

Anticenter-Symmetric Bialgebras

Abstract

This paper develops a bialgebra theory for anticenter-symmetric algebras by introducing the concept of an *anticenter-symmetric bialgebra*, equivalent to a Manin triple of anticenter-symmetric algebras. A study of this framework leads to the *anticenter-symmetric Yang-Baxter equation* in anticenter-symmetric algebras, analogous to the classical Yang-Baxter equation in Mock Lie algebras and the associative Yang-Baxter equation.

An unexpected finding is that the anticenter-symmetric and associative Yang-Baxter equations share the same form. Additionally, skew-symmetric solutions to the anticenter-symmetric Yang-Baxter equation define anticenter-symmetric bialgebras. To advance the theory, the paper introduces *\mathcal{O} -operators* and *pre-anticenter-symmetric algebras*, which facilitate the construction of these solutions and provide a foundation for further exploration.

Keywords. Anticenter-Symmetric Algebras, Pre-Anticenter-Symmetric Algebras, Matched Pairs, Manin Triples, Bialgebras, Yang-Baxter Equation and \mathcal{O} -operators

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1 Introduction

Mock-Lie algebras are commutative algebras characterized by their adherence to the Jacobi identity, with significant contributions to their study made by P. Zusmanovich in [21]. These algebras have appeared under various names, reflecting diverse mathematical perspectives.

Their earliest mention was in [19], where an infinite-dimensional solvable but non-nilpotent example was introduced, later reproduced in [20]. They are also referred to as “Jordan algebras of nil index 3” in Jordan-algebraic literature, “Lie–Jordan algebras” in [17], and “Jacobi–Jordan algebras” in recent studies [6] and [1]. The term “mock-Lie” originates from [11], where the operad appears in a classification of quadratic cyclic operads. They possess two particularly noteworthy features:

- (a) Algebras associated with the Koszul dual of the Mock-Lie operad can be equivalently characterized in three distinct ways, as detailed in [21] and [7].
- (b) As observed in [17], Mock-Lie algebras can also be constructed from antiassociative algebras, paralleling their derivation from associative algebras. This underscores a profound relationship between Mock-Lie and antiassociative algebras.

Significant progress has been made in understanding the cohomology and deformation theories of Mock-Lie algebras. A notable development is the introduction of a cohomology framework based on two operators, referred to as *zigzag cohomology*, which was explored in [4] alongside a detailed examination of low-degree cohomology spaces. Furthermore, [5] investigated Mock-Lie bialgebras, the Yang-Baxter equation, and Manin triples, broadening the algebraic and structural insights into these algebras. The study of Lie-admissible algebras has been of great significance, particularly the bialgebraic exploration of left-symmetric algebras as detailed in [2]. More recently, anti-flexible algebras, also known as center-symmetric algebras, have emerged as another class of

Lie-admissible algebras, with their bialgebraic properties investigated by [8]. In addition, we have recently introduced the concept of anticenter-symmetric Jacobi-Jordan algebras, which we refer to more succinctly as anticenter-symmetric algebras [14]. These algebras belong to the category of Mock-Lie admissible algebras.

The primary aim of our paper is to undertake an algebraic study of these structures; we establish a bialgebra theory for anti-center-symmetric algebras by defining the concept of an *anticenter-symmetric bialgebra*, linked to a Manin triple of such algebras. This framework introduces the *anticenter-symmetric Yang-Baxter equation*, paralleling the classical Yang-Baxter equation in Mock Lie algebras and the associative Yang-Baxter equation. Remarkably, the anticenter-symmetric and associative Yang-Baxter equations share the same form. Skew-symmetric solutions to the former directly define anticenter-symmetric bialgebras. To support this theory, we introduce *\mathcal{O} -operators* and *pre-anticenter-symmetric algebras*, providing tools for constructing solutions.

The paper begins in Section 2 with a review of the bimodules and matched pairs of anti-center-symmetric algebras. Section 3 then focuses on the Manin triple of anti-center-symmetric algebras, providing a deeper understanding of their bialgebraic structural aspects. Section 4 explores a special class of anticenter-symmetric bialgebras, this leads to anticenter-symmetric Yang-Baxter equation.

Section 5 develops the theory of \mathcal{O} -operators of anticenter-symmetric algebras and pre-anticenter-symmetric algebras. Finally, Section 6 concludes the paper with reflective remarks that summarize the findings.

2 Bimodules and matched pairs of anticenter-symmetric algebras

Definition 2.1 [14] (\mathcal{A}, \cdot) , is said to be an anticenter-symmetric algebra if $\forall x, y, z \in \mathcal{A}$, the antiassociator of the bilinear product \cdot defined by $(x, y, z)_{-1} := (x \cdot y) \cdot z + x \cdot (y \cdot z)$, is symmetric in x and z , i.e.,

$$(x, y, z)_{-1} = -(z, y, x)_{-1}. \quad (2.1)$$

As matter of notation simplification, we will denote $x \cdot y$ by xy if not any confusion.

Definition 2.2 [14] Let \mathcal{A} be an anticenter-symmetric algebra, V be a vector space. Suppose $l, r : \mathcal{A} \rightarrow \mathfrak{gl}(V)$ be two linear maps satisfying: $\forall x, y \in \mathcal{A}$,

$$[l_x, r_y] = -[l_y, r_x] \quad (2.2)$$

$$l_{xy} + l_x l_y = -r_{yx} - r_x r_y. \quad (2.3)$$

Then, (l, r, V) (or simply (l, r)) is called bimodule of the anticenter-symmetric algebra \mathcal{A} .

Let (\mathcal{A}, \cdot) be an anticenter-symmetric algebra. For any $x, y \in \mathcal{A}$, let L_x and R_x denote the left and right multiplication operators respectively, that is, $L_x(y) = xy$ and $R_x(y) = yx$. Let $L, R : \mathcal{A} \rightarrow \text{End}(\mathcal{A})$ be two linear maps with $x \rightarrow L_x$ and $x \rightarrow R_x$ for any $x \in \mathcal{A}$ respectively.

Example 2.3 Let (\mathcal{A}, \cdot) be an antisymmetric algebra. Then (L, R, \mathcal{A}) is a bimodule of (\mathcal{A}, \cdot) , which is called the **regular bimodule of (\mathcal{A}, \cdot)** .

Proposition 2.4 [14] Let (\mathcal{A}, \cdot) be an anticenter-symmetric algebra and V be a vector space over \mathbb{K} . Consider two linear maps, $l, r : \mathcal{A} \rightarrow \mathfrak{gl}(V)$. Then, (l, r, V) is a bimodule of \mathcal{A} if and only if, the semi-direct sum $\mathcal{A} \oplus V$ of vector spaces is turned into an anticenter-symmetric algebra by defining the multiplication in $\mathcal{A} \oplus V$ by $\forall x_1, x_2 \in \mathcal{A}, v_1, v_2 \in V$,

$$(x_1 + v_1) * (x_2 + v_2) = x_1 \cdot x_2 + (l_{x_1} v_2 + r_{x_2} v_1),$$

We denote it by $\mathcal{A} \ltimes_{l,r}^{-1} V$ or simply $\mathcal{A} \ltimes^{-1} V$.

Proof: It is obvious that the semi-direct sum of two vector spaces is also a vector space. Now suppose that (l, r, V) is a bimodule of \mathcal{A} and show that $(\mathcal{A} \oplus V, *)$ is an anticenter-symmetric algebra. Since $*$ is a bilinear product.

For all $x_1, x_2, x_3 \in \mathcal{A}$ and for all $v_1, v_2, v_3 \in V$, we have:

$$\begin{aligned}
((x_1 + v_1), (x_2 + v_2), (x_3 + v_3))_{-1} &= \{(x_1 + v_1) * (x_2 + v_2)\} * (x_3 + v_3) \\
&+ (x_1 + v_1) * \{(x_2 + v_2) * (x_3 + v_3)\} \\
&= (x_1 x_2) x_3 + l_{x_1 x_2} v_3 + r_{x_3} (l_{x_1} v_2 + r_{x_2} v_1) \\
&+ x_1 (x_2 x_3) + l_{x_1} (l_{x_2} v_3) + l_{x_1} (r_{x_3} v_2) + r_{x_2 x_3} v_1 \\
&= (x_1 x_2) x_3 + l_{x_1 x_2} v_3 + r_{x_3} (l_{x_3} v_2) + r_{x_3} (r_{x_2} v_1) \\
&+ x_1 (x_2 x_3) + l_{x_1} (l_{x_2} v_3) + l_{x_1} (r_{x_3} v_2) + r_{x_2 x_3} v_1 \\
((x_1 + v_1), (x_2 + v_2), (x_3 + v_3))_{-1} &= (x_1, x_2, x_3)_{-1} + (l_{x_1 x_2} + l_{x_1} l_{x_2}) v_3 \\
&+ (r_{x_3} l_{x_1} + l_{x_1} r_{x_3}) v_2 + (r_{x_3} r_{x_2} + r_{x_2 x_3}) v_1. \\
((x_3 + v_3), (x_2 + v_2), (x_1 + v_1))_{-1} &= (x_3, x_2, x_1)_{-1} + (l_{x_3 x_2} + l_{x_3} l_{x_2}) v_1 \\
&+ (r_{x_1} l_{x_3} + l_{x_3} r_{x_1}) v_2 + (r_{x_1} r_{x_2} + r_{x_2 x_1}) v_3.
\end{aligned}$$

Therefore, from the definition of anticenter-symmetric JJ algebra we have

$$\begin{aligned}
(x_1 + v_1, x_2 + v_2, x_3 + v_3)_{-1} &= -(x_3 + v_3, x_2 + v_2, x_1 + v_1)_{-1} \\
\iff &\begin{cases} l_{x_1 x_2} + l_{x_1} l_{x_2} = -r_{x_1} r_{x_2} - r_{x_2 x_1} \\ r_{x_3} l_{x_1} + l_{x_1} r_{x_3} = -r_{x_1} l_{x_3} - l_{x_1} r_{x_3} \\ l_{x_1 x_3} + l_{x_3} l_{x_2} = -r_{x_3} r_{x_2} - r_{x_2 x_3} \end{cases} \\
\iff &(\mathcal{A} \oplus V, *)
\end{aligned}$$

is an anticenter symmetric algebra. □

It is known that an anticenter-symmetric algebra is a Mock Lie-admissible algebra ([14]).

