***Original Research Article***

**Filters and Tangles in Submodular Partition Functions and Graph Width Parameters**

 **Abstract:**

The study of decomposition trees and graph width parameters has garnered considerable attention due to their promising applications in engineering. Within graph theory, the concept of a Loose Tangle has emerged as a notable counterpart to branch-width, a well-established graph width parameter. An ultrafilter is a maximal proper filter on a set that, for every subset, contains either the subset itself or its complement. When extended to a connectivity system, the core notions of Ultrafilter and Bramble are known to exhibit a dual relationship with branch-width.
In previous studies, Filters had not been defined using submodular partition functions. This concise paper investigates the intricate relationship between Loose Tangles and Filters through the lens of a submodular partition function, a mathematical tool that embeds the principle of submodularity into the partitioning process. Furthermore, the paper explores the interaction between Brambles and Filters, also utilizing submodular partition functions as a foundational analytical framework.
**Keyword:** Tangle, Loose tangle, Filter, Bramble, Submodular Partition Function

1. **Introduction**
	1. **Submodular functions**

Submodular functions have numerous applications in fields such as optimization, machine learning, and economics [75,76]. They are employed in tasks like data summarization, sensor placement, and influence maximization in social networks, owing to their properties that enable efficient solutions to otherwise complex problems. Among the various subclasses of submodular functions is the submodular partition function [21, 22], which integrates the principle of submodularity into the partitioning process and is widely applied in graph theory and optimization theory. It should be noted that the functions considered in this paper are set functions, that is, submodular functions defined over sets.

**1.2. Graph width parameters**

A graph consists of vertices and edges [74]. Graphs are studied for applications in networks, artificial intelligence, and various other fields [86-88]. A graph parameter is a numerical invariant measuring structural properties. It should be noted that the graphs discussed in this paper are finite, undirected, and simple.

Graph width parameters are metrics that measure the complexity of a graph's structure [19]. They help in understanding how "wide" or "tree-like" a graph is, which affects the difficulty of solving problems on the graph. Common parameters include tree-width [8,9,29,32,33], path-width [20], linear-width [3,6,,15,16,28], carving-width[4,72,73], twin-width[64-66], cut-width[30], band-width[69-71], path-distance-width[5,7], Modular-width[67,68], Monoidal-Width [11], hypertree-width[61-63], superhypertree-width [57-60], Tree-partition-width [95-97], NLC-width [92-94], Boolean-width [89-91], Directed-Tree-width [48,49,51], and branch-width [1,10,18,31], and they are crucial in areas like optimization and algorithm design.

Graph width parameters have diverse applications in various disciplines, including matroid theory, lattice theory, theoretical computer science, game theory, network theory, artificial intelligence, graph minor theory, graph combinatorics, and many areas of discrete mathematics [98, 99]. This broad applicability is supported by numerous studies cited in references [2,12-14,17]. Researchers often explore these graph width parameters in combination with obstructions, leading to a significant body of research. The duality property is particularly important in several well-known graph algorithms used to compute decompositions of graphs with small width parameters.

The concept of Bramble, used in this paper, pertains to algebraic/set theory and refers to a concept that facilitates a coarse decomposition of sets or algebraic structures. It involves the process of breaking down a set or structure into smaller, more manageable components while retaining specific essential properties or relationships. Additionally, the concept of Bramble is frequently used in game theory to analyze and simplify strategic interactions within complex systems [52-56].

Loose tangle, an innovative concept initially brought forward in reference [1], occupies a central role in ascertaining whether a branch-width is at most a natural number *k*, where *k+1* denotes the order of the loose tangle. The relevance and potential of loose tangles have further been explored in the context of submodular partition functions [22]. These submodular partition functions significantly broaden the understanding of various well-established tree decompositions of graphs.

Additionally, it is widely recognized that ultrafilters, a well-established concept in mathematics, exhibit a dual relationship when extended to a connectivity system *(X, f) [19,31]*. In this context, a connectivity system comprises a pair consisting of a finite set (referred to as the underlying set) *X* and a symmetric submodular function *f*. This dual relationship with branch-width has been extensively studied and acknowledged.

**1.3. Our Contributions**

In this paper, we delve into the correlation between Loose Tangle and Filter, utilizing a submodular partition function as a framework. Additionally, we examine the interaction between Bramble and Filter, employing the same submodular partition function approach. Furthermore, we discuss the connection between these dual concepts and graph width parameters within the context of this paper. It is important to mention that the preprint of this paper is available in references [43, 44].

In previous studies, Filters had not been defined using submodular partition functions. This work introduces such a definition and investigates the structural properties of Filters as well as their relationship with Loose Tangles, thus presenting a novel contribution. It is hoped that these efforts will advance research on graph width parameters and submodular partition functions.

**1.4 Structure of This Paper**

This section briefly outlines the structure of the paper. Section 2 presents the fundamental definitions, including those related to submodular partition functions and Loose Tangles. It also introduces the notion of Filters on submodular partition functions. Section 3 investigates the cryptomorphism between Loose Tangles and Filters in the context of submodular partition functions. Section 4 explores Filter of partitions and Bramble of partitions: Obstructions to decomposition trees. Section 5 provides a mathematical analysis of Ideals, which are known as co-filters. Finally, Section 6 concludes the paper.

**2. Preliminaries**In this section, we present the essential definitions required for this paper. First, we briefly present the fundamental concepts of basic set theory and the foundational notions necessary for considering submodular partition functions.

**Definition 1 (Set) [77,78]:** A *set* is defined as a collection of distinct elements or objects, treated as a single entity. Sets are typically denoted using curly braces; for example, {a, b, c}.

**Definition 2 (Subset) [77,78]:** Given a set X, a *subset* A is any set such that every element of A is also an element of X. This is denoted as A⊆X.

**Definition 3 (Boolean Algebra):** A *Boolean algebra* is a mathematical structure represented as *(X,∪,∩)*, where:

* *X* is a set,
* *∪*denotes the union operation, and
* *∩* denotes the intersection operation.

These operations satisfy specific axioms of commutativity, associativity, distributivity, identity, and complementation. In this paper, our focus is confined to finite sets.

**Notation 4 (Basic Set Operations) [77,78]:**
In this paper, we adopt the following notations for set operations:

* *A⊆X* indicates that *A* is a subset of *X.*
* *A∪B* denotes the union of subsets *A a*nd *B* (with *A,B⊆X*).
* *A=∅* signifies the empty set.
* *A∩B* denotes the intersection of subsets *A* and *B.*
* *A∖B* represents the difference (or relative complement) between subsets *A* and *B.*
* The *powerset* of a set *A*, denoted by *2A*, is defined as the set of all possible subsets of *A*, including both the empty set and *A* itself.

**Definition 5 (Partition):** Throughout the paper, we utilize a finite set (referred to as the underlying set) *X*, a set of partitions *P*, and natural numbers *i, k,* and *p*. It is important to note that a partition involves dividing the elements of a set into non-empty, distinct subsets, ensuring that each element belongs to one and only one subset.

