***Original Research Article***

**CONSTRUCTION OF BLOCK HYBRID METHODS OF SINGULAR SECOND ORDER ORDINARY DIFFERNETIAL EQUATION**

**ABSTRACT**

*This paper deals with the construction of block methods for solving singular Initial/Boundary value problems* (SIBVP)*. It was done by applying shift operator to two linear multi-step formulas and combined with hybrid set of formula which are developed at the first sub-interval to circumvent the singularity at the left end of the integration interval. The coefficients in our linear multi-step formulas (LMF) are determined using the method of undetermined coefficients. The fundamental properties of the proposed scheme are analyzed. The numerical implementation of the method on some* SIBVP *are reported*, *including physical model problems.*

**Keywords:** One-block methods; shift operator, undetermined coefficients, Lane-Emden-type equation, singular Initial/Boundary value problems (SIBVP), A K-step pair of hybrid techniques (KSPHT) which include one-step 2-hybrid point (1S2HP), one-step 3-hybrid point (1S3HP), one-step 4-hybrid point(1S4HP).

1. **INTRODUCTION**

In this paper, we consider the construction of block linear multi-step methods for the numerical solution of the singular Initial/Boundary value problems (SIBVP) of Lane–Emden-type [17] given as

 subject to the boundary conditions



where are real constants,  is continuous real function.

Existence and uniqueness of the solution to the problem (1) subject to any boundary conditions have been rigorously determined by [16 and 21].

The Lane-Emden-type equations are nonlinear ordinary differential equations on semi-infinite domain and categorized as SIBVP. Second-order singular boundary value problems occur in several areas of applied mathematics, physics and engineering, such as chemical kinetics, astrophysics, catalytic diffusion reactions, celestial mechanics [19]. Researchers across various fields of applied sciences and engineering have shown significant interest in solving equations (1) by trying to find better and efficient method for determining the solution. In recent times, analytical solutions have been proffered for its solution but the main difficulty arises at the point *t* = 0, called singular point, making them difficult to handle. In order to overcome these challenges and obtain meaningful solutions, numerical methods have emerged as crucial tools. Many scholars in the field of numerical analysis have proposed various numerical techniques for solving the problem (1) such as the finite difference methods proposed in [9, 15], the spline methods discussed in [6, 8], the approximation methods introduced in [3, 4], the high-order compact finite differences method in [12], among others. The focus of this paper is block methods which posses’ good stability properties for solving differential equations. They are constructed using two different LMF with aid of shift operator, which are combined with hybrid set of formulas called ad-hoc method that is applied only to the first sub interval due to the singularity at . In this way, we obtain a scheme capable of solving the problem posed effectively.

The present work is outlined as follows. In Section 2, we present the KSPHT method for solving SBVPs. The characteristics of the developed formulas are analyzed in Section 3. In Section 4, we present the numerical results of some test problems to show the efficiency and reliability of the proposed technique. Conclusion is outlined in Section 5.

1. **CONSTRUCTION OF THE METHOD**

The idea is to approximate the exact solution of (1) in the partition  of the integration interval , with constant step size  by a self-starting block method. The continuous coefficients () of the composing LMF are determined by imposing order condition on linear multi-step formula (L.M.F) and using the method of undetermined coefficients.

**2.1 PROPOSITION**

On the self-starting block methods [1-2, 5]. Let  be integers, also let  denote the number of k-step LMF in a composite scheme having order at least  then the technique of deriving block methods for solving a second order problem is given by shift-operator times, where  is given as .

**PROOF:**

Notice that the E-operator is effectively applied  times on the system of LMF . Thus there are unknown solution points captured in the block of solution . By this the block definition ; ;

is realized if the coefficient are square matrices of dimensions  for a fixed m and ;  are vectors as specified below

This simply implies that  so that k is chosen such that l is an integer given as

 ;  and .

Where





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Our interest in this paper is when  . which is the first case below.

Firstly, reformulate the equation in (1)

Thus, the singularity is transferred to the function .This block method cannot be used directly for solving a BVP problem in (5) because it is not possible to evaluate , since there is a singularity at .To overcome this drawback, we propose to develop a set of multi-step formulas to be applied at the first sub-interval with the purpose of specifically avoiding the use of . as a result of it, the method will have main formula and also the Formulas to Circumvent the Singularity.