Proposition 2.5 [14] *Let (\mathcal{A}, \cdot) be an anticenter-symmetric algebra. Define the anticommutator by*

$$[x, y] = x \cdot y + y \cdot x, \quad \forall x, y \in \mathcal{A}. \quad (2.4)$$

Then it is a Mock Lie algebra and we denote it by $(\mathcal{G}(\mathcal{A}), [\cdot, \cdot])$ or simply $\mathcal{G}(\mathcal{A})$, which is called the the sub-adjacent Mock Lie algebra of (\mathcal{A}, \cdot) .

Corollary 2.6 *Let (\mathcal{A}, \cdot) be an anticenter-symmetric algebra and V be a vector space over \mathbb{K} . Consider two linear maps, $l, r : \mathcal{A} \rightarrow \mathfrak{gl}(V)$, such that (l, r, V) is a bimodule of \mathcal{A} . Then, the map: $l + r : \mathcal{A} \rightarrow \mathfrak{gl}(V)$ $x \mapsto l_x + r_x$, is a linear representation of the sub-adjacent Mock Lie algebra of \mathcal{A} .*

Proof: Let (l, r, V) be a bimodule of the anticenter-symmetric algebra \mathcal{A} . Then, $\forall x, y \in \mathcal{A}$ $[l_x, r_y] = -[l_y, r_x]; l_{xy} + l_x l_y = -r_x r_y - r_y r_x$. Besides, it is a matter of straightforward computation to show that $l + r$ is a linear map on \mathcal{A} . Then, we have:

$$\begin{aligned}
[(l + r)(x), (l + r)(y)] &= [l_x + r_x, l_y + r_y] \\
&= [l_x, l_y] + [l_x, r_y] + [r_x, l_y] + [r_x, r_y] \\
&= [l_x, l_y] + [r_x, r_y] \\
&= l_x l_y + l_y l_x + r_x r_y + r_y r_x \\
&= \{l_x l_y + r_x r_y\} + \{l_y l_x + r_y r_x\} \\
&= \{l_{xy} + r_{yx}\} + \{l_{yx} + r_{xy}\} \\
&= (l + r)_{xy} + (l + r)_{yx} = (l + r)_{[x, y]}.
\end{aligned}$$

Therefore, (l, r, V) is a bimodule of \mathcal{A} implies that $l + r$ is a representation of the linear representation of the sub-adjacent Mock Lie algebra of \mathcal{A} . □

Theorem 2.7 [14] Let (\mathcal{A}, \cdot) and (\mathcal{B}, \circ) be two anticenter-symmetric algebras. Suppose that $(l_{\mathcal{A}}, r_{\mathcal{A}}, \mathcal{B})$ and $(l_{\mathcal{B}}, r_{\mathcal{B}}, \mathcal{A})$ are bimodules of \mathcal{A} and \mathcal{B} , respectively, obeying the relations:

$$\begin{aligned} & r_{\mathcal{A}}(x)(a \circ b) + r_{\mathcal{A}}(l_{\mathcal{B}}(b)x)a + a \circ (r_{\mathcal{A}}(x)b) \\ & + l_{\mathcal{A}}(r_{\mathcal{B}}(b)x)a + (l_{\mathcal{A}}(x)b) \circ a + l_{\mathcal{A}}(x)(b \circ a) = 0, \end{aligned} \quad (2.5)$$

$$\begin{aligned} & r_{\mathcal{B}}(a)(x \cdot y) + r_{\mathcal{B}}(l_{\mathcal{A}}(y)a)x + x \cdot (r_{\mathcal{B}}(a)y) \\ & + l_{\mathcal{B}}(r_{\mathcal{A}}(y)a)x + (l_{\mathcal{B}}(a)y) \cdot x + l_{\mathcal{B}}(a)(y \cdot x) = 0, \end{aligned} \quad (2.6)$$

$$\begin{aligned} & a \circ (l_{\mathcal{A}}(x)b) + (r_{\mathcal{A}}(x)b) \circ a + (r_{\mathcal{A}}(x)a) \circ b + l_{\mathcal{A}}(l_{\mathcal{B}}(a)x)b \\ & + r_{\mathcal{A}}(r_{\mathcal{B}}(b)x)a + l_{\mathcal{A}}(l_{\mathcal{B}}(b)x)a + b \circ (l_{\mathcal{A}}(x)a) + r_{\mathcal{A}}(r_{\mathcal{B}}(a)x)b = 0, \end{aligned} \quad (2.7)$$

$$\begin{aligned} & x \cdot (l_{\mathcal{B}}(a)y) + (r_{\mathcal{B}}(a)y) \cdot x + (r_{\mathcal{B}}(a)x) \cdot y + l_{\mathcal{B}}(l_{\mathcal{A}}(x)a)y \\ & + r_{\mathcal{B}}(r_{\mathcal{A}}(y)a)x + l_{\mathcal{B}}(l_{\mathcal{A}}(y)a)x + y \cdot (l_{\mathcal{B}}(a)x) + r_{\mathcal{B}}(r_{\mathcal{A}}(x)a)y = 0, \end{aligned} \quad (2.8)$$

for all $x, y \in \mathcal{A}$ and $a, b \in \mathcal{B}$. Then, there is an anticenter-symmetric algebra structure on $\mathcal{A} \oplus \mathcal{B}$ given by:

$$\begin{aligned} (x + a) * (y + b) &= (x \cdot y + l_{\mathcal{B}}(a)y + r_{\mathcal{B}}(b)x) \\ &+ (a \circ b + l_{\mathcal{A}}(x)b + r_{\mathcal{A}}(y)a). \end{aligned} \quad (2.9)$$

Definition 2.8 Let (\mathcal{A}, \cdot) and (\mathcal{B}, \circ) be two anticenter-symmetric algebras. Suppose that there are linear maps $l_{\mathcal{A}}, r_{\mathcal{A}} : \mathcal{A} \rightarrow \mathfrak{gl}(\mathcal{B})$ and $l_{\mathcal{B}}, r_{\mathcal{B}} : \mathcal{B} \rightarrow \mathfrak{gl}(\mathcal{A})$ such that $(l_{\mathcal{A}}, r_{\mathcal{A}}, \mathcal{B})$ is a bimodule of \mathcal{A} and $(l_{\mathcal{B}}, r_{\mathcal{B}}, \mathcal{A})$ is a bimodule of \mathcal{B} . If the conditions (2.5) - (2.8) are satisfied, then, $(\mathcal{A}, \mathcal{B}, l_{\mathcal{A}}, r_{\mathcal{A}}, l_{\mathcal{B}}, r_{\mathcal{B}})$ is called a **matched pair of anticenter-symmetric algebras**. We denote this anticenter-symmetric algebra by $\mathcal{A} \bowtie_{l_{\mathcal{B}}, r_{\mathcal{B}}}^{-1, l_{\mathcal{A}}, r_{\mathcal{A}}} \mathcal{B}$, or simply by $\mathcal{A} \bowtie^{-1} \mathcal{B}$.

Definition 2.9 Let (l, r, V) be a bimodule of an anticenter-symmetric algebra \mathcal{A} , where V is a finite dimensional vector space. The dual maps l^*, r^* of the linear maps l, r , are defined, respectively, as: $l^*, r^* : \mathcal{A} \rightarrow \mathfrak{gl}(V^*)$ such that: for all $x \in \mathcal{A}, u^* \in V^*, v \in V$,

$$\begin{aligned} l^* : \mathcal{A} &\longrightarrow \mathfrak{gl}(V^*) \\ & \quad \quad \quad V^* \longrightarrow V^* \\ x &\longmapsto l_x^* : u^* \longmapsto l_x^* u^* : V \longrightarrow \mathbb{K} \\ & \quad \quad \quad v \longmapsto \langle l_x^* u^*, v \rangle := \langle u^*, l_x v \rangle, \end{aligned} \quad (2.10)$$

$$\begin{aligned} r^* : \mathcal{A} &\longrightarrow \mathfrak{gl}(V^*) \\ & \quad \quad \quad V^* \longrightarrow V^* \\ x &\longmapsto r_x^* : u^* \longmapsto r_x^* u^* : V \longrightarrow \mathbb{K} \\ & \quad \quad \quad v \longmapsto \langle r_x^* u^*, v \rangle := \langle u^*, r_x v \rangle. \end{aligned} \quad (2.11)$$

Proposition 2.10 Let (\mathcal{A}, \cdot) be an anticenter-symmetric algebra and $l, r : \mathcal{A} \rightarrow \mathfrak{gl}(V)$ be two linear maps, where V is a finite dimensional vector space. The following conditions are equivalent:

1. (l, r, V) is a bimodule of \mathcal{A} .
2. (r^*, l^*, V^*) is a bimodule of \mathcal{A} .