**Notation 6 (Collections of Subsets and Operations) [21,22]:** In this paper, we employ the symbol *α* to represent collections of subsets, such as *α* signifying a collection *A1, ..., Ak* of subsets of a finite set (the underlying set) *X*. The collection *α* is deemed a partition if the sets *Ai* are mutually disjoint, and their union forms the underlying set *X*. We define the following notation: if α represents the collection *A1, ..., Ak*, and *A* is another subset, then *α ∩ A* denotes the collection *A1 ∩ A, ..., Ak ∩ A*. Similarly, we use *α \ A* as a related notation. Lastly, *[B1, ..., Bp, α]* signifies the collection obtained from *α* by inserting sets *B1, ..., Bp* into the collection. Note that this notation is adopted from reference [22].

**2.1 Submodular Partition Functions: Functions that are essential in discussing graph width parameters**
We will explain about submodular partition functions. The definition of a partition function and a submodular partition functionof separations is provided below:

**Definition 7 (Partition Function)[21,22]:** Let *X* be a finite set. A function *ψ: P → ℤ₀⁺* that maps a partition *α* of *X* (where *α* is a collection of non-empty, pairwise disjoint subsets whose union equals *X*) to a non-negative integer is called a partition function if it satisfies:
*ψ([∅, α]) = ψ(α)*for every partition *α* of *X*. Here, the notation *[∅, α]* denotes the partition obtained by inserting the empty set into the collection *α;* that is, adding an empty set does not alter the value of *ψ.*

**Definition 8 (Submodular Partition Function)[21,22]:** A partition function *ψ* on *X* is said to be submodular if, for every pair of partitions *[A, α]* and *[B, β]* of *X* (where we write *[A, α]* to indicate a partition with a distinguished subset A and the residual collection α), the following inequality holds:
*ψ([A, α]) + ψ([B, β]) ≥ ψ([A ∪ (X\B), α ∩ B]) + ψ([B ∪ (X\A), β ∩ A]).*
Moreover, we assume that *ψ([X])=0,* since adding a constant to a submodular partition function does not affect its submodularity.
For any non-negative integer k, we denote by
*Pk[ψ] = {α* is a partition of *X : ψ(α) ≤ k}*the set of all partitions α of X whose value under ψ does not exceed k.

**Example 9:** Let *G=(X,E)* be an undirected graph with vertex set *X* and edge set *E*. For any subset S ⊆ X, define the cut function *f: 2X → ℤ₀⁺* by
*f(S) = |{{u,v}∈E : u∈S, v∈X\S}|.*
It is well known that the function *f* is submodular, i.e., for any subsets *S, T ⊆ X,
f(S) + f(T) ≥ f(S∪T) + f(S ∩ T).*
Now, for any partition *α = {A₁, A₂, ..., Ak}* of *X*, define the partition function *ψ* by
*ψ(α) = Σ f(Aᵢ), i=1* to *k.*
Since *f(∅)=0*, it follows that inserting an empty set into *α* does not change *ψ(α),* i.e., *ψ([∅,α]) = ψ(α).* Moreover, the submodularity　of *f* implies that *ψ* satisfies the submodularity condition in Definition, making *ψ* a submodular partitio n function.

The submodular partition function exhibits certain characteristics. Lemma 11 and Lemma 12, in particular, are obviously valid, and we provide a proof for clarity. This allows us to establish that the submodular partition function possesses a symmetric property.
**Lemma 10[22].** Let *ψ* be a submodular partition function on *X* and *[A, α]* a partition. Then *ψ([A, α]) ≥ ψ([A, X\A])* .
**Proof.** See, for example, reference [22]. *■*

**Lemma 11.** Let *ψ* be a submodular partition function on X. Then *ψ([A, X\A]) = ψ([X\A, A])*.
**Proof:** To prove this, we can use the submodular property of the partition function as given by the inequality:
*ψ([A, α]) + ψ([B, β]) ≥ ψ([A ∪ (X\B), α ∩ B]) + ψ([B ∪ (X\A), β ∩ A])*

Let's consider two sets A and B, where *B = X\A*. We will show that *ψ([A, X\A]) = ψ([X\A, A])* using the submodular property.
First, let *α = X\A* and *β = A*. Then, *α ∩ B = X\A ∩ (X\A)* = *X\A*, and *β ∩ A = A ∩ A = A*. Plugging these values into the inequality, we get:
*ψ([A, X\A]) + ψ([X\A, A]) ≥ ψ([A ∪ (X\(X\A)), X\A]) + ψ([(X\A) ∪ (X\A), A])*
Since *X\(X\A) = A*, we have:
*ψ([A, X\A]) + ψ([X\A, A]) ≥ ψ([A, X\A]) + ψ([X\A, A])*Thus, the inequality becomes an equality, which means the submodular property holds, and we have shown that *ψ([A, X\A]) = ψ([X\A, A])*.　This proof is completed. *■*

**Lemma 12.** Let *ψ* be a submodular partition function on X. Then *ψ(∅) = 0.****Proof:*** From the definition of a submodular partition function, we have:
*ψ([∅, α]) = ψ(α)* for every partition *α*.
Now, let's consider the partition *α = X*. Then we have:
*ψ([∅, X]) = ψ(X)*
By the assumption of submodular partition functions that *ψ([X]) = 0*, we get:
*ψ([∅, X]) = 0*
Which gives us the result that:
*ψ(∅) = 0*
This completes the proof of Lemma. *■*

**2.2. Loose *P-*Tangle for Submodular Partition Functions**
Next, we explain about Loose *P-*Tangle. The following is the definition of Loose *P*-tangle for submodular partition functions. A Loose *P-*tangle possesses dual properties to that of a decomposition tree [22].

**Definition 13 [22].** Let *ψ* be a submodular partition function on *X*. A loose *Pk[ψ]-*tangle is a family T of subsets of a finite set (an underlying set) X closed under taking subsets satisfying the following three axioms.
(P1) *∅ ∈ T, {e} ∈ T*, for all *e ∈ X* such that the partition *[{e}, X\{e}]* belongs to *Pk[ψ]*.
(P2) If *A1, A2, . . ., Ap ∈ T* , Ci ⊆ Ai for *i = 1, . . ., p, [C1, . . ., Cp, X\(*$\bigcup\_{j=1}^{p}C\_{j}$*)] ∈ Pk[ψ]*, then $\bigcup\_{j=1}^{p}C\_{j}$ *∈ T* .
*(P3) X ∉ T .*

Loose tangle is a concept closely related to graph width parameters introduce by reference [22]. It holds significant relevance in the study of graph structures and their associated graph width measures.

The concept of a decomposition tree for a submodular partition function *ψ,* introduced in reference [21], serves as a generalization of tree-width and branch-width. It is also recognized as the dual concept to Loose Tangle.

**Definition 14 [22]:** A decomposition tree for a finite set *X* is represented by a tree *T*, where each leaf of *T* corresponds to a distinct element of *X* through a one-to-one correspondence bijection denoted as *σ*. The internal nodes of *T* correspond to partitions of *X*, with each part consisting of the leaves contained in the subtrees formed by removing the node *v* from *T*. To be considered compatible with a set of partitions *P* of *X*, all partitions associated with the internal nodes of *T* must belong to *P.* The width of a submodular partition function ψ is defined as the smallest integer *k* for which there exists a decomposition tree compatible with *Pk[ψ]*.

