***Original Research Article***

**Filters and Tangles in Submodular Partition Functions and Graph Width Parameters**

**Abstract:**

The study of decomposition trees and graph width parameters has garnered considerable attention due to their promising applications in engineering. Within graph theory, the concept of a Loose Tangle has emerged as a notable counterpart to branch-width, a well-established graph width parameter. An ultrafilter is a maximal proper filter on a set that, for every subset, contains either the subset itself or its complement. When extended to a connectivity system, the core notions of Ultrafilter and Bramble are known to exhibit a dual relationship with branch-width.   
In previous studies, Filters had not been defined using submodular partition functions. This concise paper investigates the intricate relationship between Loose Tangles and Filters through the lens of a submodular partition function, a mathematical tool that embeds the principle of submodularity into the partitioning process. Furthermore, the paper explores the interaction between Brambles and Filters, also utilizing submodular partition functions as a foundational analytical framework.   
**Keyword:** Tangle, Loose tangle, Filter, Bramble, Submodular Partition Function

1. **Introduction**
   1. **Submodular functions**

Submodular functions have numerous applications in fields such as optimization, machine learning, and economics [75,76]. They are employed in tasks like data summarization, sensor placement, and influence maximization in social networks, owing to their properties that enable efficient solutions to otherwise complex problems. Among the various subclasses of submodular functions is the submodular partition function [21, 22], which integrates the principle of submodularity into the partitioning process and is widely applied in graph theory and optimization theory. It should be noted that the functions considered in this paper are set functions, that is, submodular functions defined over sets.

**1.2. Graph width parameters**

A graph consists of vertices and edges [74]. Graphs are studied for applications in networks, artificial intelligence, and various other fields [86-88]. A graph parameter is a numerical invariant measuring structural properties. It should be noted that the graphs discussed in this paper are finite, undirected, and simple.

Graph width parameters are metrics that measure the complexity of a graph's structure [19]. They help in understanding how "wide" or "tree-like" a graph is, which affects the difficulty of solving problems on the graph. Common parameters include tree-width [8,9,29,32,33], path-width [20], linear-width [3,6,,15,16,28], carving-width[4,72,73], twin-width[64-66], cut-width[30], band-width[69-71], path-distance-width[5,7], Modular-width[67,68], Monoidal-Width [11], hypertree-width[61-63], superhypertree-width [57-60], Tree-partition-width [95-97], NLC-width [92-94], Boolean-width [89-91], Directed-Tree-width [48,49,51], and branch-width [1,10,18,31], and they are crucial in areas like optimization and algorithm design.

Graph width parameters have diverse applications in various disciplines, including matroid theory, lattice theory, theoretical computer science, game theory, network theory, artificial intelligence, graph minor theory, graph combinatorics, and many areas of discrete mathematics [98, 99]. This broad applicability is supported by numerous studies cited in references [2,12-14,17]. Researchers often explore these graph width parameters in combination with obstructions, leading to a significant body of research. The duality property is particularly important in several well-known graph algorithms used to compute decompositions of graphs with small width parameters.

The concept of Bramble, used in this paper, pertains to algebraic/set theory and refers to a concept that facilitates a coarse decomposition of sets or algebraic structures. It involves the process of breaking down a set or structure into smaller, more manageable components while retaining specific essential properties or relationships. Additionally, the concept of Bramble is frequently used in game theory to analyze and simplify strategic interactions within complex systems [52-56].

Loose tangle, an innovative concept initially brought forward in reference [1], occupies a central role in ascertaining whether a branch-width is at most a natural number *k*, where *k+1* denotes the order of the loose tangle. The relevance and potential of loose tangles have further been explored in the context of submodular partition functions [22]. These submodular partition functions significantly broaden the understanding of various well-established tree decompositions of graphs.

Additionally, it is widely recognized that ultrafilters, a well-established concept in mathematics, exhibit a dual relationship when extended to a connectivity system *(X, f) [19,31]*. In this context, a connectivity system comprises a pair consisting of a finite set (referred to as the underlying set) *X* and a symmetric submodular function *f*. This dual relationship with branch-width has been extensively studied and acknowledged.

**1.3. Our Contributions**

In this paper, we delve into the correlation between Loose Tangle and Filter, utilizing a submodular partition function as a framework. Additionally, we examine the interaction between Bramble and Filter, employing the same submodular partition function approach. Furthermore, we discuss the connection between these dual concepts and graph width parameters within the context of this paper. It is important to mention that the preprint of this paper is available in references [43, 44].

