (1.3) via ([3], Lemma 2.1). The proof is complete.

3 Applications

As applications, we can investigate the second order Hamiltonian systems as follows

$$\ddot{x}(t) + V'(t, x) = 0, \tag{3.9}$$

$$x(0)\cos\alpha - \dot{x}(0)\sin\alpha = 0, \tag{3.10}$$

$$x(1)\cos\beta - \dot{x}(1)\sin\beta = 0, \tag{3.11}$$

where $V : [0,1] \times \mathbf{R}^n \to \mathbf{R}$, $V' : [0,1] \times \mathbf{R}^n \to \mathbf{R}^n$ are continuous, V' denotes the gradient of V with respect to x, $\alpha \in [0, \pi), \beta \in (0, \pi]$. We need the following assumptions:

(V_1) $V \in C^1([0,1] \times \mathbf{R}R^n, \mathbf{R}R)$;

(V_2) $V(t, x) \le C_1$ for all $x \in \mathbf{R}^n$ and some $C_1 > 0$;

(V_3) $V(t, x + T_j e_j) = V(t, x)$ for all $(t, x) \in [0,1] \times \mathbf{R}^n$, where $T_j > 0, j = 1, 2, \cdots, n$ and $\{e_j\}_{j=1}^n$ is basis of $\mathbf{R}^n$;

(V_3') $V(t, x + T_j \bar{e}_j) = V(t, x)$ for all $(t, x) \in [0,1] \times \mathbf{R}^n$, where $T_j > 0, j = 1, \cdots, m$ and $\{\bar{e}_j\}_{j=1}^m$ is a basis of $\ker(M - I)$.

Theorem 3.1 Assume that (V_1),(V_2),(V_3) hold with $\alpha = \beta = \frac{\pi}{2}$. Then (3.9)-(3.11) has one solution.

(2) Assume that $(V_1) - (V_2)$ hold with $i_{\alpha,\beta}(0) = \nu_{\alpha,\beta}(0) = 0$. Then (3.9)-(3.11) has one solution.

Remark 1 In the assumptions we have used the index denoted by $(\nu_{\alpha,\beta}(B), i_{\alpha,\beta}(B))$ concerning the following system

$$\ddot{x}(t) + B(t)x(t) = 0,$$

$$x(0)\cos\alpha - \dot{x}(0)\sin\alpha = 0,$$

$$x(1)\cos\beta - \dot{x}(1)\sin\beta = 0,$$

where $B \in L^\infty([0,1], \mathcal{L}_s(\mathbf{R}^n))$. From ([4], Definition 7.1.2), $\nu_{\alpha,\beta}(B) =$ the dimension of the solution space of the system, $i_{\alpha,\beta}(B) = \sum_{\lambda<0} \nu_{\alpha,\beta}(B + \lambda I_n)$.

Proof of Theorem 3.1 Define $X = L^2([0,1], \mathbf{R}^n)$ and $(A_1 x)(t) = -\ddot{x}(t)$ with $D(A_1) = \{x \in H^2([0,1], \mathbf{R}^n) | x$ satisfies (3.2)-(3.3)$\}$. By ([4], Proposition 7.1.1), A_1 is self-adjoint and $\sigma(A_1)$ is discrete and bounded from below. Then $\ker(A_1) = R^n$ as $\alpha = \beta = \frac{\pi}{2}$. And define $Z_{\alpha,\beta} = H^1([0,1], \mathbf{R}^n)$ as $\alpha \in (0,\pi)$; $Z_{0,\beta} = \{x \in H^1([0,1], \mathbf{R}^n) | x(0) = 0\}$ as $\beta \in (0,\pi)$; $Z_{\alpha,\pi} = \{x \in H^1([0,1], \mathbf{R}^n) | x(1) = 0\}$ as $\alpha \in (0,\pi)$ and $Z_{0,\pi} = \{x \in H^1([0,1], \mathbf{R}^n) | x(0) = 0 = x(1)\}$. Define $Z_1 = Z_{\alpha,\beta}$ as $\alpha \in [0,\pi), \beta \in (0,\pi]$; then $Z_1 \subset C([0,1], R^n)$ and the embedding is compact. Since $i_{\alpha,\beta}(0) = \nu_{\alpha,\beta}(0) = 0$, $X = X^0 \bigoplus X^+$. By ([4], Proposition 7.3.1), $Z_1 = D(|A_1|^{\frac{1}{2}})$. Define $\Phi(x) = \int_0^1 V(t, x(t))dt, \forall x \in Z_1$. If $x_j \rightharpoonup x_0$ in Z_1, then $x_j \to x_0$ in X. Hence (Φ_1),(Φ_2),(Φ_3) follow from (V_1),(V_2),(V_3). The proof is complete.

Remark 2 During past years, K. C. Chang, P. Rabinowitz, I. Ekeland, Y. Long, J. Mawhin, M. Willem and other authors investigated periodic solutions of second Hamiltonian systems under different conditions, such as periodic conditions, coercivity conditions, convexity conditions and so on. For related material, we refer to [1, 5, 6, 8, 9, 20, 21].

Then we investigate the following Hamiltonian system

$$\begin{aligned} &\ddot{x}(t) + V'(t,x) = 0, \\ &x(1) = Mx(0), \dot{x}(1) = N\dot{x}(0), \end{aligned} \tag{3.12}$$

where $M, N \in GL(\mathbf{R}^n)$ satisfy $M^T N = I$.

Theorem 3.2 Assume that V satisfies (V_1), (V_2) and (V_3'). Then the system (3.9) and (3.12) has one solution.

Proof Define $X = L^2([0,1], \mathbf{R}^n)$ and $(A_2 x)(t) = -\ddot{x}(t)$ with $D(A_2) = \{x \in H^2([0,1], \mathbf{R}^n) | x$ satisfies (3.12)$\}$; then A_2 is self-adjoint, $\ker(A_2) = \{c \in \mathbf{R}^n | (M - I)c = 0\}$ and $Z_2 = \{x \in H^1([0,1]); \mathbf{R}^n) | x(1) = Mx(0)\} = D(|A_2|^{\frac{1}{2}})$ via [4, Proposition 7.3.2]. For any $\lambda \in R$ and $\lambda \notin \sigma_p(A_2)$, $R(A_2 - \lambda Id) = L^2([0,1], \mathbf{R}^n)$ by the general theory of ordinary differential equations. So $\sigma(A_2) = \sigma_d(A_2) \subset [0,\infty)$. Note that $\lambda \geq 0$ is a eigenvalue of A_2 if and only if $\det(M^T + M - (I_n + M^T M)\cos\sqrt{\lambda}) = 0$ and $\sigma(A_2) = \sigma_d(A_2)$ is unbounded from above, and we have $X = X^0 \bigoplus X^+$. Define $\Phi(x) = \int_0^1 V(t, x(t))dt, \forall x \in Z_2$. If $x_j \rightharpoonup x_0$ in Z_2, then $x_j \to x_0$ in $C([0,1], R^n)$. And hence, Φ is weakly continuous in Z_2. Hence (Φ_1),(Φ_2),(Φ_3) follow from (V_1),(V_2),(V_3'). The proof is complete.

4 Conclusion

In this paper we eneralized some results to the operator equation $Ax - \nabla\Phi(x) = 0$. And then we discuss the Sturm-Liouville boundary value problem and the generalized periodic boundary value problem with periodic potential.

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