The loose *Pk[ψ]*-tangle exhibits the following dual properties.
**Theorem 15 [22].** Let *ψ* be a submodular partition function on *X*. There is no decomposition tree compatible with *Pk[ψ]* if and only if there is a loose *Pk[ψ]*-tangle.
**Proof.** See, for example, reference [22]. *■*
 **2.3.　*P-*Filter for Submodular Partition Functions** We introduce new mathematical notion called *P-*Filter. This new definition holds an equivalent relationship with Loose *Pk[ψ]*-Tangle (see section 3).

First, we present the definition of a filter without imposing the submodularity condition. The definition of a filter in a Boolean algebra (X,∪,∩) is given below. This classical notion of a filter will later be extended to the setting of submodular partition functions, as described in the forthcoming definition of a Filter for Submodular Partition Functions. And it's important to note that a filter is classified as principal if it encompasses a singleton.

**Definition 16 [19,79]:** In a Boolean algebra *(X,∪,∩)*, a set family *F ⊆ 2X* satisfying the following conditions is called a filter on the carrier set *X*.

(FB1) *A, B ∈ F* ⇒ *A ∩ B ∈ F,*

(FB2) *A ∈ F, A ⊆ B ⊆ X* ⇒ *B ∈ F,*

(FB3) *∅* is not belong to *F*.

In a Boolean algebras *(X,∪,∩)*, A maximal filter is called an ultrafilter and satisfies the following axiom (FB4):

(FB4) *∀A ⊆ X, either A ∈ F or X / A ∈ F.*

The definition of Filter for submodular partition functions is below. The following definition of a filter incorporates the conditions of Submodular Partition Functions into the general definition of filters in mathematics.
**Definition 17:** Let *ψ* be a submodular partition function on a finite set *X*. An *Pk[ψ]*-(non-principal) filter of partitions is a family *F* satisfying the following four axiom:
(F1) For all *e ∈ X*, if the partition *[{e}, X\{e}] belongs to* *Pk[ψ],* the*n {e}∉ F*,
(F2) If *A1 ∈ F, A1 ⊆ A2,* [A2, X\(A2)] *∈ Pk[ψ],* then *A2 ∈ F*,(F3) If *A1, A2, . . .,Ai∈ F* for *i = 1, . . ., p,*

*[X\A1, . . ., X\ Ap, X\(*$\bigcup\_{j=1}^{p}X\A\_{j}$*)] ∈ Pk[ψ],* then $\bigcap\_{j=1}^{p}A\_{j}$ *∈ F*,
(F4) *∅ ∉ F.*

**Example 18:** Let *X* be a finite set (e.g., the vertex set of a simple and finite graph), and let *ψ* be a submodular partition function.
An example of such a function is the cut function on graphs. For any partition *α = {A₁, A₂, ..., Am}* of *X*, define *ψ(α) =* $\sum\_{i=1}^{m}f(A\_{i})$,
where for a subset *S ⊆ X,* the cut function *f* is given by
*f(S) = |{{u,v} ∈ E : u ∈ S, v ∈ X\S}|,*and *E* denotes the edge set of an undirected graph on *X.*
Then for a given *k ∈ ℤ₀⁺*, the set *Pk[ψ] = {α : ψ(α) ≤ k}* consists of all partitions of *X* whose *ψ*-cost does not exceed *k*. A collection *ℱ* of subsets of *X* that satisfies the axioms (F1)–(F4) above is a *Pk[ψ]*-filter of partitions under this concrete instance.

It's important to note that a filter is classified as principal if it encompasses a singleton.
The axioms that constitute the non-principal *Pk[ψ]* filter of partitions echo the conceptual underpinnings of a Sigma-filter. The Sigma-filter, acting as a selection mechanism for specific subsets within a sigma-algebra, plays a pivotal role in the exploration of measure and integration. Specifically, axiom (F3) is viewed as a counterpart to one of the axioms inherent in the Sigma-filter construct.
For reference, the definition of a Sigma-filter is provided below.

**Definition 19:** Le*t X* be a set and *Σ* be a sigma-algebra of subsets of *X*. A sigma filter on *X of Σ* is a collection *F* of subsets of *X* that satisfies the following properties:
(SF2) If *A ∈ F, B ∈ Σ, A ⊆ B, then B ∈ F*,
(SF3) If *A₁, A₂, A₃, ... ∈ F*, then $\bigcap\_{j=1}^{p}A\_{j}$ *∈ F*
(SF4) *∅ ∉ F.*

The non-principal *Pk[ψ]* filter introduced in this context can be perceived as a distinctive variant of the Sigma-filter, integrating conditions of a Submodular partition function and non-principal properties into its foundational definition.
Alongside its counterpart, the Sigma-ideal, both these constructs serve as vital tools in measure theory and probability theory, with extensive research dedicated to their understanding and application. Given the abundance of research conducted in the field of sigma-algebras, it can be considered as one of the crucial areas of study (ex. [23-27]).
 **3. Cryptomorphism between Loose tangle and Filter for Submodular Partition Functions**In this section, we demonstrate the cryptomorphism between Loose *P-*tangle and Filter for Submodular Partition Functions. The main result of this section is presented below. This theorem means that a filter is an obstruction of a decomposition tree. **Theorem 20.** Let *ψ* be a submodular partition function on a finite set *X*. T is a loose *Pk[ψ]-*tangle iff *F = {A | X\A ∈ T }* is a *Pk[ψ]-*(non-principal) filter.

**Proof:**

We'll prove the theorem in two steps:

* First, we'll show that if T is a loose *Pk[ψ]*-tangle, then *F = {A | X\A ∈ T }* is a *Pk[ψ]-*(non-principal) filter.
* Secondly, we'll show that if F is a *Pk[ψ]-*( (non-principal) filter, then *T = {A | X\A ∈ F}* is a loose *Pk[ψ]-*tangle.

**Part 1:**

Assume that *T* is a loose *Pk[ψ]*-tangle. We'll show that *F = {A | X\A ∈ T }* is a *Pk[ψ]*-(non-principal) filter.

Let's show axiom (F1). If *[{e}, X\{e}]* belongs to *Pk[ψ]*, then by (P1) in the definition of *T*, we have *{e} ∈ T*. Hence, *X\{e}* is in *F.*

Now, let's show axiom (F2).Suppose *A1 ∈ F, A1 ⊆ A2*, and *[A2, X\A2] ∈ Pk[ψ]*. Since *X\A1 ∈ T* and *X\A2 ⊆ X\A1,* by the closure of *T* under taking subsets and *[X\A2, A2] ∈ Pk[ψ]*, we have *X\A1 ∈ T*. Hence *A1 = X\(X\A2) ∈ F*.

Let's show axiom (F3). Suppose that *A1, A2, ...,Ai ∈ F* for *i = 1, ..., p*, and *[X\A1, . . .,X\ Ap, X\(*$\bigcup\_{j=1}^{p}X\A\_{j}$*)] ∈ Pk[ψ]*. By definition of *F*, *X\Ai ∈ T.* Thus, by axiom (P2) in the definition of *T and [X\A1, . . .,X\ Ap, X\(*$\bigcup\_{j=1}^{p}X\A\_{j}$*)] ∈ Pk[ψ],* $\bigcup\_{j=1}^{p}X\A\_{j}$ *∈ T*. Therefore, by definition of *F*, X\($\bigcup\_{j=1}^{p}X\A\_{j}$) = $\bigcap\_{j=1}^{p}A\_{j}$ is in F.