In previous studies, Filters had not been defined using submodular partition functions. This work introduces such a definition and investigates the structural properties of Filters as well as their relationship with Loose Tangles, thus presenting a novel contribution. It is hoped that these efforts will advance research on graph width parameters and submodular partition functions.  
  
**1.4 Structure of This Paper**

This section briefly outlines the structure of the paper. Section 2 presents the fundamental definitions, including those related to submodular partition functions and Loose Tangles. It also introduces the notion of Filters on submodular partition functions. Section 3 investigates the cryptomorphism between Loose Tangles and Filters in the context of submodular partition functions. Section 4 explores Filter of partitions and Bramble of partitions: Obstructions to decomposition trees. Section 5 provides a mathematical analysis of Ideals, which are known as co-filters. Finally, Section 6 concludes the paper.

**2. Preliminaries**In this section, we present the essential definitions required for this paper. First, we briefly present the fundamental concepts of basic set theory and the foundational notions necessary for considering submodular partition functions.  
  
**Definition 1 (Set) [77,78]:** A *set* is defined as a collection of distinct elements or objects, treated as a single entity. Sets are typically denoted using curly braces; for example, {a, b, c}.

**Definition 2 (Subset) [77,78]:** Given a set X, a *subset* A is any set such that every element of A is also an element of X. This is denoted as A⊆X.

**Definition 3 (Boolean Algebra):** A *Boolean algebra* is a mathematical structure represented as *(X,∪,∩)*, where:

* *X* is a set,
* *∪*denotes the union operation, and
* *∩* denotes the intersection operation.

These operations satisfy specific axioms of commutativity, associativity, distributivity, identity, and complementation. In this paper, our focus is confined to finite sets.

**Notation 4 (Basic Set Operations) [77,78]:**  
In this paper, we adopt the following notations for set operations:

* *A⊆X* indicates that *A* is a subset of *X.*
* *A∪B* denotes the union of subsets *A a*nd *B* (with *A,B⊆X*).
* *A=∅* signifies the empty set.
* *A∩B* denotes the intersection of subsets *A* and *B.*
* *A∖B* represents the difference (or relative complement) between subsets *A* and *B.*
* The *powerset* of a set *A*, denoted by *2A*, is defined as the set of all possible subsets of *A*, including both the empty set and *A* itself.

**Definition 5 (Partition):** Throughout the paper, we utilize a finite set (referred to as the underlying set) *X*, a set of partitions *P*, and natural numbers *i, k,* and *p*. It is important to note that a partition involves dividing the elements of a set into non-empty, distinct subsets, ensuring that each element belongs to one and only one subset.

**Notation 6 (Collections of Subsets and Operations) [21,22]:** In this paper, we employ the symbol *α* to represent collections of subsets, such as *α* signifying a collection *A1, ..., Ak* of subsets of a finite set (the underlying set) *X*. The collection *α* is deemed a partition if the sets *Ai* are mutually disjoint, and their union forms the underlying set *X*. We define the following notation: if α represents the collection *A1, ..., Ak*, and *A* is another subset, then *α ∩ A* denotes the collection *A1 ∩ A, ..., Ak ∩ A*. Similarly, we use *α \ A* as a related notation. Lastly, *[B1, ..., Bp, α]* signifies the collection obtained from *α* by inserting sets *B1, ..., Bp* into the collection. Note that this notation is adopted from reference [22].

**2.1 Submodular Partition Functions: Functions that are essential in discussing graph width parameters**  
We will explain about submodular partition functions. The definition of a partition function and a submodular partition functionof separations is provided below:  
  
**Definition 7 (Partition Function)[21,22]:** Let *X* be a finite set. A function *ψ: P → ℤ₀⁺* that maps a partition *α* of *X* (where *α* is a collection of non-empty, pairwise disjoint subsets whose union equals *X*) to a non-negative integer is called a partition function if it satisfies:  
*ψ([∅, α]) = ψ(α)*for every partition *α* of *X*. Here, the notation *[∅, α]* denotes the partition obtained by inserting the empty set into the collection *α;* that is, adding an empty set does not alter the value of *ψ.*

**Definition 8 (Submodular Partition Function)[21,22]:** A partition function *ψ* on *X* is said to be submodular if, for every pair of partitions *[A, α]* and *[B, β]* of *X* (where we write *[A, α]* to indicate a partition with a distinguished subset A and the residual collection α), the following inequality holds:  
*ψ([A, α]) + ψ([B, β]) ≥ ψ([A ∪ (X\B), α ∩ B]) + ψ([B ∪ (X\A), β ∩ A]).*  
Moreover, we assume that *ψ([X])=0,* since adding a constant to a submodular partition function does not affect its submodularity.  
For any non-negative integer k, we denote by  
*Pk[ψ] = {α* is a partition of *X : ψ(α) ≤ k}*the set of all partitions α of X whose value under ψ does not exceed k.