**2.2 MAIN FORMULAS **

Let us consider the Linear Multi-step method (LMM) of the form

using definition of order, This leads to the following matrix equation:



For ****

In equation (6) when is solved by Mathematica software package method to obtain the value of the continuous coefficient ****

 and its derivative as

Evaluating (7) at the points *t* = 3 gives the method and its derivative



Applying the theory in equation (3) on the equation (8), the equations obtained required the shift operator once. The coefficients of the resultant block method after the shift operator application in vector form are below



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**2.3 FORMULAS TO CIRCUMVENT THE SINGULARITY**

Considering different intermediate points using undetermined coefficient .This off-step point is carefully selected to guarantee zero stability condition as circumvent the singularity at the left end of the integration interval, as a result we will developed a set of multi-step formulas specially designed for the sub-interval , where the value is absent,

Let us consider the Hybrid Linear Multi-step method (HLMM) of the form



This leads to the following matrix equation:



**2.3.1 ONE-STEP METHOD WITH TWO OFF-STEP POINTS**

Applying equation (10), two off-step points are introduced, with When  Equation (11) is solved by Mathematica software package method to obtain the continuous coefficient **** expressed as functions of t (whose expressions are not included), and can be written as

and its derivative with respect to t gives



Evaluating (12) at the points  gives the continuous form of the method and the addition method which implied that



**2.3.2 ONE-STEP METHOD WITH THREE OFF-STEP POINTS**

Similarly to the above, applying equation (10), three off-step points are introduced, with When  Equation (11) is solved by Mathematica software package method to obtain the continuous coefficient **** expressed as functions of *t* (whose expressions are not included), and can be written as

and its derivative with respect to t gives



Evaluating (14) at the points  gives the continuous form of the method and the addition method which implied that



**2.3.3 ONE-STEP METHOD WITH FOUR OFF-STEP POINTS**

Applying equation (10), four off-step points are introduced, with When  Equation (11) is solved by Mathematica software package method to obtain the continuous coefficients ****expressed as functions of *t* (whose expressions are not included), and can be written as

and its derivative with respect to t gives

Evaluating (16) at the points  gives the continuous form of the method and the addition method which implied that



1. **ANALYSIS OF THE METHODS**

The linear difference operator  associated with the block (6) is defined;

Expanding (18) using Taylor series, we obtained

So that

Here *p* is the order and  is the error constant ([10]). The following table shows the error constant

**Table 1. Error constants of the composing LMFs**

|  |  |  |  |
| --- | --- | --- | --- |
| Formulae |  | formulae |  |
|  |  |  |  |
|  |  |  |  |

where the formulas , represent  , From (20), it follows that for all the formulas, the order 

Since the order of the formulas is greater than one, they are consistent. For the ad-hoc formulas used for the first step, it is easy to see that they are also consistent

Zero Stability

Definition 1: The implicit block method (10) is said to be zero stable if the roots of the first characteristic polynomial defined by



satisfies and every root with has multiplicity not exceeding two in the limit as . Using the definitions, the method in (10) may be rewritten in a more appropriate vector form to study zero-stability as



The same procedure are done for the ad hoc formulas used for the first step,(whose expressions are not included), its was proofed to be zero stable.

1. **NUMERICAL ILLUSTRATIONS**

In order to study the efficiency of the developed methods, K-step pair of hybrid techniques (KSPHT) which include 1S2HP, 1S3HP, 1S4HP were applied to solve the following test problems: The following notations are used in the tables: x - Point of Evaluation, YEX - Exact solution, 1S2HP - one step, Two off step points, 1S3HP -one step, Three off step points, 1S4HP - one step, Four off step points. Erc - Absolute error. TWS-Taylor wavelet solution [7], AADM - Advanced Adomian decomposition method [20], NA-Numerical Approach [14]. HNT - Hybrid Nyström Techniques [18] The computed results for the four problems using the methods proposed are presented in tables and also shown graphically.

Problem 1. Consider the following physical model SBVP problem of the isothermal gas sphere equilibrium, as described in [7,20]:

In this study, the problem is solved for m = 5 and the results are compared both with the other numerical results and the analytical solution to show the efficiency and validity of the method. with analytical solution