Finally, let's show axiom (F4). By axiom (P3) in the definition of T, we have X ∉ T. Therefore, X\X = ∅ is in F.

Thus, if T is a loose *Pk[ψ]-t*angle, then *F = {A | X\A ∈ T }* is a *Pk[ψ]-* (non-principal) filter.

**Part 2:**

Now, assume that *F* is a *Pk[ψ]-*(non-principal) filter. We'll show that *T = {A | X\A ∈ F}* is a loose *Pk[ψ]*-tangle.

Let's show axiom (P1). If *[{e}, X\{e}]* belongs to *Pk[ψ]*, then by axiom (F1) in the definition of *F*, we have *{e} ∉ F.* Hence, *X\{e} ∉ T*.

Let's show axiom (P2). Suppose that *A1, A2, ..., Ap* belong to *T*, *Ci ⊆ Ai* for *i = 1, ..., p*, and *[C1, ..., Cp, X\(*$\bigcup\_{j=1}^{p}C\_{j}$*)]* belongs to *Pk[ψ]*. By definition of *T*, we have *X\Ai ∈ F* and *X\Ai ⊆ X\Ci*. Thus, by axiom (F2) in the definition of *F, X\C1, X\C2, …, X\Cp* is in *F*. By axiom (F3) in the definition of F, $\bigcap\_{j=1}^{p}X\C\_{j}$*=* $\bigcup\_{j=1}^{p}C\_{j}$is in *T*. So axiom (P2) holds.

Finally, let's show (P3). By (F4) in the definition of *F*, we have ∅ ∉ F. Therefore, *X\∅ = X ∉ T.*

Thus, if *F* is a *Pk[ψ]-* (non-principal) filter, then *T = {A | X\A ∈ F}* is a loose *Pk[ψ]*-tangle.

Hence, based on parts 1 and 2, the theorem is proven, thus concluding the proof. *■*

**4. Filter of partitions and Bramble of partitions: Obstructions to Decomposition tree**

In this section, we discuss about filter of partitions and bramble of partitions. Inspired by reference [22], we redefine the concept of Bramble for submodular partition function, which serves as a fundamental dual concept to width parameters such as Tree-width and branch-width, and tree-cut width [29-35].

**Definition 20[21]:**  Let *ψ* be a submodular partition function on a finite set *X*. A (non-principal) *k*-bramble, denoted as *L*, is a nonempty family of subsets of *X* satisfying the following conditions:
(B1)For any *A* and *B* belonging to *L*, their intersection *A ∩ B* is not empty.
(B2)For every *[A1, . . . , An] ∈ Pk[ψ]*, there exists *Ai* in *L*.
(B3)For all *e ∈ X*, if the partition *[{e}, X\{e}] belongs to* *Pk[ψ],* the*n {e}∉ L*,

In the case of a non-principal *k*-bramble for submodular partition function, the following holds true.
**Lemma 21:** Let *X* be a finite set. A (non-principal) *k*-bramble satisfies following conditions:
(B4) If *A1 ∈ L, A1 ⊆ A2,* *[A2, X\(A2)] ∈ Pk[ψ],* then *A2 ∈ L,*
(B5) *∅ ∉ L.***Proof:** (B4): Suppose *A1 ∈ L* and *A1 ⊆ A2 ⊆ X* and *α = [A2, X\A2] ∈ Pk[ψ]*. By axiom (B2), one part must be in *L*. If *X\A2 ∈ L*, then *A1 ∩ (X\A2) ≠ ∅* contradicts *A1 ⊆ A2.* Thus, *A2 ∈ L.*

(B5): If *∅ ∈ L, t*hen for an*y A ∈ L, ∅ ∩ A = ∅,* contradicting (B1). Hence *∅ ∉ L.* This completes the proof of Lemma. *■*
Furthermore, it is known that the following holds true in the context of Bramble.
**Lemma 22[21]:** Let *L* be a *k-*bramble corresponding to the partition function. For every *A, B, C* in *L*, the intersection *A ∩ B ∩ C* is non-empty.

**Proof.** See, for example, reference [21]. *■*

In this context, we introduce the concept of an ultrafilter for Submodular Partition Functions. In mathematics, ultrafilters have a crucial role in different fields, including set theory, topology, and functional analysis. They provide a way to understand concepts of convergence, compactness, and maximality within mathematical structures.
In this paper, we introduce an additional axiom (F5) for the *Pk[ψ]*-(non-principal) filter of partitions. We refer to this as a *Pk[ψ]*-(non-principal) ultrafilter. (F5) If *[A1, X\A1] ∈ Pk[ψ], either A1 ∈ F* or *X* ***\*** *A1 ∈ F.*
The following theorem discusses the maximality of ultrafilters. **Theorem 23:** Let *ψ* be a submodular partition function on a finite set *X*. Maximal *Pk[ψ]*-(non-principal) filter satisfies axiom (F5).(F5) If *[A1, X\A1] ∈ Pk[ψ], either A1 ∈ F* or *X* ***\*** *A1 ∈ F.***Proof:** Assume that *F* is a maximal *Pk[ψ]*-(non-principal) filter, but that *F* does not satisfy axiom (F5). This means there exists a partition *[A1, X \ A1]* such that neither *A1 ∈ F* nor *X \ A1 ∈ F. ψ* is a submodular partition function on *X*, meaning *ψ([A, α]) + ψ([B, β]) ≥ ψ([A ∪ (X \ B), α ∩ B]) + ψ([B ∪ (X \ A), β ∩ A])* for any partitions *[A, α]* and *[B, β]*.
*F* is a *Pk[ψ]-(*non-principal) filter, meaning it satisfies axioms (F1) through (F4).
Assume *[A1, X \ A1]* ∈ *Pk[ψ]*.By our assumption, neither *A1 ∈ F* nor *X \ A1 ∈ F*. Since *F* is a maximal filter, it contains the largest possible subsets that can be accommodated while still satisfying the conditions *Pk[ψ].*
If neither *A1* nor *X \ A1* is in *F,* then *F* is missing these significant partitions, suggesting *F* is not maximal, contradicting our assumption.

The function *ψ* should satisfy the submodular inequality for the partitions *[A1, X \ A1]* and their complements. So we obtain *ψ([A1, X \ A1]) + ψ([X \ A1, A1]) ≥ ψ([A1, X \ A1]) + ψ([X \ A1, A1]).* This inequality holds trivially, reinforcing that the partitions *A1* and *X \ A1* should be consistent with *ψ*'s submodularity.
Since neither *A1 ∈ F* nor *X \ A1 ∈ F,* this implies that adding either to *F* must violate a filter axiom. Specifically, if *A1* is not in *F* and X \ *A1* is not in F, then for any partition *[A1, X \ A1],* its complement is also not in *F,* which violates the maximality of *F.*This contradiction arises from our initial assumption that *F* does not satisfy (F5).Therefore, a family F must satisfy axiom (F5) to maintain its maximal *Pk[ψ]*-(non-principal) filter status. *■*We can infer from the given lemma that there appears to be a close relationship between Bramble and non-principal ultrafilters, specifically within the context of *Pk[ψ]*. **Lemma 24:** Let *L* be a *Pk[ψ]*-(non-principal) ultrafilter corresponding to the partition function. For every *A, B, C* in *L*, the intersection *A ∩ B ∩ C* is non-empty. **Proof of:** Proof of this lemma can be established similarly to Lemma 22. *■*The main result of this section is presented below. **Theorem 25.** Let *ψ* be a submodular partition function on a finite set *X*. *T* is a k-Bramble if *T* is a *Pk[ψ]-*(non-principal) ultrafilter.