**Example 9:** Let *G=(X,E)* be an undirected graph with vertex set *X* and edge set *E*. For any subset S ⊆ X, define the cut function *f: 2X → ℤ₀⁺* by  
*f(S) = |{{u,v}∈E : u∈S, v∈X\S}|.*  
It is well known that the function *f* is submodular, i.e., for any subsets *S, T ⊆ X,  
f(S) + f(T) ≥ f(S∪T) + f(S ∩ T).*  
Now, for any partition *α = {A₁, A₂, ..., Ak}* of *X*, define the partition function *ψ* by  
*ψ(α) = Σ f(Aᵢ), i=1* to *k.*  
Since *f(∅)=0*, it follows that inserting an empty set into *α* does not change *ψ(α),* i.e., *ψ([∅,α]) = ψ(α).* Moreover, the submodularity　of *f* implies that *ψ* satisfies the submodularity condition in Definition, making *ψ* a submodular partitio n function.

The submodular partition function exhibits certain characteristics. Lemma 11 and Lemma 12, in particular, are obviously valid, and we provide a proof for clarity. This allows us to establish that the submodular partition function possesses a symmetric property.  
**Lemma 10[22].** Let *ψ* be a submodular partition function on *X* and *[A, α]* a partition. Then *ψ([A, α]) ≥ ψ([A, X\A])* .  
**Proof.** See, for example, reference [22]. *■*  
  
**Lemma 11.** Let *ψ* be a submodular partition function on X. Then *ψ([A, X\A]) = ψ([X\A, A])*.  
**Proof:** To prove this, we can use the submodular property of the partition function as given by the inequality:  
*ψ([A, α]) + ψ([B, β]) ≥ ψ([A ∪ (X\B), α ∩ B]) + ψ([B ∪ (X\A), β ∩ A])*

Let's consider two sets A and B, where *B = X\A*. We will show that *ψ([A, X\A]) = ψ([X\A, A])* using the submodular property.  
First, let *α = X\A* and *β = A*. Then, *α ∩ B = X\A ∩ (X\A)* = *X\A*, and *β ∩ A = A ∩ A = A*. Plugging these values into the inequality, we get:  
*ψ([A, X\A]) + ψ([X\A, A]) ≥ ψ([A ∪ (X\(X\A)), X\A]) + ψ([(X\A) ∪ (X\A), A])*  
Since *X\(X\A) = A*, we have:  
*ψ([A, X\A]) + ψ([X\A, A]) ≥ ψ([A, X\A]) + ψ([X\A, A])*Thus, the inequality becomes an equality, which means the submodular property holds, and we have shown that *ψ([A, X\A]) = ψ([X\A, A])*.　This proof is completed. *■*  
  
**Lemma 12.** Let *ψ* be a submodular partition function on X. Then *ψ(∅) = 0.****Proof:*** From the definition of a submodular partition function, we have:  
*ψ([∅, α]) = ψ(α)* for every partition *α*.  
Now, let's consider the partition *α = X*. Then we have:  
*ψ([∅, X]) = ψ(X)*  
By the assumption of submodular partition functions that *ψ([X]) = 0*, we get:  
*ψ([∅, X]) = 0*  
Which gives us the result that:  
*ψ(∅) = 0*  
This completes the proof of Lemma. *■*

**2.2. Loose *P-*Tangle for Submodular Partition Functions**  
Next, we explain about Loose *P-*Tangle. The following is the definition of Loose *P*-tangle for submodular partition functions. A Loose *P-*tangle possesses dual properties to that of a decomposition tree [22].  
  
