**On the properties of frames in 2-Hilbert spaces**

**Abstract:** 2-frames in 2-Hilbert spaces are studied, and several related results are presented. A definition of a frame associated with a fixed element in 2-Hilbert spaces is introduced and illustrated through examples. Various properties of the corresponding frame operator are investigated. Furthermore, several results from the theory of frames in Hilbert spaces are extended to the setting of 2-Hilbert spaces.

Key words: 2-norm, 2-inner product, frame, 2-Hilbert space.

2000 Mathematics subject classification: 46C05, 46C99,

**1. Introduction**

“The concept of frames in Hilbert spaces was first introduced by Duffin and Schaeffer in 1952 to address certain deep problems in nonharmonic Fourier series [9,10]. Later, D. Han and D.R. Larson” [4] “developed several foundational aspects of the operator-theoretic approach to frame theory in Hilbert spaces. Peter G. Casazza” [2] provided a comprehensive tutorial on frame theory and identified key directions for future research in the field.

“The concept of 2-frames in 2-inner product spaces was introduced by Ali Akbar Arefijammaal and Ghadir Sadeghi” [1], who also established several fundamental properties. Y.J. Cho, S.S. Dragomir, A. White, and S.S. Kim [3] “presented various inequalities in the context of 2-inner product spaces, while additional results were contributed by H. Mazaheri and R. Kazemi” [6]. “Furthermore, the tensor product of 2-frames in 2-Hilbert spaces was introduced by G.U. Reddy” [7], who demonstrated that the tensor product of two 2-frames forms a 2-frame for the tensor product of the corresponding Hilbert spaces. Chetna Mehra, Narendra Biswas and Mahesh C. Joshi [5] were introduced 2-woven frames in 2- Hilbert spaces and explore some of their properties.

In this paper 2-frames in 2-Hilbert spaces are studied, and several related results are presented. A definition of a frame associated with a fixed element in 2-Hilbert spaces is introduced and illustrated through examples. Various properties of the corresponding frame operator are investigated. Furthermore, several results from the theory of frames in Hilbert spaces are extended to the setting of 2-Hilbert spaces.Key words: 2-norm, 2-inner product, frame, 2-Hilbert space.

**2. Preliminaries**

The following basic definitions of 2-normed spaces and 2-inner product spaces from[6] are usefull in our discussion.

**Definition2.1.** Let H be a complex vector space of dimension greater than 1 and let  be a real-valued function on HxH satisfying the following conditions: 

a)  and  if and only if a and b are linearly dependent vectors.

b) 

c) 

d)

Then  is called 2-norm on H and  called a linear 2-normed space.

We can consider a 2-norm on H defined by an inner product  on H as follows.

for all 

M. Eshaghi Gordji, A. Divandari, M.R. Safi, and Cho[8] were define a new definition of inner product

space as follows.

**Definition2.2 .** A Complex vector space H is called a 2-inner product space if there exists a complex-valued function  on such that for all a,b,c,d,e  H and 

1. If a and b are linearly Independent in H, then 
2. 
3. 
4. 
5. 

**Remerark. 2.3.** By using the above we have the following axioms for a,b,c,d, and 

1. 
2. 
3. 
4. 
5. are linearly dependent
6. 

**Theorem. 2.4[8].** Let H be a 2- inner product space. Then the real valued function defined by is a 2-norm on H.

**Proof:** Given H is a 2-inner product space for 

1. 
2. 
3. 
4. 











 [By Schwartz inequality)





For all 

 H is a 2- normad space.

**Example 2.5.** Let H be a complex vector space with inner product <.,.>, we define 

For all Then H is a 2-inner product space.

**Theorem. 2.6[8].** [The Schwartz inequality] Let H be a 2 – inner product space. Then  for all .

**Proof:** For any Complex number , we have









Putting 







Mazahen and Kazemin [6,8 ] introduced concept of cauchy sequence, where  is non-zero vector in 2-innerproduct space H.

**Definition2.7[8].** Let  be a 2-inner product space and .

1. A sequence  in H is a Cauchy sequence if for any there exists a positive integer N such that



1. If every Cauchy sequence converges a point in a semi -2-normad space 

Then H is called a Hilbert space.

if H is a - Hilbert space for any  then we say that H is called 2 – Hilbert space.