Proof:

(1) \Rightarrow (2) Suppose that (l, r, V) is a bimodule of (\mathcal{A}, \cdot) and show that (r^*, l^*, V^*) is also a bimodule of (\mathcal{A}, \cdot) . We have:

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$$\begin{aligned}
& \langle (r_{xy}^* + r_x^* r_y^*) u^*, v \rangle \\
&= \langle r_{xy}^* u^*, v \rangle + \langle (r_x^* r_y^*) u^*, v \rangle = \langle r_{xy}(v), u^* \rangle + \langle r_y(r_x(v)), u^* \rangle \\
&= \langle (r_{xy} + r_y r_x)(v), u^* \rangle = \langle -(l_{yx} + l_y l_x)(v), u^* \rangle \\
&= -\langle l_{yx}(v), u^* \rangle - \langle (l_y l_x)(v), u^* \rangle \\
&= -\langle l_{yx}^* u^*, v \rangle - \langle (l_x^* l_y^*) u^*, v \rangle \\
&= \langle -(l_{yx}^* + l_x^* l_y^*) u^*, v \rangle.
\end{aligned}$$

Therefore,

$$l_{yx}^* + l_x^* l_y^* = -r_{xy}^* - r_x^* r_y^*, \quad \forall x, y \in \mathcal{A} \quad (2.12)$$

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$$\begin{aligned}
& \langle [l_x^*, r_y^*] u^*, v \rangle \\
&= \langle l_x^*(r_y^*) u^*, v \rangle + \langle r_y^*(l_x^*) u^*, v \rangle = \langle l_x(v), r_y^* u^* \rangle + \langle r_y v, l_x^* u^* \rangle \\
&= \langle r_y(l_x(v)), u^* \rangle + \langle l_x(r_y(v)), u^* \rangle = \langle [r_y, l_x] v, u^* \rangle \\
&= \langle -[r_x, l_y] v, u^* \rangle = \langle -(r_x(l_y) + l_y(r_x)) v, u^* \rangle \\
&= \langle -(l_y^* r_x^* + r_x^* l_y^*) u^*, v \rangle = \langle -[l_y^*, r_x^*] u^*, v \rangle
\end{aligned}$$

Therefore

$$[l_x^*, r_y^*] = -[l_y^*, r_x^*], \quad \forall x, y \in \mathcal{A}. \quad (2.13)$$

By considering the relations (2.12) and (2.13), we conclude that (r^*, l^*, V) is a bimodule of (\mathcal{A}, \cdot) .

(2) \Rightarrow (1) The converse, (i.e., by supposing that (r^*, l^*, V) is a bimodule of (\mathcal{A}, \cdot) then (l, r, V) is also a bimodule of (\mathcal{A}, \cdot)), can be proved by direct calculations by using similar relations as for the first part of the proof. □

3 Manin triple of anticenter-symmetric algebras

In this section, [similarly in \[8\] and \[14\]](#), we first give the definition of Manin triple of an anticenter-symmetric algebra and investigate its main properties.

Definition 3.1 *A Manin triple of anticenter-symmetric algebras is a triple $(\mathcal{A}, \mathcal{A}^+, \mathcal{A}^-)$ equipped with a nondegenerate symmetric bilinear form $\mathfrak{B}(\cdot, \cdot)$ on \mathcal{A} which is invariant, i.e., $\forall x, y, z \in \mathcal{A}$, $\mathfrak{B}(x * y, z) = \mathfrak{B}(x, y * z)$, satisfying:*

1. $\mathcal{A} = \mathcal{A}^+ \oplus \mathcal{A}^-$ as \mathbb{K} -vector space;
2. \mathcal{A}^+ and \mathcal{A}^- are anticenter-symmetric subalgebras of \mathcal{A} ;
3. \mathcal{A}^+ and \mathcal{A}^- are isotropic with respect to $\mathfrak{B}(\cdot, \cdot)$, that is $\mathfrak{B}(\mathcal{A}^+, \mathcal{A}^+) = \mathfrak{B}(\mathcal{A}^-, \mathcal{A}^-) = 0$.

Definition 3.2 *Two Manin triples $(\mathcal{A}_1, \mathcal{A}_1^+, \mathcal{A}_1^-, \mathfrak{B}_1)$ and $(\mathcal{A}_2, \mathcal{A}_2^+, \mathcal{A}_2^-, \mathfrak{B}_2)$ of anticenter-symmetric algebras \mathcal{A}_1 and \mathcal{A}_2 are homomorphic (isomorphic) if there is a homomorphism (isomorphism) $\varphi: \mathcal{A}_1 \rightarrow \mathcal{A}_2$ such that: $\varphi(\mathcal{A}_1^+) \subset \mathcal{A}_2^+$, $\varphi(\mathcal{A}_1^-) \subset \mathcal{A}_2^-$, $\mathfrak{B}_1(x, y) = \mathfrak{B}_2(\varphi(x), \varphi(y))$.*

In particular, if (\mathcal{A}, \cdot) is an anticommutative-symmetric algebra, and if there exists an anticommutative-symmetric algebra structure on its dual space \mathcal{A}^* denoted (\mathcal{A}^*, \circ) , then there is a anticommutative-symmetric algebra structure on the direct sum of the underlying vector spaces of \mathcal{A} and \mathcal{A}^* (see Theorem 2.7) such that $(\mathcal{A} \oplus \mathcal{A}^*, \mathcal{A}, \mathcal{A}^*)$ is the associated Manin triple with the invariant bilinear symmetric form given by

$$\mathfrak{B}_d(x + a^*, y + b^*) = \langle x, b^* \rangle + \langle y, a^* \rangle, \quad \forall x, y \in \mathcal{A}; a^*, b^* \in \mathcal{A}^*, \quad (3.1)$$

called the standard Manin triple of the anticommutative-symmetric algebra \mathcal{A} .

Theorem 3.3 *Let (\mathcal{A}, \cdot) and (\mathcal{A}^*, \circ) be two anticommutative-symmetric algebras. Then, the sextuple $(\mathcal{A}, \mathcal{A}^*, R^*, L^*; R_\circ^*, L_\circ^*)$ is a matched pair of anticommutative-symmetric algebras \mathcal{A} and \mathcal{A}^* if and only if $(\mathcal{A} \oplus \mathcal{A}^*, \mathcal{A}, \mathcal{A}^*)$ is their standard Manin triple.*

Proof:

By considering that $(\mathcal{A}, \mathcal{A}^*, R^*, L^*; R_\circ^*, L_\circ^*)$ is a matched pair of anticommutative-symmetric algebras, it follows that the bilinear product $*$ defined in the Theorem 2.7 is anticommutative-symmetric on the direct sum of underlying vectors spaces, $\mathcal{A} \oplus \mathcal{A}^*$.

We have $\forall x, y, z \in \mathcal{A}; a, b, c \in \mathcal{A}^*$.

$$\begin{aligned} \mathfrak{B}_d((x + a) * (y + b), z + c) &= \langle xy + R_\circ^*(a)y + L_\circ^*(b)x, c \rangle + \langle z, a \circ b + R^*(x)b + L^*(y)a \rangle \\ &= \langle xy, c \rangle + \langle R_\circ^*(a)y, c \rangle + \langle L_\circ^*(b)x, c \rangle + \langle z, a \circ b \rangle + \langle z, R^*(x)b \rangle \\ &+ \langle z, L^*(y)a \rangle = \langle xy, c \rangle + \langle y, R_a(c) \rangle + \langle x, L_b(c) \rangle + \langle z, a \circ b \rangle \\ &+ \langle R_x(z), b \rangle + \langle L_y(z), a \rangle = \langle xy, c \rangle + \langle y, c \circ a \rangle \\ &+ \langle x, b \circ c \rangle + \langle z, a \circ b \rangle + \langle zx, b \rangle + \langle yz, a \rangle. \end{aligned}$$

$$\begin{aligned} \mathfrak{B}_d((x + a), (y + b) * (z + c)) &= \langle x, b \circ c + R^*(y)c + L^*(z)b \rangle + \langle yz + R_\circ^*(b)z \\ &+ L_\circ^*(c)y, a \rangle + \langle x, b \circ c \rangle + \langle x, R^*(y)c \rangle + \langle x, L^*(z)b \rangle \\ &+ \langle yz, a \rangle + \langle R_\circ^*(b)z, a \rangle + \langle L_\circ^*(c)y, a \rangle \\ &= \langle x, b \circ c \rangle + \langle R_y(x), c \rangle + \langle L_z(x), b \rangle \\ &+ \langle yz, a \rangle + \langle z, R_b(a) \rangle + \langle y, L_c(a) \rangle \\ &= \langle x, b \circ c \rangle + \langle xy, c \rangle + \langle zx, b \rangle + \langle yz, a \rangle \\ &+ \langle z, a \circ b \rangle + \langle y, c \circ a \rangle. \end{aligned}$$