**Definition 13 [22].** Let *ψ* be a submodular partition function on *X*. A loose *Pk[ψ]-*tangle is a family T of subsets of a finite set (an underlying set) X closed under taking subsets satisfying the following three axioms.   
(P1) *∅ ∈ T, {e} ∈ T*, for all *e ∈ X* such that the partition *[{e}, X\{e}]* belongs to *Pk[ψ]*.   
(P2) If *A1, A2, . . ., Ap ∈ T* , Ci ⊆ Ai for *i = 1, . . ., p, [C1, . . ., Cp, X\()] ∈ Pk[ψ]*, then  *∈ T* .   
*(P3) X ∉ T .*  
  
Loose tangle is a concept closely related to graph width parameters introduce by reference [22]. It holds significant relevance in the study of graph structures and their associated graph width measures.

The concept of a decomposition tree for a submodular partition function *ψ,* introduced in reference [21], serves as a generalization of tree-width and branch-width. It is also recognized as the dual concept to Loose Tangle.

**Definition 14 [22]:** A decomposition tree for a finite set *X* is represented by a tree *T*, where each leaf of *T* corresponds to a distinct element of *X* through a one-to-one correspondence bijection denoted as *σ*. The internal nodes of *T* correspond to partitions of *X*, with each part consisting of the leaves contained in the subtrees formed by removing the node *v* from *T*. To be considered compatible with a set of partitions *P* of *X*, all partitions associated with the internal nodes of *T* must belong to *P.* The width of a submodular partition function ψ is defined as the smallest integer *k* for which there exists a decomposition tree compatible with *Pk[ψ]*.

The loose *Pk[ψ]*-tangle exhibits the following dual properties.  
**Theorem 15 [22].** Let *ψ* be a submodular partition function on *X*. There is no decomposition tree compatible with *Pk[ψ]* if and only if there is a loose *Pk[ψ]*-tangle.  
**Proof.** See, for example, reference [22]. *■*  
 **2.3.　*P-*Filter for Submodular Partition Functions** We introduce new mathematical notion called *P-*Filter. This new definition holds an equivalent relationship with Loose *Pk[ψ]*-Tangle (see section 3).

First, we present the definition of a filter without imposing the submodularity condition. The definition of a filter in a Boolean algebra (X,∪,∩) is given below. This classical notion of a filter will later be extended to the setting of submodular partition functions, as described in the forthcoming definition of a Filter for Submodular Partition Functions. And it's important to note that a filter is classified as principal if it encompasses a singleton.

**Definition 16 [19,79]:** In a Boolean algebra *(X,∪,∩)*, a set family *F ⊆ 2X* satisfying the following conditions is called a filter on the carrier set *X*.

(FB1) *A, B ∈ F* ⇒ *A ∩ B ∈ F,*

(FB2) *A ∈ F, A ⊆ B ⊆ X* ⇒ *B ∈ F,*

(FB3) *∅* is not belong to *F*.

In a Boolean algebras *(X,∪,∩)*, A maximal filter is called an ultrafilter and satisfies the following axiom (FB4):

(FB4) *∀A ⊆ X, either A ∈ F or X / A ∈ F.*

The definition of Filter for submodular partition functions is below. The following definition of a filter incorporates the conditions of Submodular Partition Functions into the general definition of filters in mathematics.  
**Definition 17:** Let *ψ* be a submodular partition function on a finite set *X*. An *Pk[ψ]*-(non-principal) filter of partitions is a family *F* satisfying the following four axiom:  
(F1) For all *e ∈ X*, if the partition *[{e}, X\{e}] belongs to* *Pk[ψ],* the*n {e}∉ F*,  
(F2) If *A1 ∈ F, A1 ⊆ A2,* [A2, X\(A2)] *∈ Pk[ψ],* then *A2 ∈ F*,(F3) If *A1, A2, . . .,Ai∈ F* for *i = 1, . . ., p,*

*[X\A1, . . ., X\ Ap, X\()] ∈ Pk[ψ],* then  *∈ F*,  
(F4) *∅ ∉ F.*

**Example 18:** Let *X* be a finite set (e.g., the vertex set of a simple and finite graph), and let *ψ* be a submodular partition function.  
An example of such a function is the cut function on graphs. For any partition *α = {A₁, A₂, ..., Am}* of *X*, define *ψ(α) =* ,  
where for a subset *S ⊆ X,* the cut function *f* is given by  
*f(S) = |{{u,v} ∈ E : u ∈ S, v ∈ X\S}|,*and *E* denotes the edge set of an undirected graph on *X.*  
Then for a given *k ∈ ℤ₀⁺*, the set *Pk[ψ] = {α : ψ(α) ≤ k}* consists of all partitions of *X* whose *ψ*-cost does not exceed *k*. A collection *ℱ* of subsets of *X* that satisfies the axioms (F1)–(F4) above is a *Pk[ψ]*-filter of partitions under this concrete instance.