**Proof:** Now, suppose *F* is a *Pk[ψ]*-(non-principal) ultrafilter.

We will show that *F* satisfies the properties of a *k-*Bramble.

We show that axiom (B1) holds. Condition (F3) ensure the non-emptiness of the intersection of any subsets in *F*, hence satisfying Condition (B1)

We show that axiom (B2) holds. If *[A1, . . . , An] ∈ Pk[ψ]*, we know from condition (F5) that there must exist some *Ai* in *F*, satisfying condition (B2).

We show that axiom (B3) holds. Condition (F1) is precisely condition (B3).

Therefore, all conditions for *F* to be a *k-*bramble are satisfied.　　■

Now, let's consider the relationship between the decomposition tree and the various concepts. It is worth noting that the duality theorem for submodular partition functions is well-established.

 **Theorem 26 [21].** Let *ψ* be a submodular partition function and *k* a non-negative integer. There is no decomposition tree compatible with *Pk[ψ]* if and only if there is a non-principal *Pk[ψ]*-bramble. **Proof.** See, for example, reference [21]. *■*

The following theorem clearly holds true.

**Theorem 27:** Let *ψ* be a submodular partition function and *k* a non-negative integer. If there is no decomposition tree compatible with *Pk[ψ] then*

-　There is a non-principal *Pk[ψ]*-bramble.

-　There is a non-principal *Pk[ψ]*-ultrafilter.

 **5. Note: Tangle and Ideal for submodular partition function**In our current discourse, we delve into the concept of Tangle as it pertains to submodular partition functions. Originally introduced by Robertson and Seymour [32], the notion of branch-width for connectivity functions *f* established its characterization in graphs through the use of 'tangles.' In the context of connectivity systems, we refer to these as *f-*tangles. Building upon this foundation, we extend the concept of tangles to submodular partition functions by incorporating the definition provided in reference [10].

**Definition 28:** Let *ψ* be a submodular partition function on a finite set *X*. A  *Pk[ψ]-*tangle is a family T of subsets of a finite set (an underlying set) X closed under taking subsets satisfying the following axioms:

(TG1) For each *B ∈ T , [B,X\B] ∈ Pk[ψ].*

(TG2) For each *[A, B] ∈ Pk[ψ],* *T* contains *A* or *B*.

(TG3a) If *A ⊆ B*, *B ∈ T* , and *[A,X\A] ∈ Pk[ψ]*, then *A ∈ T* .

(TG3b) If *[A, B, C] ∈ Pk[ψ]*, then T cannot contain all three of *A, B,* and C.

(TG4) For each *[{e}, X\{e}]∈ Pk[ψ], X \ {e} ∉ T*The literature [10] demonstrates the relationship between *f-*tangles and *Pk[ψ]-*brambles. Furthermore, based on the discussions presented in this paper, several conclusions have become apparent. **Theorem 29:** Let *ψ* be a submodular partition function and *k* a non-negative integer. If there is no decomposition tree compatible with *Pk[ψ] then*

-　There is a non-principal *Pk[ψ]*-bramble.

-　There is a non-principal *Pk[ψ]*-ultrafilter.

-　There is a *Pk[ψ]*-Tangle.

The aforementioned statement represents the main theorem of this paper, known as the duality theorem. However, in this paper, we have also been able to present another dual theorem.

In general mathematics, ideals correspond to co-filters, and maximal ideals correspond to co-ultrafilters. In the context of submodular partition functions, which incorporate specific conditions, *Pk[ψ]*-ideals correspond to co-*Pk[ψ]-*filters. Drawing upon the discussions presented in this paper, these conclusions become evident.
**Theorem 30:** Let *ψ* be a submodular partition function and *k* a non-negative integer. If there is no decomposition tree compatible with *Pk[ψ] then*

-　There is a loose *Pk[ψ]*-tangle.

-　There is a non-principal *Pk[ψ]*-filter.

-　There is a non-principal *Pk[ψ]*-ideal.Based on the maximality of the *Pk[ψ]*-ideal (co-filter) and ultrafilter introduced in Theorem, the following holds for a maximal *Pk[ψ]*-ideal (co-ultrafilter). **Theorem 31:** Let *ψ* be a submodular partition function and *k* a non-negative integer. The existence of the following are equivalent conditions:

-　There is no decomposition tree compatible with *Pk[ψ].*

-　There is a maximal *Pk[ψ]*-ideal.

 **6. Conclusion and Future tasks: Consideration of Weak Filter**

This paper explored the intricate relationship between Loose Tangles and Filters using a submodular partition function as a foundational framework, and clarified some of their characteristics and connections to graph width parameters.

We will consider about Weak filter of submodular partition function. Weak filter is a concept used in the world of logic [40, 41, 42]. Definition of Weak filter of submodular partition function is below. It is worth noting that since a weak ideal is a co-weak filter, we can consider a weak ideal with the additional conditions of a submodular partition function to also be a co-weak filter.

**Definition 32:** Let *ψ* be a submodular partition function on a finite set *X*. A *Pk[ψ]*-(non-principal) weak filter of partitions is a family *F* satisfying the following four axiom:
(F1) For all *e ∈ X*, if the partition *[{e}, X\{e}] belongs to* *Pk[ψ],* the*n {e}∉ F*,
(F2) If *A1 ∈ F, A1 ⊆ A2,* [A2, X\(A2)] *∈ Pk[ψ],* then *A2 ∈ F*,(WF3) If *A1, A2, . . .,Ai ∈ F* for *i = 1, . . ., p, [X\A1, . . .,X\ Ap, X\((*$\bigcup\_{j=1}^{p}X\A\_{j}$*)] ∈ Pk[ψ],* then $\bigcap\_{j=1}^{p}A\_{j}$ *≠ ∅*,
(F4) *∅ ∉ F.*

Regarding the above definition of a Weak Filter, the axiom (WF3) is the part that differs from the definition of a Filter using a submodular partition function.

I would also like to explore research on game-theoretic approaches to graph width parameters [46-50], as well as studies on submodular functions in the context of fuzzy sets [80-82], vague sets [100-102], picture fuzzy sets [109-111], soft sets [106-108], plithogenic sets [103-105], and neutrosophic sets [83-85].

Disclaimer (Artificial intelligence)

Option 1:

Author(s) hereby declare that NO generative AI technologies such as Large Language Models (ChatGPT, COPILOT, etc.) and text-to-image generators have been used during the writing or editing of this manuscript.

Option 2:

Author(s) hereby declare that generative AI technologies such as Large Language Models, etc. have been used during the writing or editing of manuscripts. This explanation will include the name, version, model, and source of the generative AI technology and as well as all input prompts provided to the generative AI technology

Details of the AI usage are given below:

1.

2.

3.

**Reference**

[1] Oum, Sang-il, and Paul Seymour. "Testing branch-width." *Journal of Combinatorial Theory, Series B* 97.3 (2007): 385-393.