Therefore, the following relation

$$\mathfrak{B}_d((x + a) * (y + b), (z + c)) = \mathfrak{B}_d((x + a), (y + b) * (z + c)) \quad (3.2)$$

holds, which expresses the invariance of the standard bilinear form on $\mathcal{A} \oplus \mathcal{A}^*$. Therefore, $(\mathcal{A} \oplus \mathcal{A}^*, \mathcal{A}, \mathcal{A}^*)$ is the standard Manin triple of the anticommutative-symmetric algebras \mathcal{A} and \mathcal{A}^* . \square

Proposition 3.4 *Let (\mathcal{A}, \cdot) be an anticommutative-symmetric algebra. Suppose that there exists an anticommutative-symmetric algebra structure “ \circ ” on the dual space \mathcal{A}^* .*

Then, $(\mathcal{A}, \mathcal{A}^, R^*, L^*; R_\circ^*, L_\circ^*)$ is a matched pair of anticommutative-symmetric algebras if and only if for any $x, y \in \mathcal{A}, a \in \mathcal{A}^*$,*

$$R_\circ^*(a)(x \cdot y) + L_\circ^*(a)(y \cdot x) + L_\circ^*(R^*(x)a)y + y \cdot (L_\circ^*(a)x) + R_\circ^*(L^*(x)a)y + (R_\circ^*(a)x) \cdot y = 0, \quad (3.3)$$

$$\begin{aligned} y \cdot (R_\circ^*(a)x) + x \cdot (R_\circ^*(a)y) + (L_\circ^*(a)x) \cdot y + (L_\circ^*(a)y) \cdot x \\ + L_\circ^*(L^*(x)a)y + R_\circ^*(R^*(y)a)x + R_\circ^*(R^*(x)a)y + L_\circ^*(L^*(y)a)x = 0. \end{aligned} \quad (3.4)$$

Proof: Obviously, Eq. (3.3) is exactly Eq. (2.6) and Eq. (3.4) is exactly Eq. (2.8) in the case $l_A = R^*, r_A = L^*, l_B = l_{A^*} = R_o^*, r_B = r_{A^*} = L_o^*$. For any $x, y \in A, a, b \in A^*$, we have:

$$\begin{aligned} \langle R_o^*(a)(x \cdot y), b \rangle &= \langle x \cdot y, R_o(a)b \rangle = \langle x \cdot y, b \circ a \rangle = \langle L_o(x)y, b \circ a \rangle = \langle y, L_o^*(x)(b \circ a) \rangle; \\ \langle L_o^*(a)(y \cdot x), b \rangle &= \langle y \cdot x, L_o(a)b \rangle = \langle y \cdot x, a \circ b \rangle = \langle R_o(x)y, a \circ b \rangle = \langle y, R_o^*(x)(a \circ b) \rangle; \\ \langle L_o^*(R_o^*(x)a)y, b \rangle &= \langle y, L_o(R_o^*(x)a)b \rangle = \langle y, (R_o^*(x)a) \circ b \rangle; \\ \langle y \cdot (L_o^*(a)x), b \rangle &= \langle R_o(L_o^*(a)x)y, b \rangle = \langle y, R_o^*(L_o^*(a)x)b \rangle; \\ \langle R_o^*(L_o^*(x)a)y, b \rangle &= \langle y, R_o(L_o^*(x)a)b \rangle = \langle y, b \circ (L_o^*(x)a) \rangle; \\ \langle (R_o^*(a)x) \cdot y, b \rangle &= \langle L_o(R_o^*(a)x)y, b \rangle = \langle y, L_o^*(R_o^*(a)x)b \rangle. \end{aligned}$$

Then Eq. (2.5) holds if and only if Eq. (2.6) holds. Similarly, Eq. (2.7) holds if and only if Eq. (2.8) holds. Therefore the conclusion holds. \square

Let V be a vector space. Let $\sigma : V \otimes V \rightarrow V \otimes V$ be the *flip* defined as

$$\sigma(x \otimes y) = y \otimes x, \quad \forall x, y \in V. \quad (3.5)$$

Theorem 3.5 *Let (\mathcal{A}, \cdot) be an anticommutative algebra. Suppose there is an anticommutative algebra structure “ \circ ” on its dual space \mathcal{A}^* given by a linear map $\Delta^* : \mathcal{A}^* \otimes \mathcal{A}^* \rightarrow \mathcal{A}^*$. Then $(\mathcal{A}, \mathcal{A}^*, R_o^*, L_o^*, R_o, L_o)$ is a matched pair of anticommutative algebras if and only if $\Delta : \mathcal{A} \rightarrow \mathcal{A} \otimes \mathcal{A}$ satisfies the following two conditions:*

$$\Delta(x \cdot y) + \sigma \Delta(y \cdot x) = -(\sigma(\text{id} \otimes L_o(y)) + R_o(y) \otimes \text{id})\Delta(x) - (\sigma(R_o(x) \otimes \text{id}) + \text{id} \otimes L_o(x))\Delta(y), \quad (3.6)$$

$$\begin{aligned} &(\sigma(\text{id} \otimes R_o(y)) + \text{id} \otimes R_o(y) + \sigma(L_o(y) \otimes \text{id}) + L_o(y) \otimes \text{id})\Delta(x) = \\ &(-\sigma(\text{id} \otimes R_o(x)) - \text{id} \otimes R_o(x) - \sigma(L_o(x) \otimes \text{id}) - L_o(x) \otimes \text{id})\Delta(y), \end{aligned} \quad (3.7)$$

for any $x, y \in \mathcal{A}$.

Proof: For any $x, y \in \mathcal{A}$ and any $a, b \in \mathcal{A}^*$, we have

$$\begin{aligned} \langle \Delta(x \cdot y), a \otimes b \rangle &= \langle x \cdot y, a \cdot b \rangle = \langle L_o^*(a)(x \cdot y), b \rangle, \\ \langle \sigma \Delta(y \cdot x), a \otimes b \rangle &= \langle y \cdot x, b \circ a \rangle = \langle R_o^*(a)(y \cdot x), b \rangle, \\ \langle \sigma(\text{id} \otimes L_o(y))\Delta(x), a \otimes b \rangle &= \langle x, b \circ (L_o^*(y)a) \rangle = \langle R_o^*(L_o^*(y)a)x, b \rangle, \\ \langle (R_o(y) \otimes \text{id})\Delta(x), a \otimes b \rangle &= \langle x, (R_o^*(y)a) \circ b \rangle = \langle L_o^*(R_o^*(y)a)x, b \rangle, \\ \langle \sigma(R_o(x) \otimes \text{id})\Delta(y), a \otimes b \rangle &= \langle y, (R_o^*(x)b) \circ a \rangle = \langle (R_o^*(a)y) \cdot x, b \rangle, \\ \langle (\text{id} \otimes L_o(x))\Delta(y), a \otimes b \rangle &= \langle y, a \circ (L_o^*(x)b) \rangle = \langle x \cdot (L_o^*(a)y), b \rangle. \end{aligned}$$

Then Eq. (3.3) is equivalent to Eq. (3.6). Moreover, we have

$$\begin{aligned} \langle \sigma(\text{id} \otimes R_o(y))\Delta(x), a \otimes b \rangle &= \langle x, b \circ (R_o^*(y)a) \rangle = \langle R_o^*(R_o^*(y)a)x, b \rangle, \\ \langle (\text{id} \otimes R_o(y))\Delta(x), a \otimes b \rangle &= \langle x, a \circ (R_o^*(y)b) \rangle = \langle (L_o^*(a)x) \cdot y, b \rangle, \\ \langle \sigma(L_o(y) \otimes \text{id})\Delta(x), a \otimes b \rangle &= \langle x, (L_o^*(y)b) \circ a \rangle = \langle y \cdot (R_o^*(a)x), b \rangle, \\ \langle (L_o(y) \otimes \text{id})\Delta(x), a \otimes b \rangle &= \langle x, (L_o^*(y)a) \circ b \rangle = \langle L_o^*(L_o^*(y)a)x, b \rangle. \end{aligned}$$

Then Eq. (3.4) is equivalent to Eq. (3.7). Hence the conclusion holds. \square

Remark 3.6 *From the symmetry of the anticommutative algebras (\mathcal{A}, \cdot) and (\mathcal{A}^*, \circ) in the standard Manin triple of anticommutative algebras associated to \mathfrak{B}_d , we also can consider a linear map $\gamma : \mathcal{A}^* \rightarrow \mathcal{A}^* \otimes \mathcal{A}^*$ such that $\gamma^* : \mathcal{A} \otimes \mathcal{A} \rightarrow \mathcal{A}$ gives the anticommutative algebra structure “ \cdot ” on \mathcal{A} . It is straightforward to show that Δ satisfies Eqs. (3.6) and (3.7) if and only if γ satisfies*

$$\gamma(a \circ b) + \sigma \gamma(b \circ a) = (\sigma(\text{id} \otimes L_o(b)) + R_o(b) \otimes \text{id})\gamma(a) + (\sigma(R_o(a) \otimes \text{id}) + \text{id} \otimes L_o(a))\gamma(b), \quad (3.8)$$

$$\begin{aligned} &(\sigma(\text{id} \otimes R_o(b)) + \text{id} \otimes R_o(b) + \sigma(L_o(b) \otimes \text{id}) + (L_o(b) \otimes \text{id}))\gamma(a) + \\ &((L_o(a) \otimes \text{id}) + \sigma(L_o(a) \otimes \text{id}) + \sigma(\text{id} \otimes R_o(a)) + (\text{id} \otimes R_o(a)))\gamma(b) = 0, \end{aligned} \quad (3.9)$$

for any $a, b \in \mathcal{A}^*$.