**3. Frames**

The following definitions from [2,4] are essential to our discussion.

***Definition3.1.*** *A sequence**of vectors in a Hilbert space H is called a frame if there exist constants 0 < A ≤ B <∝ such that*

*A  ≤  ≤ B  for all x ∈* H.

The above inequality is called the frame inequality. The numbers A and B are called lower and upper frame bounds respectively. If A=B then **is called tight frame, if A=B=1 then ** is called normalized tight frame.

**Definition3.2.** Let  be a frame for H, a synthesis operator T : *l*2 →H is defined as .

**Definition3.3.** Let  be a frame for H, the analysis operator T : H → *l*2 is the adjoint of synthesis operator T and is defined as  for all x ∈ H.

**Definition3.4.** Let  be a frame for the Hilbert space H. A frame operator S = T T: H→ H is defined as  for all x ∈ H.

**4. Frames in 2-Hilbert spaces**

In this section, we introduce the definition of a frame associated with a fixed element in 2-Hilbert spaces and illustrate it through examples. Various properties of the corresponding frame operator are examined. Furthermore, several results from the theory of frames in Hilbert spaces are extended to the framework of 2-Hilbert spaces.

**Definition4.1.** Let H be a 2-Hilbert space and . A Sequence  of elements of H is called a frame associated to  for 2-Hilbert pace H if there exist 0<A<B< such that .

The above inequality is called the frame inequality. The numbers A and B are called lower and upper frame bounds respectively. If A=B then **is called tight frame associated to , if A=B=1 then ** is called normalized tight frame associated to .

A frame associated with a fixed element in 2-Hilbert spaces is illustrated through the following examples.

**Example: 4.2:** Let  be a 2-Hilbert space. Consider the set of vectors  associated to  is a tight frame for with frame bound 3.

Solution: Let 

Consider 

**Example: 4.3:** Let  be a 2-Hilbert space. Consider the set of vectors  associated to  is a tight frame for with frame bound .

Solution: Let 

Consider 

**Definition4.4.** Let be a frame associated to for 2-Hilbert space H. Then the Synthesis

operator associated to is  is defined as 

**Definition4.5.** Let  be a frame associated to for 2-Hilbert space H. Then the 2-Anaiysis

operaor  is defined as 

**Definition4.6.** Let  be a frame associated to for 2-Hilbert space H. Then the frame Operator

 is defined as 

**Result 4.7**. Consider









**Result 4.8.** Consider







 

Which shows that is self – adjoint operator

**Theorem 4.9.** Suppose is a frame associated to  for 2-Hilbert space 

**Proof:** Suppose  is a frame associated to for 2-Hilbert space H, so we have



Consider 







Conversly suppose 



  
i sequence is a frame associated to  for 2-Hilbert space H.

**Theorem 4.10.** Suppose is a sequence in 2-Hilbert space H, with  holds for all  if and only if  is a normalized tight frame associated to  for 2-Hilbert space H.

**Proof:** Suppose that  is a normalized tight frame associated to  for 2-Hilbert space H, for all  













**Theorem4.11.** Suppose is a frame associated to  for 2-Hilbert space H and T is co-isometry. thenis a frame associated to  for 2-Hilbert space H.

**Proof:** Given  is a normalized tight frame associated to  for 2-Hilbert space H, by definition we have  (1)

Since  is an operator for all  , we have  therefore equation (1) is true for 



 

Since T is co-isometry so we have

 

Which shows that is a frame associated to  for 2-Hilbert space H.

### 5.Conclusion & Future work

In this study, we explored the concept of 2-frames in 2-Hilbert spaces and introduced a definition of a frame associated with a fixed element. We illustrated this concept through examples and examined various properties of the corresponding frame operator. Additionally, we extended several known results from the classical theory of frames in Hilbert spaces to the framework of 2-Hilbert spaces. These developments provide a foundation for further investigations into frame theory in more generalized and structured inner product spaces.

The several potential directions for future research work are investigating the concept of dual 2-frames in 2-Hilbert spaces and their reconstruction properties. Further examining the role of tensor products of 2-frames and their results in 2-Hilbert spaces.

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1.

2.

3.

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