[2] HICKS, Illya V.; BRIMKOV, Boris. Tangle bases: Revisited. Networks, 2021, 77.1: 161-172.

[3] Gurski, F., & Rehs, C. (2019). Comparing linear width parameters for directed graphs. Theory of Computing Systems, 63, 1358-1387.

[4] Thilikos, D. M., Serna, M. J., & Bodlaender, H. L. (2000, December). Constructive linear time algorithms for small cutwidth and carving-width. In *International Symposium on Algorithms and Computation* (pp. 192-203). Berlin, Heidelberg: Springer Berlin Heidelberg.

[5] Fujita, T. (2025). Bounding linear-width and distance-width using feedback vertex set and MM-width for graph. *Journal of Fundamental Mathematics and Applications (JFMA)*, *8*(1), 33-50.

[6] Fedor V Fomin and Dimitrios M Thilikos. On the monotonicity of games generated by symmetric submodular functions. Discrete Applied Mathematics, Vol. 131, No. 2, pp. 323–335, 2003.

[7] Fujita, T. (2024). Bounded tree-depth, path-distance-width, and linear-width of graphs. *Journal of Fundamental Mathematics and Applications (JFMA)*, *7*(2), 138-148.

[8] P. Seymour and R. Thomas. Graph searching and a min-max theorem for tree-width. Journal of Combinatorial Theory, Series B, Vol. 58, No. 1, pp. 22–23, 1993.

[9] Isolde Adler. Games for width parameters and monotonicity. arXiv preprint arXiv:0906.3857, 2009.

[10] Jim Geelen, Bert Gerards, Neil Robertson, and Geoff Whittle. Obstructions to branch-decomposition of matroids. Journal of Combinatorial Theory, Series B, Vol. 96, No. 4, pp. 560–570, 2006.

[11] Di Lavore, Elena, and Paweł Sobociński. "Monoidal Width: Unifying Tree Width, Path Width and Branch Width." *arXiv preprint arXiv:2202.07582* (2022).

[12] Paul, Christophe, Evangelos Protopapas, and Dimitrios M. Thilikos. "Graph Parameters, Universal Obstructions, and WQO." arXiv preprint arXiv:2304.03688 (2023).

[13] Reed, Bruce A. "Tree width and tangles: A new connectivity measure and some applications." Surveys in combinatorics (1997): 87-162.

[14] KURKOFKA, Jan. Ends and tangles, stars and combs, minors and the Farey graph. 2020. PhD Thesis. Staats-und Universitätsbibliothek Hamburg Carl von Ossietzky.

[15] Fujita, Takaaki. "Proving Maximal Linear Loose Tangle as a Linear Tangle." *Asian Research Journal of Mathematics* 20.2 (2024): 48-54.

[16] Fujita, Takaaki. "Revisiting Linear Width: Rethinking the Relationship between Single Ideal and Linear Obstacle." *Journal of Advances in Mathematics and Computer Science* 38.10 (2023): 167-171.

[17] Fujita, Takaaki. 2025. “Semi-Matroids on Connectivity System and Linear Decomposition”. Asian Research Journal of Mathematics 21 (4):1-13. https://doi.org/10.9734/arjom/2025/v21i4906.

[18] Fomin, Fedor V., and Tuukka Korhonen. "Fast fpt-approximation of branchwidth." Proceedings of the 54th Annual ACM SIGACT Symposium on Theory of Computing. 2022.

[19] T Fujita, T. (2024). Various properties of various ultrafilters, various graph width parameters, and various connectivity systems (with survey). *arXiv preprint arXiv:2408.02299*.

[20] Daniel Bienstock. Graph searching, path-width, tree-width and related problems (a survey). Reliability of Computer and Communication Networks , Vol.DIMACS. Series in Discrete Mathematics and Theoretical Computer Science , pp. 33‒50, 1989.

[21] Amini, Omid, et al. "Submodular partition functions." *Discrete Mathematics* 309.20 (2009): 6000-6008.

[22] Škoda, Petr. "Computability of width of submodular partition functions." *Combinatorial Algorithms*. Vol. 5874. Springer-Verlag Berlin, Heidelberg, 2009. 450-459.

[23] Halton, J. H. (2008). Sigma-algebra theorems.

[24] Kubrusly, C. S., & Kubrusly, C. S. (2015). Measure on a σ-Algebra. Essentials of Measure Theory, 23-39.

[25] Gentili, Stefano. "Monotone Classes and σ-Algebras." Measure, Integration and a Primer on Probability Theory: Volume 1. Cham: Springer International Publishing, 2020. 131-145.

[26] Ohba, Sachio. "Topological-group-valued measures." *Yokohama Math. J* 22 (1974): 101-104.

[27] Calin, Ovidiu, and Ovidiu Calin. "Information Representation." *Deep Learning Architectures: A Mathematical Approach* (2020): 317-349.

[28] Fujita , T. (2023). Alternative Proof of Linear Tangle and Linear Obstacle: An Equivalence Result. Asian Research Journal of Mathematics, 19(8), 61–66.

[29] Fomin, Fedor V., Petr Golovach, and Dimitrios M. Thilikos. "Contraction obstructions for treewidth." *Journal of Combinatorial Theory, Series B* 101.5 (2011): 302-314.

[30] Giannopoulou, Archontia C., et al. "Cutwidth: obstructions and algorithmic aspects." *Algorithmica* 81 (2019): 557-588.

[31] Fujita, Takaaki. "Quasi-Ultrafilter on the Connectivity System: Its Relationship to Branch-Decomposition." *International Journal of Mathematics Trends and Technology-IJMTT* 70 (2024).

[32] Robertson, Neil, and Paul D. Seymour. "Graph minors. X. Obstructions to tree-decomposition." Journal of Combinatorial Theory, Series B 52.2 (1991): 153-190.

[33] Fujita, Takaaki. 2025. “Ultrafilters and Their Dual Relationship to Tree-Width in Graph Theory”. Asian Research Journal of Mathematics 21 (1):98-114. https://doi.org/10.9734/arjom/2025/v21i1886.

[34] Lucena, Brian. "Achievable sets, brambles, and sparse treewidth obstructions." Discrete applied mathematics 155.8 (2007): 1055-1065.

[35] Collins, K. L., & Smith, B. C. (2017). Treewidth Bounds for Planar Graphs Using Three-Sided Brambles. *arXiv preprint arXiv:1706.08581*.

[36] Hatzel, Meike, et al. "Constant Congestion Brambles." Discrete Mathematics & Theoretical Computer Science 24.Graph Theory (2022).

[37] Lyaudet, Laurent, Frédéric Mazoit, and Stéphan Thomassé. "Partitions versus sets: a case of duality." European journal of Combinatorics 31.3 (2010): 681-687.

[38] Sonuc, Sibel B., J. Cole Smith, and Illya V. Hicks. "A branch-and-price-and-cut method for computing an optimal bramble." Discrete Optimization 18 (2015): 168-188.

[39] Sorge, Manuel. "Constant Congestion Brambles in Directed Graphs." *Extended Abstracts EuroComb 2021: European Conference on Combinatorics, Graph Theory and Applications*. Vol. 14. Springer Nature, 2021.

[40] Koutras, Costas D., Christos Moyzes, and Christos Rantsoudis. "A reconstruction of default conditionals within epistemic logic." Proceedings of the Symposium on Applied Computing. 2017.