Definition 3.7 Let (\mathcal{A}, \cdot) be an anticenter-symmetric algebra. An **anticenter-symmetric bialgebra** structure on \mathcal{A} is a linear map $\Delta : \mathcal{A} \rightarrow \mathcal{A} \otimes \mathcal{A}$ such that

1. $\Delta^* : \mathcal{A}^* \otimes \mathcal{A}^* \rightarrow \mathcal{A}^*$ defines an anticenter-symmetric algebra structure on \mathcal{A}^* ;
2. Δ satisfies Eqs. (3.6) and (3.7).

We denote it by (\mathcal{A}, Δ) or $(\mathcal{A}, \mathcal{A}^*)$.

Example 3.8 Let (\mathcal{A}, Δ) be an anticenter-symmetric bialgebra on an anticenter-symmetric algebra \mathcal{A} . Then (\mathcal{A}^*, γ) is an anticenter-symmetric bialgebra on the anticenter-symmetric algebra \mathcal{A}^* , where γ is given in Remark 3.6.

Combining Proposition 3.4 and Theorem 3.5 together, we have the following conclusion.

Theorem 3.9 Let (\mathcal{A}, \cdot) be an anticenter-symmetric algebra. Suppose that there is an anticenter-symmetric algebra structure on its dual space \mathcal{A}^* denoted “ \circ ” which is defined by a linear map $\Delta : \mathcal{A} \rightarrow \mathcal{A} \otimes \mathcal{A}$. Then the following conditions are equivalent.

1. $(\mathcal{A} \oplus \mathcal{A}^*, \mathcal{A}, \mathcal{A}^*)$ is a standard Manin triple of anticenter-symmetric algebras associated to \mathfrak{B}_d defined by Eq. (3.1).
2. $(\mathcal{A}, \mathcal{A}^*, R^*, L^*, R_\circ^*, L_\circ^*)$ is a matched pair of anticenter-symmetric algebras.
3. (\mathcal{A}, Δ) is an anticenter-symmetric bialgebra.

Recall a Mock Lie bialgebra structure on a Mock Lie algebra \mathcal{G} is a linear map $\delta : \mathcal{G} \rightarrow \mathcal{G} \otimes \mathcal{G}$ such that $\delta^* : \mathcal{G}^* \otimes \mathcal{G}^* \rightarrow \mathcal{G}^*$ defines a Mock Lie algebra structure on \mathcal{G}^* and δ satisfies

$$\delta[x, y] = -(\text{ad}(x) \otimes \text{id} + \text{id} \otimes \text{ad}(x))\delta(y) - (\text{ad}(y) \otimes \text{id} + \text{id} \otimes \text{ad}(y))\delta(x), \quad \forall x, y \in \mathcal{G}, \quad (3.10)$$

where $\text{ad}(x)(y) = [x, y]$ for any $x, y \in \mathcal{G}$. We denoted it by (\mathcal{G}, δ) .

Proposition 3.10 Let (\mathcal{A}, Δ) be an anticenter-symmetric bialgebra. Then $(\mathcal{G}(\mathcal{A}), \delta)$ is a Mock Lie bialgebra, where $\delta = \Delta + \sigma\Delta$.

Proof: It is straightforward. □

4 A special class of anticenter-symmetric bialgebras

In this section, we consider a special class of anticenter-symmetric bialgebras, that is, the anticenter-symmetric bialgebra (\mathcal{A}, Δ) on an anti-flexible algebra (\mathcal{A}, \cdot) , with the linear map Δ defined by

$$\Delta(x) = -(\text{id} \otimes L.(x))r - (R.(x) \otimes \text{id})\sigma r, \quad \forall x \in \mathcal{A}, \quad (4.1)$$

where $r \in \mathcal{A} \otimes \mathcal{A}$.

Proposition 4.1 Let (\mathcal{A}, \cdot) be an anticenter-symmetric algebra and $r \in \mathcal{A} \otimes \mathcal{A}$. Let $\Delta : \mathcal{A} \rightarrow \mathcal{A} \otimes \mathcal{A}$ be a linear map defined by Eq. (4.1). Eq. (3.6) holds if and only if

$$(L.(y) \otimes R.(x) + R.(y) \otimes L.(x))(r + \sigma r) = 0, \quad \forall x, y \in \mathcal{A}. \quad (4.2)$$

Proof: Let $r = \sum_i u_i \otimes v_i \in \mathcal{A} \otimes \mathcal{A}$. Then, Eq. (4.1) becomes

$$\Delta(x) = \sum_i (-u_i \otimes xv_i - v_i x \otimes u_i),$$

and

$$\sigma\Delta(x) = \sum_i (-xv_i \otimes u_i - u_i \otimes v_i x).$$

We have:

$$\mathbf{A} = \Delta(xy) + \sigma\Delta(yx) = \sum_i (-u_i \otimes (xy)v_i - v_i(xy) \otimes u_i - (yx)v_i \otimes u_i - u_i \otimes v_i(yx));$$

and

$$\begin{aligned} \mathbf{B} &= -(\sigma(\text{id} \otimes L.(y)) + R.(y) \otimes \text{id})\Delta(x) - (\sigma(R.(x) \otimes \text{id}) + \text{id} \otimes L.(x))\Delta(y) \\ &= \sum_i \left[-(\sigma(\text{id} \otimes L.(y)) + R.(y) \otimes \text{id})(-u_i \otimes xv_i - v_ix \otimes u_i) \right. \\ &\quad \left. - (\sigma(R.(x) \otimes \text{id}) + \text{id} \otimes L.(x))(-u_i \otimes yv_i - v_iy \otimes u_i) \right] \\ &= \mathbf{A} + \sum_i (yu_i \otimes v_ix + u_iy \otimes xv_i + yv_i \otimes u_ix + v_iy \otimes xu_i) \\ &= \mathbf{A} + (L.(y) \otimes R.(x) + R.(y) \otimes L.(x))(r + \sigma r). \end{aligned}$$

By setting $\mathbf{B} = \mathbf{A}$, Eq. (4.2) is established. \square

Proposition 4.2 *Let (\mathcal{A}, \cdot) be an anticommutative algebra and $r \in \mathcal{A} \otimes \mathcal{A}$. Let $\Delta : \mathcal{A} \rightarrow \mathcal{A} \otimes \mathcal{A}$ be a linear map defined by Eq. (4.1). Eq. (3.7) holds if and only if*

$$(R.(x) \otimes R.(y) + R.(y) \otimes R.(x) + L.(x) \otimes L.(y) + L.(y) \otimes L.(x))(r + \sigma r) = 0, \quad \forall x, y \in \mathcal{A}. \quad (4.3)$$

Proof: In this proof, for simplicity, we take $r = u_i \otimes v_i \in \mathcal{A} \otimes \mathcal{A}$.

On the one hand, the left-hand side of Eq. (3.7) is given by:

$$\begin{aligned} \mathbf{A} &= (\sigma(\text{id} \otimes R.(y)) + \text{id} \otimes R.(y) + \sigma(L.(y) \otimes \text{id}) + L.(y) \otimes \text{id})\Delta(x) \\ &= -(xv_i)y \otimes u_i - u_iy \otimes v_ix - u_i \otimes (xv_i)y - v_ix \otimes u_iy - xv_i \otimes yu_i \\ &\quad - u_i \otimes y(v_ix) - yu_i \otimes xv_i - y(v_ix) \otimes u_i. \end{aligned}$$

On the other hand, the right-hand side of Eq. (3.7) is:

$$\begin{aligned} \mathbf{B} &= (-\sigma(\text{id} \otimes R.(x)) - \text{id} \otimes R.(x) - \sigma(L.(x) \otimes \text{id}) - L.(x) \otimes \text{id})\Delta(y) \\ &= (yv_i)x \otimes u_i + u_ix \otimes v_iy + u_i \otimes (yv_i)x + v_iy \otimes u_ix + yv_i \otimes xu_i \\ &\quad + u_i \otimes x(v_iy) + xu_i \otimes yv_i + x(v_iy) \otimes u_i. \end{aligned}$$

By setting $\mathbf{A} = \mathbf{B}$, we obtain:

$$\begin{aligned} &u_iy \otimes v_ix + v_ix \otimes u_iy + xv_i \otimes yu_i + yu_i \otimes xv_i \\ &+ u_ix \otimes v_iy + v_iy \otimes u_ix + yv_i \otimes xu_i + xu_i \otimes yv_i = 0. \end{aligned}$$

This establishes Eq. (4.3). \square

Lemma 4.3 *Let \mathcal{A} be a vector space and $\Delta : \mathcal{A} \rightarrow \mathcal{A} \otimes \mathcal{A}$ be a linear map. Then the dual map $\Delta^* : \mathcal{A}^* \otimes \mathcal{A}^* \rightarrow \mathcal{A}^*$ defines an anticommutative algebra structure on \mathcal{A}^* if and only if $H_\Delta = 0$, where*

$$H_\Delta = (\Delta \otimes \text{id})\Delta + (\text{id} \otimes \Delta)\Delta + ((\sigma\Delta) \otimes \text{id})(\sigma\Delta) + (\text{id} \otimes (\sigma\Delta))(\sigma\Delta). \quad (4.4)$$