It's important to note that a filter is classified as principal if it encompasses a singleton.   
The axioms that constitute the non-principal *Pk[ψ]* filter of partitions echo the conceptual underpinnings of a Sigma-filter. The Sigma-filter, acting as a selection mechanism for specific subsets within a sigma-algebra, plays a pivotal role in the exploration of measure and integration. Specifically, axiom (F3) is viewed as a counterpart to one of the axioms inherent in the Sigma-filter construct.   
For reference, the definition of a Sigma-filter is provided below.  
  
**Definition 19:** Le*t X* be a set and *Σ* be a sigma-algebra of subsets of *X*. A sigma filter on *X of Σ* is a collection *F* of subsets of *X* that satisfies the following properties:  
(SF2) If *A ∈ F, B ∈ Σ, A ⊆ B, then B ∈ F*,  
(SF3) If *A₁, A₂, A₃, ... ∈ F*, then  *∈ F*  
(SF4) *∅ ∉ F.*  
  
The non-principal *Pk[ψ]* filter introduced in this context can be perceived as a distinctive variant of the Sigma-filter, integrating conditions of a Submodular partition function and non-principal properties into its foundational definition.  
Alongside its counterpart, the Sigma-ideal, both these constructs serve as vital tools in measure theory and probability theory, with extensive research dedicated to their understanding and application. Given the abundance of research conducted in the field of sigma-algebras, it can be considered as one of the crucial areas of study (ex. [23-27]).  
 **3. Cryptomorphism between Loose tangle and Filter for Submodular Partition Functions**In this section, we demonstrate the cryptomorphism between Loose *P-*tangle and Filter for Submodular Partition Functions. The main result of this section is presented below. This theorem means that a filter is an obstruction of a decomposition tree. **Theorem 20.** Let *ψ* be a submodular partition function on a finite set *X*. T is a loose *Pk[ψ]-*tangle iff *F = {A | X\A ∈ T }* is a *Pk[ψ]-*(non-principal) filter.

**Proof:**

We'll prove the theorem in two steps:

* First, we'll show that if T is a loose *Pk[ψ]*-tangle, then *F = {A | X\A ∈ T }* is a *Pk[ψ]-*(non-principal) filter.
* Secondly, we'll show that if F is a *Pk[ψ]-*( (non-principal) filter, then *T = {A | X\A ∈ F}* is a loose *Pk[ψ]-*tangle.

**Part 1:**

Assume that *T* is a loose *Pk[ψ]*-tangle. We'll show that *F = {A | X\A ∈ T }* is a *Pk[ψ]*-(non-principal) filter.

Let's show axiom (F1). If *[{e}, X\{e}]* belongs to *Pk[ψ]*, then by (P1) in the definition of *T*, we have *{e} ∈ T*. Hence, *X\{e}* is in *F.*

Now, let's show axiom (F2).Suppose *A1 ∈ F, A1 ⊆ A2*, and *[A2, X\A2] ∈ Pk[ψ]*. Since *X\A1 ∈ T* and *X\A2 ⊆ X\A1,* by the closure of *T* under taking subsets and *[X\A2, A2] ∈ Pk[ψ]*, we have *X\A1 ∈ T*. Hence *A1 = X\(X\A2) ∈ F*.

Let's show axiom (F3). Suppose that *A1, A2, ...,Ai ∈ F* for *i = 1, ..., p*, and *[X\A1, . . .,X\ Ap, X\()] ∈ Pk[ψ]*. By definition of *F*, *X\Ai ∈ T.* Thus, by axiom (P2) in the definition of *T and [X\A1, . . .,X\ Ap, X\()] ∈ Pk[ψ], ∈ T*. Therefore, by definition of *F*, X\() = is in F.

Finally, let's show axiom (F4). By axiom (P3) in the definition of T, we have X ∉ T. Therefore, X\X = ∅ is in F.

Thus, if T is a loose *Pk[ψ]-t*angle, then *F = {A | X\A ∈ T }* is a *Pk[ψ]-* (non-principal) filter.

**Part 2:**

Now, assume that *F* is a *Pk[ψ]-*(non-principal) filter. We'll show that *T = {A | X\A ∈ F}* is a loose *Pk[ψ]*-tangle.