Table 2. Comparing the absolute errors in the new methods to errors of others

|  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- |
| x | Error in 1S2HP | Error in 1S3HP | Error in 1S4HP | AADM[20] | TWS[17] |
| 0.1 | 7.78932376e-7 | 5.294675e-9 | 3.62155e-10 | 1.65000e-6 | 6.46000 e- 6 |
| 0.2 | 7.5386121e-7 | 5.614592e-9 | 3.46792e-10 | 6.63000e−6 | 6.30000e- 6 |
| 0.3 | 7.07239719e-7 | 5.959704e-9 | 3.17325e-10 | 1.59000e-6 | 6.05000e- 6 |
| 0.4 | 6.3830484e-7 | 6.161494e-9 | 2.74391e-10 | 1.53000e-6 | 5.70000 e- 6 |
| 0.5 | 5.49157323e-7 | 6.062213e-9 | 2.21806e-10 | 1.44000e -6 | 5.30000 e- 6 |
| 0.6 | 4.44458267e-7 | 5.561083e-9 | 1.653e-10 | 1.34000e-6 | 4.84000 e- 6 |
| 0.7 | 3.30551613e-7 | 4.633646e-9 | 1.10888e-10 | 1.10000e-6 | 4.33000 e- 6 |
| 0.8 | 2.14367027e-7 | 3.327415e-9 | 6.3487e-11 | 9.58000e-7 | 3.86000e- 6 |
| 0.9 | 1.02404574e-7 | 1.741484e-9 | 2.616e-11 | 7.30000e-7 | 3.24000 e- 6 |
| 1 | 0 | 0 | 0 | 1.89000e-14 | 1.45000 e- 13 |



Figure 1. Plots of exact and KSPHT solution for Problem (1). It show good agreement between the numerical and exact solutions.

Problem 2. The boundary value problem (1)–(2) with  and  arise in the study of various tumor growth problems [3–8] with linear G(x,y) and with non-linear G(x,y) of the form and ,in the study of steady state oxygen diffusion in a spherical cell with Michaelis–Menten uptake kinetics [11,13]. The next problem models the oxygen diffusion in a spherical cell [16]

This equation is also used in the modeling of heat conduction through a solid. Here, is the heat generating function.

**Table 3. Comparison of KSPHT and the exact solution on test 2 with *h* = 1/10**

|  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- |
| t | 1S2HP | 1S3HP | 1S4HP | AADM | NA[14] |  |
| 0 | 0.828483291663973 | 0.828483290349303 | 0.828483290359505 | 0.8284832870 | 0.82848328186193 |  |
| 0.1 | 0.829706093736771 | 0.829706092423853 | 0.829706092433623 | 0.8297060890 | 0.82970609243390 |  |
| 0.2 | 0.833374734887335 | 0.833374733581639 | 0.833374733590825 | 0.8333747302 | 0.83337473359110 |  |
| 0.3 | 0.839489915231093 | 0.839489913945234 | 0.839489913953530 | 0.8394899106 | 0.83948991395381 |  |
| 0.4 | 0.848052786236763 | 0.848052784988731 | 0.848052784995897 | 0.8480527816 | 0.84805278499617 |  |
| 0.5 | 0.859064928350919 | 0.859064927163187 | 0.859064927169075 | 0.8590649238 | 0.85906492716933 |  |
| 0.6 | 0.872528321055240 | 0.872528319953576 | 0.872528319958143 | 0.8725283166 | 0.87252831995828 |  |
| 0.7 | 0.888445306607624 | 0.888445305619761 | 0.888445305623070 | 0.8884453022 | 0.88844530562329 |  |
| 0.8 | 0.888445306607624 | 0.906818548064489 | 0.906818548066702 | 0.9068185447 | 0.90681854806690 |  |
| 0.9 | 0.927650989040035 | 0.927650988364132 | 0.927650988365501 | 0.9276509852 | 0.92765098836568 |  |
| 1 | 0.950945798975819 | 0.950945798495581 | 0.950945798496426 | 0.9509457960 | 0.95094579849657 |  |

 Figure 2. Plots of exact and KSPHT solution for Problem 2.

Problem 3.

We firstly consider the following scalar Lane–Emden singular equation (SCLSE), which corresponds to the reaction–diffusion process in a spherical permeable catalyst as reported in [3,18],

The general analytical solution of problem (4.1.3) is unknown, but its analytical solution for *n* = 1, is given by

whererepresents the Thiele modulus. The value ofis determined by the ratio of the reaction rate at the catalyst surface to the diffusion rate through the catalyst pores.