Proof: Denote by \circ the product on \mathcal{A}^* defined by Δ^* . Specifically,

$$\langle a \circ b, x \rangle = \langle \Delta^*(a \otimes b), x \rangle = \langle a \otimes b, \Delta(x) \rangle, \quad \forall x \in \mathcal{A}, a, b \in \mathcal{A}^*.$$

For all $a, b, c \in \mathcal{A}^*$ and $x \in \mathcal{A}$, we have:

$$\begin{aligned}
\langle (a, b, c), x \rangle &= \langle (a \circ b) \circ c + a \circ (b \circ c), x \rangle \\
&= \langle (\Delta^*(\Delta^* \otimes \text{id}) + \Delta^*(\text{id} \otimes \Delta^*)) (a \otimes b \otimes c), x \rangle \\
&= \langle ((\Delta \otimes \text{id})\Delta + (\text{id} \otimes \Delta)\Delta)(x), a \otimes b \otimes c \rangle; \\
\langle -(c, b, a), x \rangle &= \langle -(c \circ b) \circ a - c \circ (b \circ a), x \rangle \\
&= \langle (-\Delta^*(\Delta^* \otimes \text{id}) - \Delta^*(\text{id} \otimes \Delta^*)) (c \otimes b \otimes a), x \rangle \\
&= \langle (-\Delta^*\sigma^*)((\Delta^*\sigma^*) \otimes \text{id}) - (\Delta^*\sigma^*)(\text{id} \otimes (\Delta^*\sigma^*)) (a \otimes b \otimes c), x \rangle \\
&= \langle (-((\sigma\Delta) \otimes \text{id})(\sigma\Delta) - (\text{id} \otimes (\sigma\Delta))(\sigma\Delta))(x), a \otimes b \otimes c \rangle.
\end{aligned}$$

Thus, (\mathcal{A}^*, \circ) is an anticenter-symmetric algebra if and only if $H_\Delta = 0$. \square

Now, let (\mathcal{A}, \cdot) be an anticenter-symmetric algebra and let

$$r = \sum_i u_i \otimes v_i \in \mathcal{A} \otimes \mathcal{A}.$$

Define:

$$\begin{aligned}
r_{12} &= \sum_i u_i \otimes v_i \otimes 1, & r_{13} &= \sum_i u_i \otimes 1 \otimes v_i, & r_{23} &= \sum_i 1 \otimes u_i \otimes v_i, \\
r_{21} &= \sum_i v_i \otimes u_i \otimes 1, & r_{31} &= \sum_i v_i \otimes 1 \otimes u_i, & r_{32} &= \sum_i 1 \otimes v_i \otimes u_i,
\end{aligned}$$

where 1 denotes the unit if (\mathcal{A}, \cdot) has a unit. Otherwise, it is a symbol that serves a similar role to a unit. The operation between two rs is then defined in an obvious manner. For example,

$$r_{12}r_{13} = \sum_{i,j} u_i \cdot u_j \otimes v_i \otimes v_j, \quad r_{13}r_{23} = \sum_{i,j} u_i \otimes u_j \otimes v_i \cdot v_j, \quad r_{23}r_{12} = \sum_{i,j} u_j \otimes u_i \cdot v_j \otimes v_i, \quad (4.5)$$

and so on.

Theorem 4.4 *Let (\mathcal{A}, \cdot) be an anticenter-symmetric algebra and $r \in \mathcal{A} \otimes \mathcal{A}$. Let $\Delta : \mathcal{A} \rightarrow \mathcal{A} \otimes \mathcal{A}$ be a linear map defined by Eq. (4.1). Then, Δ^* defines an anticenter-symmetric algebra structure on \mathcal{A}^* if and only if, for any $x \in \mathcal{A}$, the following holds:*

$$\begin{aligned}
&(\text{id} \otimes \text{id} \otimes L(x))(M(r)) + (\text{id} \otimes \text{id} \otimes R(x))(P(r)) \\
&+ (L(x) \otimes \text{id} \otimes \text{id})(-N(r)) + (R(x) \otimes \text{id} \otimes \text{id})(-Q(r)) = 0,
\end{aligned} \quad (4.6)$$

where:

$$\begin{aligned}
M(r) &= r_{23}r_{12} + r_{21}r_{13} - r_{13}r_{23}, & N(r) &= r_{31}r_{21} - r_{21}r_{32} - r_{23}r_{31}, \\
P(r) &= r_{13}r_{21} + r_{12}r_{23} - r_{23}r_{13}, & Q(r) &= r_{21}r_{31} - r_{31}r_{23} - r_{32}r_{21}.
\end{aligned}$$

Proof. Let $r = \sum_i u_i \otimes v_i \in \mathcal{A} \otimes \mathcal{A}$. Then:

$$\begin{aligned}
&((\Delta \otimes \text{id})\Delta + (\text{id} \otimes \Delta)\Delta)(x) \\
&= \sum_{i,j} (u_j \otimes u_i v_j \otimes x v_i + v_j u_i \otimes u_j \otimes x v_i + u_j \otimes (v_i x) v_j \otimes u_i + v_j (v_i x) \otimes u_j \otimes u_i \\
&\quad + u_i \otimes u_j \otimes (x v_i) v_j + u_i \otimes v_j (x v_i) \otimes u_j + v_i x \otimes u_j \otimes u_i v_j + v_i x \otimes v_j u_i \otimes u_j) \\
&= (\text{id} \otimes \text{id} \otimes L(x))(r_{23}r_{12}) + (\text{id} \otimes \text{id} \otimes L(x))(r_{21}r_{13}) - (R(x) \otimes \text{id} \otimes \text{id})(r_{21}r_{31}) \\
&\quad - (\text{id} \otimes \text{id} \otimes L(x))(r_{13}r_{23}) + (R(x) \otimes \text{id} \otimes \text{id})(r_{31}r_{23}) + (R(x) \otimes \text{id} \otimes \text{id})(r_{32}r_{21}) \\
&\quad + \sum_{i,j} (u_j \otimes (v_i x) v_j \otimes u_i + u_i \otimes v_j (x v_i) \otimes u_j).
\end{aligned}$$

Similarly:

$$\begin{aligned}
& (((\sigma\Delta) \otimes \text{id})(\sigma\Delta) + (\text{id} \otimes (\sigma\Delta))(\sigma\Delta))(x) \\
&= \sum_{i,j} ((xv_i)v_j \otimes u_j \otimes u_i + u_j \otimes v_j(xv_i) \otimes u_i + u_i v_j \otimes u_j \otimes v_i x + u_j \otimes v_j u_i \otimes v_i x \\
&\quad + xv_i \otimes u_i v_j \otimes u_j + xv_i \otimes u_j \otimes v_j u_i + u_i \otimes (v_i x)v_j \otimes u_j + u_i \otimes u_j \otimes v_j(v_i x)) \\
&= -(L.(x) \otimes \text{id} \otimes \text{id})(r_{31}r_{21}) - (\text{id} \otimes \text{id} \otimes R.(x))(r_{23}r_{13}) + (\text{id} \otimes \text{id} \otimes R.(x))(r_{13}r_{21}) \\
&\quad + (\text{id} \otimes \text{id} \otimes R.(x))(r_{12}r_{23}) + (L.(x) \otimes \text{id} \otimes \text{id})(r_{21}r_{32}) + (L.(x) \otimes \text{id} \otimes \text{id})(r_{23}r_{31}) \\
&\quad + \sum_{i,j} (u_j \otimes v_j(xv_i) \otimes u_i + u_i \otimes (v_i x)v_j \otimes u_j).
\end{aligned}$$

By exchanging the indices i and j , we obtain:

$$\sum_{i,j} (u_j \otimes (v_i x)v_j \otimes u_i + u_i \otimes v_j(xv_i) \otimes u_j) + \sum_{i,j} (u_j \otimes v_j(xv_i) \otimes u_i + u_i \otimes (v_i x)v_j \otimes u_j) = 0.$$

Thus, it follows that:

$$\begin{aligned}
& (L.(x) \otimes \text{id} \otimes \text{id})(r_{21}r_{32} + r_{23}r_{31} - r_{31}r_{21}) \\
&+ (\text{id} \otimes \text{id} \otimes L.(x))(r_{23}r_{12} + r_{21}r_{13} - r_{13}r_{23}) \\
&+ (R.(x) \otimes \text{id} \otimes \text{id})(r_{31}r_{23} + r_{32}r_{21} - r_{21}r_{31}) \\
&+ (\text{id} \otimes \text{id} \otimes R.(x))(r_{13}r_{21} + r_{12}r_{23} - r_{23}r_{13}) = 0.
\end{aligned}$$

This establishes Eq. (4.6). □

Remark 4.5 [8] For any $r \in \mathcal{A} \otimes \mathcal{A}$, the following holds:

$$N(r) = -\sigma_{13}M(r), \quad P(r) = \sigma_{12}M(r), \quad Q(r) = -\sigma_{12}\sigma_{13}M(r),$$

where $\sigma_{12}(x \otimes y \otimes z) = y \otimes x \otimes z$ and $\sigma_{13}(x \otimes y \otimes z) = z \otimes y \otimes x$, for any $x, y, z \in \mathcal{A}$.

Combining Proposition 4.1, Proposition 4.2, Theorem 4.4, and Remark 4.5, we arrive at the following result.