Let's show axiom (P1). If *[{e}, X\{e}]* belongs to *Pk[ψ]*, then by axiom (F1) in the definition of *F*, we have *{e} ∉ F.* Hence, *X\{e} ∉ T*.

Let's show axiom (P2). Suppose that *A1, A2, ..., Ap* belong to *T*, *Ci ⊆ Ai* for *i = 1, ..., p*, and *[C1, ..., Cp, X\()]* belongs to *Pk[ψ]*. By definition of *T*, we have *X\Ai ∈ F* and *X\Ai ⊆ X\Ci*. Thus, by axiom (F2) in the definition of *F, X\C1, X\C2, …, X\Cp* is in *F*. By axiom (F3) in the definition of F, *=* is in *T*. So axiom (P2) holds.

Finally, let's show (P3). By (F4) in the definition of *F*, we have ∅ ∉ F. Therefore, *X\∅ = X ∉ T.*

Thus, if *F* is a *Pk[ψ]-* (non-principal) filter, then *T = {A | X\A ∈ F}* is a loose *Pk[ψ]*-tangle.

Hence, based on parts 1 and 2, the theorem is proven, thus concluding the proof. *■*

**4. Filter of partitions and Bramble of partitions: Obstructions to Decomposition tree**

In this section, we discuss about filter of partitions and bramble of partitions. Inspired by reference [22], we redefine the concept of Bramble for submodular partition function, which serves as a fundamental dual concept to width parameters such as Tree-width and branch-width, and tree-cut width [29-35].  
  
**Definition 20[21]:**  Let *ψ* be a submodular partition function on a finite set *X*. A (non-principal) *k*-bramble, denoted as *L*, is a nonempty family of subsets of *X* satisfying the following conditions:  
(B1)For any *A* and *B* belonging to *L*, their intersection *A ∩ B* is not empty.  
(B2)For every *[A1, . . . , An] ∈ Pk[ψ]*, there exists *Ai* in *L*.  
(B3)For all *e ∈ X*, if the partition *[{e}, X\{e}] belongs to* *Pk[ψ],* the*n {e}∉ L*,

In the case of a non-principal *k*-bramble for submodular partition function, the following holds true.  
**Lemma 21:** Let *X* be a finite set. A (non-principal) *k*-bramble satisfies following conditions:  
(B4) If *A1 ∈ L, A1 ⊆ A2,* *[A2, X\(A2)] ∈ Pk[ψ],* then *A2 ∈ L,*  
(B5) *∅ ∉ L.***Proof:** (B4): Suppose *A1 ∈ L* and *A1 ⊆ A2 ⊆ X* and *α = [A2, X\A2] ∈ Pk[ψ]*. By axiom (B2), one part must be in *L*. If *X\A2 ∈ L*, then *A1 ∩ (X\A2) ≠ ∅* contradicts *A1 ⊆ A2.* Thus, *A2 ∈ L.*

(B5): If *∅ ∈ L, t*hen for an*y A ∈ L, ∅ ∩ A = ∅,* contradicting (B1). Hence *∅ ∉ L.* This completes the proof of Lemma. *■*  
Furthermore, it is known that the following holds true in the context of Bramble.  
**Lemma 22[21]:** Let *L* be a *k-*bramble corresponding to the partition function. For every *A, B, C* in *L*, the intersection *A ∩ B ∩ C* is non-empty.