**Table 4. Comparing the absolute errors in the new methods to the exact solution**

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
| x | HNT[18] | Error in 1S2HP | Error in 1S3HP | Error in 1S4HP |
| 0.1 | 1.87474e-9 | 3.36833567e-7 | 6.17621e-9 | 4.069691e-9 |
| 0.2 | 9.16894 e-10 | 3.67734455e-7 | 6.861048e-9 | 3.8896e-10 |
| 0.3 | 1.91515 e-9 | 4.19915345e-7 | 7.966245e-9 | 2.9896e-10 |
| 0.4 | 2.61712 e-9 | 4.92803425e-7 | 9.473596e-9 | 2.671458e-9 |
| 0.5 | 3.29006e-9 | 5.83054692e-7 | 1.1316567e-8 | 5.88818e-10 |
| 0.6 | 3.95964e-9 | 6.80232762e-7 | 1.3291835e-8 | 7.4041e-10 |
| 0.7 | 4.50469e-9 | 7.58832348e-7 | 1.4898722e-8 | 5.092317e-9 |
| 0.8 | 4.59898e-9 | 7.63576405e-7 | 1.5045103e-8 | 2.062058e-9 |
| 0.9 | 3.55123e-9 | 5.82698005e-7 | 1.1513203e-8 | 1.513203e-9 |
| 1 | 0 | 0 | 0 | 0 |

To analyze the impact of the Thiele modulus ) on the concentration profile (y(x)),

we also considered other values of and . Figure 3 displays the numerical outcomes for various values of and . We observed that in Figure 3, the concentration profile increases when diminishes.



Figure 3. Plots of exact and KSPHT solution for Problem 3.

Problem 4

A similar equation for m = 2 and a = 0 arise in the study of the distribution of heat sources in the human head [18], This problem is a non-linear heat conduction model of the human head, which corresponds to (1-2) with  The general analytical solution of problem is unknown (Umesh,(2020).The problem is solved and result is presented in table 4 and figure 4.

**Table 5. Comparison of KSPHT and the exact solution on test 4**

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
| X | 1S2HP | 1S3HP | 1S4HP | AADM[20] |
| 0 | 1.147039046998964 | 1.147039018988580 | 1.147039019331250 | 1.1470390193 |
| 0.1 | 1.146509670092678 | 1.146509642069824 | 1.146509642412113 | 1.1465096423 |
| 0.2 | 1.144920529809256 | 1.144920501751584 | 1.144920502093611 | 1.1449205020 |
| 0.3 | 1.142268591336077 | 1.142268563230935 | 1.142268563572533 | 1.1422685635 |
| 0.4 | 1.138548776182400 | 1.138548748025628 | 1.138548748366582 | 1.1385487483 |
| 0.5 | 1.133753931190800 | 1.133753902987215 | 1.133753903327258 | 1.1337539033 |
| 0.6 | 1.127874784605714 | 1.127874756369671 | 1.127874756708472 | 1.1278747566 |
| 0.7 | 1.120899888633847 | 1.120899860389977 | 1.120899860727127 | 1.1208998607 |
| 0.8 | 1.112815547750723 | 1.112815519534882 | 1.112815519869877 | 1.1128155198 |
| 0.9 | 1.103605731808831 | 1.103605703669293 | 1.103605704001510 | 1.1036057039 |
| 1 | 1.093251972782518 | 1.093251944781447 | 1.093251945110118 | 1.0932519450 |



Figure 4. Plots of exact and KSPHT solution .

**4.1 DISCUSSION OF RESULTS**

The results obtained from the four test problems considered are summarized in Tables 2 - 5 and figures 1 - 4. In Table 2, we compare the solution of problem 1 using the proposed

Methods (KSPHT) which include 1S2HP,1S3HP, 1S4HP at the points *x* = 0(0.1)1.0 with Umesh (2021), Gumgum (2020). Overall, the KSPHT performed better compared to other methods. It is observed from the tables 2 and table 4 that the results obtained from the methods converged faster when the number of off step points were increased. Tables 4 and 5, and figure 2 and 4, represent the comparison of these approximate solutions for problems 2 and 4 . We observe that our results are in very good agreement up to 6 to 7 places of decimal with the results obtained by advanced Adomian decomposition method (umesh 2020) and also we observe that our results are in very good agreement up to 12 to 13 places of decimal with the results obtained by (A numerical approach for the solution of a class of singular boundary value problems arising in physiology Mohamadreza (2015)). In general it compares favourably with the existing methods despite their different methods.

1. **CONCLUSION**

This paper has demonstrated how self-starting block methods for the solution of second order SIBVP can be constructed by applying shift operator on two different linear multi-step formulas and combined with hybrid set of formulas which are developed at the first sub-interval. The continuous coefficients of the linear multi-step methods are derived using the order definition. Four real-world model problems in literature were used to show the efficiency of the methods, accuracy in terms of the errors obtained, when compared to other methods. It was observed that the proposed methods compare favourably to some existing methods cited in the literature and are even superior to some.

Disclaimer (Artificial intelligence)

Author(s) hereby declare that NO generative AI technologies such as Large Language Models (ChatGPT, COPILOT, etc.) and text-to-image generators have been used during the writing or editing of this manuscript.

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