Theorem 4.6 Let (\mathcal{A}, \cdot) be an anticommutative algebra and $r \in \mathcal{A} \otimes \mathcal{A}$. Let $\Delta : \mathcal{A} \rightarrow \mathcal{A} \otimes \mathcal{A}$ be a linear map defined by Eq. (4.1). Then (\mathcal{A}, Δ) is an anticommutative bialgebra if and only if r satisfies Eqs. (4.2), (4.3), and

$$\begin{aligned}
& ((\text{id} \otimes \text{id} \otimes L.(x)) + (R.(x) \otimes \text{id} \otimes \text{id})\sigma_{12}\sigma_{13} \\
&\quad + ((\text{id} \otimes \text{id} \otimes R.(x))\sigma_{12} + (L.(x) \otimes \text{id} \otimes \text{id})\sigma_{13}))(M(r)) = 0,
\end{aligned} \tag{4.7}$$

where $M(r) = r_{23}r_{12} + r_{21}r_{13} - r_{13}r_{23}$.

As a direct consequence of Theorem 4.6, we have the following corollary.

Corollary 4.7 Let (\mathcal{A}, \cdot) be an anticommutative algebra and $r \in \mathcal{A} \otimes \mathcal{A}$. Let $\Delta : \mathcal{A} \rightarrow \mathcal{A} \otimes \mathcal{A}$ be a linear map defined by Eq. (4.1). If, in addition, r is skew-symmetric and satisfies

$$r_{12}r_{13} - r_{23}r_{12} + r_{13}r_{23} = 0, \tag{4.8}$$

then (\mathcal{A}, Δ) is an anticommutative bialgebra.

Definition 4.8 Let (\mathcal{A}, \cdot) be an anticommutative algebra and $r \in \mathcal{A} \otimes \mathcal{A}$. Eq. (4.8) is called the **anticommutative Yang-Baxter equation (ACSYBE)** in (\mathcal{A}, \cdot) .

Remark 4.9 The term "antcenter-symmetric Yang-Baxter equation" reflects its analogy with the classical Yang-Baxter equation in a Mock Lie algebra (see [5]). Notably, the antcenter-symmetric Yang-Baxter equation in an antcenter-symmetric algebra, the anti-flexible Yang-Baxter equation in an anti-flexible algebra, and the associative Yang-Baxter equation (see [3, 8]) in an associative algebra all share the same form as Eq. (4.8). Thus, these three equations exhibit common properties.

At the end of this section, we highlight two properties of the antcenter-symmetric Yang-Baxter equation. The proofs are omitted since they mirror the proofs in the case of the associative Yang-Baxter equation.

Let \mathcal{A} be a vector space. For any $r \in \mathcal{A} \otimes \mathcal{A}$, r can be regarded as a linear map from \mathcal{A}^* to \mathcal{A} as follows:

$$\langle r, u^* \otimes v^* \rangle = \langle r(u^*), v^* \rangle, \quad \forall u^*, v^* \in \mathcal{A}^*. \quad (4.9)$$

Proposition 4.10 Let (\mathcal{A}, \cdot) be an antcenter-symmetric algebra and $r \in \mathcal{A} \otimes \mathcal{A}$ be skew-symmetric. Then r is a solution of the antcenter-symmetric Yang-Baxter equation if and only if r satisfies

$$r(a) \cdot r(b) = r(R^*(r(a))b + L^*(r(b))a), \quad \forall a, b \in \mathcal{A}^*. \quad (4.10)$$

Theorem 4.11 Let (\mathcal{A}, \cdot) be an antcenter-symmetric algebra and $r \in \mathcal{A} \otimes \mathcal{A}$. Suppose that r is antisymmetric and nondegenerate. Then r is a solution of the antcenter-symmetric Yang-Baxter equation in (\mathcal{A}, \cdot) if and only if the inverse of the isomorphism $\mathcal{A}^* \rightarrow \mathcal{A}$ induced by r , regarded as a bilinear form ω on \mathcal{A} (i.e., $\omega(x, y) = \langle r^{-1}x, y \rangle$ for any $x, y \in \mathcal{A}$), satisfies

$$\omega(x \cdot y, z) + \omega(y \cdot z, x) + \omega(z \cdot x, y) = 0, \quad \forall x, y, z \in \mathcal{A}. \quad (4.11)$$

5 \mathcal{O} -operators of antcenter-symmetric algebras and pre-antcenter-symmetric algebras

In this section, we introduce the notions of \mathcal{O} -operators for antcenter-symmetric algebras and pre-antcenter-symmetric algebras, which are used to construct skew-symmetric solutions of the antcenter-symmetric Yang-Baxter equation and, consequently, to construct antcenter-symmetric bialgebras.

Definition 5.1 Let (l, r, V) be a bimodule of an antcenter-symmetric algebra (\mathcal{A}, \cdot) . A linear map $T : V \rightarrow \mathcal{A}$ is called an \mathcal{O} -operator associated with (l, r, V) if T satisfies

$$T(u) \cdot T(v) = T(l(T(u))v + r(T(v))u), \quad \forall u, v \in V.$$

Example 5.2 Let (\mathcal{A}, \cdot) be an antcenter-symmetric algebra. An \mathcal{O} -operator R_B associated with the regular bimodule (L, R, \mathcal{A}) is called a **Rota-Baxter operator of weight zero**. In this case, R_B satisfies

$$R_B(x) \cdot R_B(y) = R_B(R_B(x) \cdot y + x \cdot R_B(y)), \quad \forall x, y \in \mathcal{A}.$$

Example 5.3 Let (\mathcal{A}, \cdot) be an antcenter-symmetric algebra, and let $r \in \mathcal{A} \otimes \mathcal{A}$. If r is skew-symmetric, then by Proposition 4.10, r is a solution of the antcenter-symmetric Yang-Baxter equation if and only if r , regarded as a linear map from \mathcal{A}^* to \mathcal{A} , is an \mathcal{O} -operator associated with the bimodule $(R^*, L^*, \mathcal{A}^*)$.

There is the following construction of (skew-symmetric) solutions of antcenter-symmetric Yang-Baxter equation in a semi-direct product antcenter-symmetric algebra from an \mathcal{O} -operator of an antcenter-symmetric algebra which is similar as for associative algebras ([3, Theorem 2.5.5], hence the proof is omitted).

Theorem 5.4 Let (l, r, V) be a bimodule of an anticommutative algebra (\mathcal{A}, \cdot) , and let $T : V \rightarrow \mathcal{A}$ be a linear map. Identifying T as an element in $(\mathcal{A} \ltimes_{r^*, l^*} V^*) \oplus (\mathcal{A} \ltimes_{r^*, l^*} V^*)$, $r = T - \sigma(T)$ is a skew-symmetric solution of the anticommutative Yang-Baxter equation in $\mathcal{A} \ltimes_{r^*, l^*} V^*$ if and only if T is an \mathcal{O} -operator associated with the bimodule (l, r, V) .

Definition 5.5 Let \mathcal{A} be a vector space with two bilinear products $\prec, \succ : \mathcal{A} \otimes \mathcal{A} \rightarrow \mathcal{A}$. The pair $(\mathcal{A}, \prec, \succ)$ is called a **pre-anticenter-symmetric algebra** if, for any $x, y, z \in \mathcal{A}$, the following conditions hold:

$$\begin{aligned}(x, y, z)_m &= -(z, y, x)_m, \\ (x, y, z)_l &= -(z, y, x)_r,\end{aligned}$$

where:

$$\begin{aligned}(x, y, z)_m &:= (x \succ y) \prec z + x \succ (y \prec z), \\ (x, y, z)_l &:= (x * y) \succ z + x \succ (y \succ z), \\ (x, y, z)_r &:= (x \prec y) \prec z + x \prec (y * z),\end{aligned}$$

and $x * y = x \prec y + x \succ y$.

Proposition 5.6 Let $(\mathcal{A}, \prec, \succ)$ be a pre-anticenter-symmetric algebra. Define a bilinear product $* : \mathcal{A} \otimes \mathcal{A} \rightarrow \mathcal{A}$ by

$$x * y = x \prec y + x \succ y, \quad \forall x, y \in \mathcal{A}. \quad (5.1)$$

Then $(\mathcal{A}, *)$ is an anticommutative algebra, referred to as the **associated anticommutative algebra** of $(\mathcal{A}, \prec, \succ)$.

Proof: Set $(x, y, z)_* = (x * y) * z + x * (y * z)$. For any $x, y, z \in \mathcal{A}$, we have:

$$(x, y, z)_* = (x, y, z)_m + (x, y, z)_l + (x, y, z)_r = -(z, y, x)_m - (z, y, x)_r - (z, y, x)_l = -(z, y, x)_*.$$

Hence, $(\mathcal{A}, *)$ is an anticommutative algebra. \square

Let $(\mathcal{A}, \prec, \succ)$ be a pre-anticenter-symmetric algebra. For any $x \in \mathcal{A}$, let $L_\succ(x), R_\prec(x)$ denote the left multiplication operator of (\mathcal{A}, \prec) and the right multiplication operator of (\mathcal{A}, \succ) respectively, that is, $L_\succ(x)(y) = x \succ y$, $R_\prec(x)(y) = y \prec x$, $\forall x, y \in \mathcal{A}$. Moreover, let $L_\succ, R_\prec : \mathcal{A} \rightarrow \mathfrak{gl}(\mathcal{A})$ be two linear maps with $x \rightarrow L_\succ(x)$ and $x \rightarrow R_\prec(x)$ respectively.