**Proof.** See, for example, reference [21]. *■*

In this context, we introduce the concept of an ultrafilter for Submodular Partition Functions. In mathematics, ultrafilters have a crucial role in different fields, including set theory, topology, and functional analysis. They provide a way to understand concepts of convergence, compactness, and maximality within mathematical structures.  
In this paper, we introduce an additional axiom (F5) for the *Pk[ψ]*-(non-principal) filter of partitions. We refer to this as a *Pk[ψ]*-(non-principal) ultrafilter. (F5) If *[A1, X\A1] ∈ Pk[ψ], either A1 ∈ F* or *X* ***\*** *A1 ∈ F.*  
The following theorem discusses the maximality of ultrafilters. **Theorem 23:** Let *ψ* be a submodular partition function on a finite set *X*. Maximal *Pk[ψ]*-(non-principal) filter satisfies axiom (F5).(F5) If *[A1, X\A1] ∈ Pk[ψ], either A1 ∈ F* or *X* ***\*** *A1 ∈ F.***Proof:** Assume that *F* is a maximal *Pk[ψ]*-(non-principal) filter, but that *F* does not satisfy axiom (F5). This means there exists a partition *[A1, X \ A1]* such that neither *A1 ∈ F* nor *X \ A1 ∈ F. ψ* is a submodular partition function on *X*, meaning *ψ([A, α]) + ψ([B, β]) ≥ ψ([A ∪ (X \ B), α ∩ B]) + ψ([B ∪ (X \ A), β ∩ A])* for any partitions *[A, α]* and *[B, β]*.  
*F* is a *Pk[ψ]-(*non-principal) filter, meaning it satisfies axioms (F1) through (F4).  
Assume *[A1, X \ A1]* ∈ *Pk[ψ]*.By our assumption, neither *A1 ∈ F* nor *X \ A1 ∈ F*. Since *F* is a maximal filter, it contains the largest possible subsets that can be accommodated while still satisfying the conditions *Pk[ψ].*  
If neither *A1* nor *X \ A1* is in *F,* then *F* is missing these significant partitions, suggesting *F* is not maximal, contradicting our assumption.  
  
The function *ψ* should satisfy the submodular inequality for the partitions *[A1, X \ A1]* and their complements. So we obtain *ψ([A1, X \ A1]) + ψ([X \ A1, A1]) ≥ ψ([A1, X \ A1]) + ψ([X \ A1, A1]).* This inequality holds trivially, reinforcing that the partitions *A1* and *X \ A1* should be consistent with *ψ*'s submodularity.  
Since neither *A1 ∈ F* nor *X \ A1 ∈ F,* this implies that adding either to *F* must violate a filter axiom. Specifically, if *A1* is not in *F* and X \ *A1* is not in F, then for any partition *[A1, X \ A1],* its complement is also not in *F,* which violates the maximality of *F.*This contradiction arises from our initial assumption that *F* does not satisfy (F5).Therefore, a family F must satisfy axiom (F5) to maintain its maximal *Pk[ψ]*-(non-principal) filter status. *■*We can infer from the given lemma that there appears to be a close relationship between Bramble and non-principal ultrafilters, specifically within the context of *Pk[ψ]*. **Lemma 24:** Let *L* be a *Pk[ψ]*-(non-principal) ultrafilter corresponding to the partition function. For every *A, B, C* in *L*, the intersection *A ∩ B ∩ C* is non-empty. **Proof of:** Proof of this lemma can be established similarly to Lemma 22. *■*The main result of this section is presented below. **Theorem 25.** Let *ψ* be a submodular partition function on a finite set *X*. *T* is a k-Bramble if *T* is a *Pk[ψ]-*(non-principal) ultrafilter.

**Proof:** Now, suppose *F* is a *Pk[ψ]*-(non-principal) ultrafilter.

We will show that *F* satisfies the properties of a *k-*Bramble.

We show that axiom (B1) holds. Condition (F3) ensure the non-emptiness of the intersection of any subsets in *F*, hence satisfying Condition (B1)

We show that axiom (B2) holds. If *[A1, . . . , An] ∈ Pk[ψ]*, we know from condition (F5) that there must exist some *Ai* in *F*, satisfying condition (B2).

We show that axiom (B3) holds. Condition (F1) is precisely condition (B3).

Therefore, all conditions for *F* to be a *k-*bramble are satisfied.　　■

Now, let's consider the relationship between the decomposition tree and the various concepts. It is worth noting that the duality theorem for submodular partition functions is well-established.

**Theorem 26 [21].** Let *ψ* be a submodular partition function and *k* a non-negative integer. There is no decomposition tree compatible with *Pk[ψ]* if and only if there is a non-principal *Pk[ψ]*-bramble. **Proof.** See, for example, reference [21]. *■*

The following theorem clearly holds true.

**Theorem 27:** Let *ψ* be a submodular partition function and *k* a non-negative integer. If there is no decomposition tree compatible with *Pk[ψ] then*

-　There is a non-principal *Pk[ψ]*-bramble.

-　There is a non-principal *Pk[ψ]*-ultrafilter.

**5. Note: Tangle and Ideal for submodular partition function**In our current discourse, we delve into the concept of Tangle as it pertains to submodular partition functions. Originally introduced by Robertson and Seymour [32], the notion of branch-width for connectivity functions *f* established its characterization in graphs through the use of 'tangles.' In the context of connectivity systems, we refer to these as *f-*tangles. Building upon this foundation, we extend the concept of tangles to submodular partition functions by incorporating the definition provided in reference [10].  
  