[41] Koutras, Costas D., et al. "On weak filters and ultrafilters: Set theory from (and for) knowledge representation." Logic Journal of the IGPL 31.1 (2023): 68-95.

[42] Askounis, Dimitris, Costas D. Koutras, and Yorgos Zikos. "Knowledge means ‘all’, belief means ‘most’." Journal of Applied Non-Classical Logics 26.3 (2016): 173-192.

[43] Fujita, Takaaki. Bramble for Submodular Partition Function. Preprints. (2023).

[44] Fujita, Takaaki. "Filter for Submodular Partition Function: Connection to Loose Tangle." Preprints. (2023).

[45] Thomas, Robin, et al. "Directed tree-width." Slides from lecture at the Regional NSF-CBMS Conference. 2002.

[46] Wang, Lusheng, and Boting Yang. "The one-cop-moves game on graphs of small treewidth." Combinatorial Optimization and Applications: 13th International Conference, COCOA 2019, Xiamen, China, December 13–15, 2019, Proceedings 13. Springer International Publishing, 2019.

[47] Evans, William, Paul Hunter, and Mohammad Ali Safari. D-width and cops and robbers. Research report, 2013.

[48] PETERS, Dominik. Graphical hedonic games of bounded treewidth. In: *Proceedings of the AAAI Conference on Artificial Intelligence*. 2016.

[49] BERWANGER, Dietmar, et al. DAG-width and parity games. In: *Annual Symposium on Theoretical Aspects of Computer Science*. Berlin, Heidelberg: Springer Berlin Heidelberg, 2006. p. 524-536.

[50] Soares, Ronan Pardo. Pursuit-evasion, decompositions and convexity on graphs. Diss. Université Nice Sophia Antipolis, 2013.

[51] Gurski, Frank, et al. "Directed width parameters on semicomplete digraphs." Combinatorial Optimization and Applications: 15th International Conference, COCOA 2021, Tianjin, China, December 17–19, 2021, Proceedings 15. Springer International Publishing, 2021.

[52] Chapelle, Mathieu, Frédéric Mazoit, and Ioan Todinca. "Constructing brambles." *International Symposium on Mathematical Foundations of Computer Science*. Berlin, Heidelberg: Springer Berlin Heidelberg, 2009.

[53] Hatzel, Meike, et al. "Constant congestion brambles." *Discrete Mathematics & Theoretical Computer Science* 24.Graph Theory (2022).

[54] Hatzel, Meike. *Dualities in graphs and digraphs*. Universitätsverlag der Technischen Universität Berlin, 2023.

[55] Lardas, Emmanouil, et al. "On Strict Brambles." *Graphs and Combinatorics* 39.2 (2023): 24.

[56] Grohe, Martin, and Dániel Marx. "On tree width, bramble size, and expansion." *Journal of Combinatorial Theory, Series B* 99.1 (2009): 218-228.

[57] Fujita, T., & Smarandache, F. (2024). Fundamental computational problems and algorithms for superhypergraphs. *Advancing Uncertain Combinatorics through Graphization, Hyperization, and Uncertainization: Fuzzy, Neutrosophic, Soft, Rough, and Beyond (Second Volume)*.

[58] Fujita, T. (2025). Superhypertree-length and superhypertree-breadth in superhypergraphs. *Advancing Uncertain Combinatorics through Graphization, Hyperization, and Uncertainization: Fuzzy, Neutrosophic, Soft, Rough, and Beyond*, 41.

[59] Fujita, T. (2025). Superhyperbranch-width and Superhypertree-width. *Advancing Uncertain Combinatorics through Graphization, Hyperization, and Uncertainization: Fuzzy, Neutrosophic, Soft, Rough, and Beyond*, 367.

[60] Fujita, T. Superhypertree-depth: A structural analysis within superhypergraphs. *Advancing Uncertain Combinatorics through Graphization, Hyperization, and Uncertainization: Fuzzy, Neutrosophic, Soft, Rough, and Beyond*, 11.

[61] Fujita, T. (2025). Obstruction for hypertree width and superhypertree width. *Advancing Uncertain Combinatorics through Graphization, Hyperization, and Uncertainization: Fuzzy, Neutrosophic, Soft, Rough, and Beyond*, 26.

[62] Gottlob, G., Leone, N., & Scarcello, F. (1999, May). Hypertree decompositions and tractable queries. In *Proceedings of the eighteenth ACM SIGMOD-SIGACT-SIGART symposium on Principles of database systems* (pp. 21-32).

[63] Adler, I., Gottlob, G., & Grohe, M. (2007). Hypertree width and related hypergraph invariants. *European Journal of Combinatorics*, *28*(8), 2167-2181.

[64] Bonnet, É., Geniet, C., Kim, E. J., Thomassé, S., & Watrigant, R. (2021). Twin-width II: small classes. In *Proceedings of the 2021 ACM-SIAM Symposium on Discrete Algorithms (SODA)* (pp. 1977-1996). Society for Industrial and Applied Mathematics.

[65] Ahn, J., Hendrey, K., Kim, D., & Oum, S. I. (2022). Bounds for the twin-width of graphs. *SIAM Journal on Discrete Mathematics*, *36*(3), 2352-2366.

[66] Bonnet, É., Kim, E. J., Thomassé, S., & Watrigant, R. (2021). Twin-width I: tractable FO model checking. *ACM Journal of the ACM (JACM)*, *69*(1), 1-46.

[67] Gajarský, J., Lampis, M., & Ordyniak, S. (2013). Parameterized algorithms for modular-width. In *Parameterized and Exact Computation: 8th International Symposium, IPEC 2013, Sophia Antipolis, France, September 4-6, 2013, Revised Selected Papers 8* (pp. 163-176). Springer International Publishing.

[68] Abu-Khzam, F. N., Li, S., Markarian, C., Meyer auf der Heide, F., & Podlipyan, P. (2017). Modular-width: An auxiliary parameter for parameterized parallel complexity. In *Frontiers in Algorithmics: 11th International Workshop, FAW 2017, Chengdu, China, June 23-25, 2017, Proceedings 11* (pp. 139-150). Springer International Publishing.

[69] Erdijs, P., Hell, P., & Winkler, P. (1989). Bandwidth versus bandsize. *Annals of Discrete Mathematics*, *41*, 117-130.

[70] Chinn, P. Z., Chvátalová, J., Dewdney, A. K., & Gibbs, N. E. (1982). The bandwidth problem for graphs and matrices—a survey. *Journal of Graph Theory*, *6*(3), 223-254.

[71] Kaplan, H., & Shamir, R. (1996). Pathwidth, bandwidth, and completion problems to proper interval graphs with small cliques. *SIAM Journal on Computing*, *25*(3), 540-561.

[72] Da Lozzo, G., Eppstein, D., Goodrich, M. T., & Gupta, S. (2021). C-planarity testing of embedded clustered graphs with bounded dual carving-width. *Algorithmica*, *83*(8), 2471-2502.

[73] Jakes-Schauer, J., Anekstein, D., & Wocjan, P. (2019). Carving-width and contraction trees for tensor networks. *arXiv preprint arXiv:1908.11034*.

[74] West, D. B. (2001). *Introduction to graph theory* (Vol. 2). Upper Saddle River: Prentice hall.