Proposition 5.7 Let $(\mathcal{A}, \prec, \succ)$ be a pre-anticenter-symmetric algebra. Then $(L_\succ, R_\prec, \mathcal{A})$ is a bimodule of the associated anti-flexible algebra $(\mathcal{A}, *)$, where $*$ is defined by Eq. (5.1).

Proof: For any $x, y, z \in \mathcal{A}$, we have

$$\begin{aligned}(L_\succ(x * y) + L_\succ(x)L_\succ(y))(z) &= (x * y) \succ z + x \succ (y \succ z) = (x, y, z)_l, \\ (-R_\prec(x)R_\prec(y) - R_\prec(y * x))(z) &= -(z \prec y) \prec x - z \prec (y * x) = -(z, y, x)_r, \\ (L_\succ(x)R_\prec(y) + R_\prec(y)L_\succ(x))(z) &= x \succ (z \prec y) + (x \succ z) \prec y = (x, z, y)_m, \\ (-L_\succ(y)R_\prec(x) - R_\prec(x)L_\succ(y))(z) &= -y \succ (z \prec x) - (y \succ z) \prec x = -(y, z, x)_m.\end{aligned}$$

Hence $(L_\succ, R_\prec, \mathcal{A})$ is a bimodule of $(\mathcal{A}, *)$. \square

Corollary 5.8 Let $(\mathcal{A}, \prec, \succ)$ be a pre-anticenter-symmetric algebra. Then the identity map id is an \mathcal{O} -operator of the associated anticommutative algebra $(\mathcal{A}, *)$ associated with the bimodule $(L_\succ, R_\prec, \mathcal{A})$.

Theorem 5.9 Let (l, r, V) be a bimodule of an anticommutative algebra (\mathcal{A}, \cdot) . Let $T : V \rightarrow \mathcal{A}$ be an \mathcal{O} -operator associated with (l, r, V) . Then, there exists a pre-anticenter-symmetric algebra structure on V given by

$$u \succ v = l(T(u))v, \quad u \prec v = r(T(v))u, \quad \forall u, v \in V. \quad (5.2)$$

Consequently, there is an associated anticommutative algebra structure on V given by Eq. (5.1), and T is a homomorphism of anticommutative algebras. Moreover, $T(V) = \{T(v) \mid v \in V\} \subset \mathcal{A}$ is an anticommutative subalgebra of (\mathcal{A}, \cdot) , and there is an induced pre-anticenter-symmetric algebra structure on $T(V)$ given by

$$T(u) \succ T(v) = T(u \succ v), \quad T(u) \prec T(v) = T(u \prec v), \quad \forall u, v \in V.$$

The corresponding associated anticommutative algebra structure on $T(V)$, as given by Eq. (5.1), is precisely the anticommutative subalgebra structure of (\mathcal{A}, \cdot) , and T is a homomorphism of pre-anticenter-symmetric algebras.

Proof: For all $u, v, w \in V$, we have

$$\begin{aligned} (u, v, w)_m &= (u \succ v) \prec w + u \succ (v \prec w) = r(T(w))l(T(u))v + l(T(u))r(T(w))v \\ &= -r(T(u))l(T(w))v - l(T(u))r(T(w))v = -(w, v, u)_m, \\ (u, v, w)_i &= (u \succ v + u \prec v) \succ w + u \succ (v \succ w) \\ &= (l(T(l(T(u))v + r(T(v))u)) + l(T(u))l(T(v)))w \\ &= (l(T(u) \cdot T(v)) + l(T(u))l(T(v)))w = -(r(T(u))r(T(v)) - r(T(v) \cdot T(u)))w \\ &= -(r(T(u))r(T(v)) - r(T(u \succ v + u \prec v)))w \\ &= -(w \prec v) \prec u - w \prec (u \succ v + u \prec v) \\ &= -(w, v, u)_r \end{aligned}$$

Therefore, (V, \prec, \succ) is a pre-anticenter-symmetric algebra. For $T(V)$, we have

$$T(u) * T(v) = T(u \succ v + u \prec v) = T(u * v) = T(u) \cdot T(v), \quad \forall u, v \in V.$$

The rest is straightforward. \square

Corollary 5.10 Let (\mathcal{A}, \cdot) be an anticommutative algebra. Then there exists a pre-anticenter-symmetric algebra structure on \mathcal{A} such that its associated anticommutative algebra is (\mathcal{A}, \cdot) if and only if there exists an invertible \mathcal{O} -operator.

Proof: Suppose that there exists an invertible \mathcal{O} -operator $T : V \rightarrow \mathcal{A}$ associated to a bimodule (l, r, V) . Then the products “ \succ, \prec ” given by Eq. (5.2) defines a pre-anticenter-symmetric algebra structure on V . Moreover, there is a pre-anticenter-symmetric algebra structure on $T(V) = \mathcal{A}$, that is,

$$x \succ y = T(l(x)T^{-1}(y)), \quad x \prec y = T(r(y)T^{-1}(x)), \quad \forall x, y \in \mathcal{A}.$$

Moreover, for any $x, y \in \mathcal{A}$, we have

$$x \succ y + x \prec y = T(l(x)T^{-1}(y) + r(y)T^{-1}(x)) = T(T^{-1}(x)) \cdot T(T^{-1}(y)) = x \cdot y.$$

Hence the associated anticommutative algebra of $(\mathcal{A}, \succ, \prec)$ is (\mathcal{A}, \cdot) .

Conversely, let $(\mathcal{A}, \succ, \prec)$ be pre-anticenter-symmetric algebra such that its associated anticommutative algebra is (\mathcal{A}, \cdot) . Then by Corollary 5.8, the identity map id is an \mathcal{O} -operator of (\mathcal{A}, \cdot) associated to the bimodule $(L_\succ, R_\prec, \mathcal{A})$. \square

Corollary 5.11 Let (\mathcal{A}, \cdot) be an anticommutative algebra and ω be a nondegenerate skew-symmetric bilinear form satisfying Eq. (4.11). Then there exists a pre-anticenter-symmetric algebra structure \succ, \prec on \mathcal{A} given by

$$\omega(x \succ y, z) = \omega(y, z \cdot x), \quad \omega(x \prec y, z) = \omega(x, y \cdot z), \quad \forall x, y, z \in \mathcal{A}, \quad (5.3)$$

such that the associated anticommutative algebra is (\mathcal{A}, \cdot) .

Proof: Define a linear map $T : \mathcal{A} \rightarrow \mathcal{A}^*$ by

$$\langle T(x), y \rangle = \omega(x, y), \quad \forall x, y \in \mathcal{A}.$$

Then T is invertible and T^{-1} is an \mathcal{O} -operator of the anticenter-symmetric algebra (\mathcal{A}, \cdot) associated to the bimodule (R^*, L^*, A^*) . By Corollary 5.10, there is a pre-anticenter-symmetric algebra structure \succ, \prec on $(\mathcal{A}, *)$ given by

$$x \succ y = T^{-1}R^*(x)T(y), \quad x \prec y = T^{-1}L^*(y)T(x), \quad \forall x, y \in \mathcal{A},$$

which gives exactly Eq. (5.3) such that the associated anticenter-symmetric algebra is (\mathcal{A}, \cdot) . \square

Finally we give the following construction of skew-symmetric solutions of anticenter-symmetric Yang-Baxter equation (hence anticenter-symmetric bialgebras) from a pre-anticenter-symmetric algebra.

Proposition 5.12 *Let $(\mathcal{A}, \succ, \prec)$ be a pre-anticenter-symmetric algebra. Then*

$$r = \sum_i^n (e_i \otimes e_i^* - e_i^* \otimes e_i) \tag{5.4}$$

is a solution of anticenter-symmetric Yang-Baxter equation in $\mathcal{A} \times_{R^, L^*} \mathcal{A}^*$, where $\{e_1, \dots, e_n\}$ is a basis of \mathcal{A} and $\{e_1^*, \dots, e_n^*\}$ is its dual basis.*

Proof: Note that the identity map $\text{id} = \sum_{i=1}^n e_i \otimes e_i^*$. Hence the conclusion follows from Theorem 5.4 and Corollary 5.8. \square

6 Concluding remarks

We established a bialgebra theory for anticenter-symmetric algebras, introducing the notion of an anticenter-symmetric bialgebra and its equivalence to a Manin triple of anticenter-symmetric algebras. A key result is the formulation of the anticenter-symmetric Yang-Baxter equation in anticenter-symmetric algebras, an analogue to the classical Yang-Baxter equation in Mock Lie algebras and the associative Yang-Baxter equation, with the unexpected finding that they share the same formal structure.

We showed that skew-symmetric solutions to this equation define anticenter-symmetric bialgebras. Additionally, the notions of \mathcal{O} -operators and pre-anticenter-symmetric algebras were introduced as tools to construct such solutions, providing a foundation for further exploration of anticenter-symmetric algebraic structures.

Investigating integrable systems in the context of anticenter-symmetric algebras is an interesting avenue for further study. Rota-Baxter-type operators are naturally investigated in this context, motivated by their importance in deformation and integrability theory, given the structural similarities between the anticenter-symmetric and associative Yang-Baxter equations [12]. Furthermore, Rota-Baxter and Nijenhuis operators can be used to deform (Lie) algebras, which makes it possible to apply these methods to anticenter-symmetric and their bialgebraic structures [9, 10, 13, 15, 16, 18]. These links imply possible uses in noncommutative geometry and mathematical physics in addition to enhancing the algebraic landscape of anticenter-symmetric bialgebras.

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