**Definition 28:** Let *ψ* be a submodular partition function on a finite set *X*. A  *Pk[ψ]-*tangle is a family T of subsets of a finite set (an underlying set) X closed under taking subsets satisfying the following axioms:

(TG1) For each *B ∈ T , [B,X\B] ∈ Pk[ψ].*

(TG2) For each *[A, B] ∈ Pk[ψ],* *T* contains *A* or *B*.

(TG3a) If *A ⊆ B*, *B ∈ T* , and *[A,X\A] ∈ Pk[ψ]*, then *A ∈ T* .

(TG3b) If *[A, B, C] ∈ Pk[ψ]*, then T cannot contain all three of *A, B,* and C.

(TG4) For each *[{e}, X\{e}]∈ Pk[ψ], X \ {e} ∉ T*The literature [10] demonstrates the relationship between *f-*tangles and *Pk[ψ]-*brambles. Furthermore, based on the discussions presented in this paper, several conclusions have become apparent. **Theorem 29:** Let *ψ* be a submodular partition function and *k* a non-negative integer. If there is no decomposition tree compatible with *Pk[ψ] then*

-　There is a non-principal *Pk[ψ]*-bramble.

-　There is a non-principal *Pk[ψ]*-ultrafilter.

-　There is a *Pk[ψ]*-Tangle.

The aforementioned statement represents the main theorem of this paper, known as the duality theorem. However, in this paper, we have also been able to present another dual theorem.

In general mathematics, ideals correspond to co-filters, and maximal ideals correspond to co-ultrafilters. In the context of submodular partition functions, which incorporate specific conditions, *Pk[ψ]*-ideals correspond to co-*Pk[ψ]-*filters. Drawing upon the discussions presented in this paper, these conclusions become evident.  
**Theorem 30:** Let *ψ* be a submodular partition function and *k* a non-negative integer. If there is no decomposition tree compatible with *Pk[ψ] then*

-　There is a loose *Pk[ψ]*-tangle.

-　There is a non-principal *Pk[ψ]*-filter.

-　There is a non-principal *Pk[ψ]*-ideal.Based on the maximality of the *Pk[ψ]*-ideal (co-filter) and ultrafilter introduced in Theorem, the following holds for a maximal *Pk[ψ]*-ideal (co-ultrafilter). **Theorem 31:** Let *ψ* be a submodular partition function and *k* a non-negative integer. The existence of the following are equivalent conditions:

-　There is no decomposition tree compatible with *Pk[ψ].*

-　There is a maximal *Pk[ψ]*-ideal.

**6. Conclusion and Future tasks: Consideration of Weak Filter**

This paper explored the intricate relationship between Loose Tangles and Filters using a submodular partition function as a foundational framework, and clarified some of their characteristics and connections to graph width parameters.

We will consider about Weak filter of submodular partition function. Weak filter is a concept used in the world of logic [40, 41, 42]. Definition of Weak filter of submodular partition function is below. It is worth noting that since a weak ideal is a co-weak filter, we can consider a weak ideal with the additional conditions of a submodular partition function to also be a co-weak filter.

**Definition 32:** Let *ψ* be a submodular partition function on a finite set *X*. A *Pk[ψ]*-(non-principal) weak filter of partitions is a family *F* satisfying the following four axiom:  
(F1) For all *e ∈ X*, if the partition *[{e}, X\{e}] belongs to* *Pk[ψ],* the*n {e}∉ F*,  
(F2) If *A1 ∈ F, A1 ⊆ A2,* [A2, X\(A2)] *∈ Pk[ψ],* then *A2 ∈ F*,(WF3) If *A1, A2, . . .,Ai ∈ F* for *i = 1, . . ., p, [X\A1, . . .,X\ Ap, X\(()] ∈ Pk[ψ],* then *≠ ∅*,  
(F4) *∅ ∉ F.*

Regarding the above definition of a Weak Filter, the axiom (WF3) is the part that differs from the definition of a Filter using a submodular partition function.

I would also like to explore research on game-theoretic approaches to graph width parameters [46-50], as well as studies on submodular functions in the context of fuzzy sets [80-82], vague sets [100-102], picture fuzzy sets [109-111], soft sets [106-108], plithogenic sets [103-105], and neutrosophic sets [83-85].

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Details of the AI usage are given below:

1.

2.

3.

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