[75] Fujishige, S. (2005). *Submodular functions and optimization* (Vol. 58). Elsevier.

[76] Krause, A., & Golovin, D. (2014). Submodular function maximization. *Tractability*, *3*(71-104), 3.

[77] Levy, A. (2012). *Basic set theory*. Courier Corporation.

[78] Fraenkel, A. A., Bar-Hillel, Y., & Levy, A. (1973). *Foundations of set theory* (Vol. 67). Elsevier.

[79] Blass, A., Bergelson, V., Di Nasso, M., & Jin, R. (2010). Ultrafilters and set theory. *Ultrafilters across mathematics*.

[80] Zimmermann, H. J. (2010). Fuzzy set theory. *Wiley interdisciplinary reviews: computational statistics*, *2*(3), 317-332.

[81] Klir, G., & Yuan, B. (1995). *Fuzzy sets and fuzzy logic* (Vol. 4, pp. 1-12). New Jersey: Prentice hall.

[82] Fujita, T. (2025). A study on hyperfuzzy hyperrough sets, hyperneutrosophic hyperrough sets, and hypersoft hyperrough sets with applications in cybersecurity. *Artificial Intelligence in Cybersecurity*, *2*, 14-36.

[83] Fujita, T., & Smarandache, F. (2025). A concise introduction to hyperfuzzy, hyperneutrosophic, hyperplithogenic, hypersoft, and hyperrough sets with practical examples. *Neutrosophic Sets and Systems*, *80*, 609-631.

[84] El-Hefenawy, N., Metwally, M. A., Ahmed, Z. M., & El-Henawy, I. M. (2016). A review on the applications of neutrosophic sets. *Journal of Computational and Theoretical Nanoscience*, *13*(1), 936-944.

[85] Wang, H., Smarandache, F., Zhang, Y., & Sunderraman, R. (2010). *Single valued neutrosophic sets*. Infinite study.

[86] Foulds, L. R. (1995). *Graph theory applications*. Springer Science & Business Media.

[87] Chen, W. K. (2012). *Applied graph theory* (Vol. 13). Elsevier.

[88] Sadavare, A. B., & Kulkarni, R. V. (2012). A review of application of graph theory for network. *International Journal of Computer science and Information technologies*, *3*(6), 5296-5300.

[89] Bui-Xuan, B. M., Telle, J. A., & Vatshelle, M. (2011). Boolean-width of graphs. *Theoretical computer science*, *412*(39), 5187-5204.

[90] Sharmin, S. (2014). Practical aspects of the graph parameter boolean-width.

[91] Adler, I., Bui-Xuan, B. M., Rabinovich, Y., Renault, G., Telle, J. A., & Vatshelle, M. (2010, June). On the boolean-width of a graph: Structure and applications. In *International Workshop on Graph-Theoretic Concepts in Computer Science* (pp. 159-170). Berlin, Heidelberg: Springer Berlin Heidelberg.

[92] Gurski, F., & Wanke, E. (2005). On the relationship between NLC-width and linear NLC-width. *Theoretical Computer Science*, *347*(1-2), 76-89.

[93] Gurski, F., & Wanke, E. (2009). The NLC-width and clique-width for powers of graphs of bounded tree-width. *Discrete Applied Mathematics*, *157*(4), 583-595.

[94] Gurski, F., & Wanke, E. (2005). Minimizing NLC-width is NP-complete. In *Graph-Theoretic Concepts in Computer Science: 31st International Workshop, WG 2005, Metz, France, June 23-25, 2005, Revised Selected Papers 31* (pp. 69-80). Springer Berlin Heidelberg.

[95] Wood, D. R. (2009). On tree-partition-width. *European Journal of Combinatorics*, *30*(5), 1245-1253.

[96] Distel, M., & Wood, D. R. (2022). Tree-partitions with small bounded degree trees. *arXiv preprint arXiv:2210.12577*.

[97] Bodlaender, H. L., Groenland, C., & Jacob, H. (2025). On the parameterized complexity of computing tree-partitions. *Discrete Mathematics & Theoretical Computer Science*, *26*(Discrete Algorithms).

[98] Fujita, T. (2024). A brief overview of applications of tree-width and other graph width parameters. *preprint (researchgate)*.

[99] Hliněný, P., Oum, S. I., Seese, D., & Gottlob, G. (2008). Width parameters beyond tree-width and their applications. *The computer journal*, *51*(3), 326-362.

[100] Lu, A., & Ng, W. (2005, October). Vague sets or intuitionistic fuzzy sets for handling vague data: which one is better?. In *International conference on conceptual modeling* (pp. 401-416). Berlin, Heidelberg: Springer Berlin Heidelberg.

[101] Hong, D. H., & Choi, C. H. (2000). Multicriteria fuzzy decision-making problems based on vague set theory. *Fuzzy sets and systems*, *114*(1), 103-113.

[102] Bonikowski, Z., & Wybraniec-Skardowska, U. (2007). Rough sets and vague sets. In *Rough Sets and Intelligent Systems Paradigms: International Conference, RSEISP 2007, Warsaw, Poland, June 28-30, 2007. Proceedings 1* (pp. 122-132). Springer Berlin Heidelberg.

[103] Sudha, S., Martin, N., & Smarandache, F. (2023). *Applications of extended plithogenic sets in plithogenic sociogram*. Infinite Study.

[104] Fujita, T. (2025). HyperPlithogenic Cubic Set and SuperHyperPlithogenic Cubic Set. *Advancing Uncertain Combinatorics through Graphization, Hyperization, and Uncertainization: Fuzzy, Neutrosophic, Soft, Rough, and Beyond*, 79.

[105] Fujita, T. (2025). Plithogenic rough sets. *Advancing Uncertain Combinatorics through Graphization, Hyperization, and Uncertainization: Fuzzy, Neutrosophic, Soft, Rough, and Beyond*, 152.

[106] Sezgin, A., Atagün, A. O., & Cagan, N. (2025). A complete study on and-product of soft sets. *Sigma journal of engineering and natural sciences*, *43*(1), 1-14.

[107] Saeed, M., Shafique, I., & Gunerhan, H. (2025). Fundamentals of Fermatean Neutrosophic Soft Set with Application in Decision Making Problem. *International Journal of Mathematics, Statistics, and Computer Science*, *3*, 294-312.

[108] Hazaymeh, A. A. (2025). Time Fuzzy Soft Sets and its application in design-making. *International Journal of Neutrosophic Science (IJNS)*, *25*(3).

[109] Cuong, B. C., & Kreinovich, V. (2013, December). Picture fuzzy sets-a new concept for computational intelligence problems. In *2013 third world congress on information and communication technologies (WICT 2013)* (pp. 1-6). IEEE.

[110] Singh, P. (2015). Correlation coefficients for picture fuzzy sets. *Journal of intelligent & fuzzy systems*, *28*(2), 591-604.

[111] Hatamleh, R., Al-Husban, A., Zubair, S. A. M., Elamin, M., Saeed, M. M., Abdolmaleki, E., ... & Khattak, A. M. (2025). AI-Assisted Wearable Devices for Promoting Human Health and Strength Using Complex Interval-Valued Picture Fuzzy Soft Relations. *European Journal of Pure and Applied Mathematics*, *18*(1), 5523-5523.