SOME EXACT SOLUTIONS FOR THE FIFTH-ORDER KdV EQUATION WITH CONSTANT COEFFICIENTS BY THE LIE SYMMETRIES

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ABSTRACT

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| The importance of the nonlinear equation on which the Lie method is applied lies in its ability to solve complex problems that cannot be addressed by classical linear methods. In this work, we use the Lie symmetries to obtain particular solutions for the fifth-order Korteweg de Vries equation. Our original equation is a nonlinear partial differential evolution equation (PDEE). We convert this equation to the nonlinear ordinary differential equation by reducing the variables. We determine the solution of the initial equation by using the solution obtained via the Lie symmetry method of the nonlinear ordinary differential equation. |

*Keywords: Fourth-Order Nonlinear Ordinary Differential Equation, Characteristic Variable, Travelling Wave, Invariant Solution, fifth-order Korteweg de Vries Equation, Lie Symmetries.*

1. INTRODUCTION

In nature and in several fields of science and technology, many physics problems involve nonlinear evolution. These include propagation waves and the theory of gravitational fields [1, 2]. Nonlinear partial differential evolution equations (NPDEEs) serve as mathematical models for describing many physical phenomena. Among these equations are the Korteweg de Vries (KdV) equations and their high-order type( such as the fifth-order KdV and seventh-order KdV), the nonlinear Schrödinger equations (NLS), the Sine Gordon equations (SG), etc.

Nonlinear equations present a complexity that often makes them difficult to solve. Fortunately, several analytical and numerical methods are now available to overcome these challenges. Among them are techniques such as the tanh, sine-cosine, and tanh-coth methods, as well as Hirota's bilinear method, the Exp function method [3], and the improved Bernoulli sub-equation method (IBSEFM) [4]. Additionally, auxiliary function methods [5] and Lie symmetry also provide effective means to obtain exact solutions.

The KdV-type equations have many applications in both the physical sciences and engineering fields [6]. For example, they describe long waves in shallow waters and deep oceans [7, 8]. The following fifth-order KdV (fKdV) equation

describes, particularly the propagation of shallow water waves over a flat surface and gravity-capillary waves [9]. This equation contains more terms than the fifth-order KdV equation in [10,11], whose Lie symmetry reduction with respect to an optimal set of one-dimensional subgroups of its full symmetry Lie group was carried out in [11]. However, this last equation contains the more general terms

Equation (1.0) includes one evolution term, four nonlinear terms (*)* and two dispersive terms and. The coefficients andare all nonzero constants. is a differentiable function of time and space which are independent variables.

In [9], a soliton solution is sought by transforming the equation into a nonlinear ordinary differential equation (ODE), followed by the formulation of ansatz. The main objective of this work is to obtain particular solutions to this equation, referred to as invariant solutions.

In general, equation (1.0) is not integrable [9, 12]. Consequently, we cannot seek a soliton invariant solution using the Lie symmetry method. Instead, our aim is to explore alternative solutions via this method.

The paper is organized as follows. Section 2 covers the transformation of the initial equation into a nonlinear ordinary differential equation (ODE). In section 3, we use the Lie symmetry method to solve the nonlinear ODE. In section 4, we solve the characteristic equation for a generator, leading to a particular solution. In section 5, Section 5 presents some illustrative examples. Lastly, Section 6 concludes the paper.

2. TRANSFORMATION TO THE NONLINEAR ODE

To transform equation (1.0) into a nonlinear ODE, we introduce infinitesimal generators, which help define the characteristic variable [6, 12]. The Lie symmetry analysis of KdV equations with constant coefficients has at least two vector fields generating the group of symmetries of space and time translation |1, 13]. By combining these symmetries, we can construct a new variable that links the spatial and temporal variables. As an example, see references [1, 12-15].

These symmetries are defined by the following infinitesimal generators:

Taking a linear combination of these two generators [13-16], we obtain:

whereis a constant.

The corresponding characteristic equation for (2.1) is given by [6, 14]:

Solving this equation provides the invariant solution, which introduces the characteristic variable [12]

Here, is known asthe wave variable [17], and is a constant representing the wave speed.

Applying this transformation, we derive the travelling wave reductions [1, 11, 12, 17and 18].

Now we can transform equation (1.0) into a nonlinear ODE.

Applying the characteristic variablefrom equation (2.3) the following derivatives hold:

that is,

Consequently, the derivatives of with respect to time and space transform as:

Substituting these into equation (1.0), we obtain the fifth-order nonlinear ODE:

We can reduce the order of the fifth-order ODE by applying known derivative relations. Specifically, we have:

it follows that

Using (2.9), we can rewrite equation (1.0) in the form

Using (2.10) with (2.7), (2.11) becomes

We can then rearrange the equation, moving the operator to the right-hand side:

By integrating equation (2.13) with respect to, we derive the fourth-order nonlinear ODE:

whereis the integration constant.

**Remark 1**

Since equation (1.0) is not integrable. If, this would imply that equation (1.0) admits an invariant solution corresponding to a solitary wave [9].

In its reduced form, equation (2.14) becomes

Or, in a simplified form

where the constants are:

that is

**3. SOME LIE POINT SYMMETRIES OF THE FOURTH-ORDER ODE**

Let us consider a one-parameter Lie group of infinitesimal transformations on the space of independent and dependent variables:

where is the group parameter.

**3.1. General Vector Field**

The general vector field associated with these transformations is given by:

**3.2. Prolongation of the Vector Field**

The prolongation formula of the vector fields [21] is defined as:

Which explicitly expands as:

where are specific functions ofand.

**3.3. Recursive Formula for Higher-Order Prolongations**

Generally, the expression for the th-order [19-21] is as follows:

Here, denotes the total derivative operator [16, 20, 21]:

**3.4. Explicit Expressions for Fourth-Order Prolongations**

For 4th order, we have the following expressions:

Sinceanddepends only on and, we use:

The calculation of the functionsgives

Using the reduced form (2.15), the symmetry condition gives:

where

Using (3.11), (3.12), (3.13), (3.14) and (3.16) the equality (3.15) becomes:

By Identifying (3.17), we obtain the following determining equations:

From equations (3.27), (3.31), (3.32) and (3.33), we obtain:

Integrating equation (3.34) with respect to, we get:

whereis the function to be determined

Consequently, the system is reduced to:

Using equations (3.40), (3.44), (3.45), and (3.46), we obtain:

The function takes the form:

whereandare functions to be determined.

Consequently, the system is reduced to:

Equation (3.53) gives:

that is

From equation (3.51), we have:

which simplifies to:

Thus

We substitute in (3.48) for (3.59) and obtain:

Deriving (3.59) with respect to results in

Identifying (3.56) by (3.61), we find:

Thus, solving for:

where is constant and consequently, the function is constant.

The system simplifies to:

Solving (3.66), we obtain:

and using (3.59), we have:

This simplifies to:

Therefore, we obtain the expression of

**3.5. Final Expression for**

Substituting into (3.60):

We see those expressions (3.35) and (3.71) satisfy (3.65).

For (3.64), we replace the functionwith the expression (3.71). Here, andare zero. We obtain the following equation

Expanding (3.72) and using and substituting the expression for, we obtain:

By simplifyingand regrouping the terms of and, we obtain:

We assume that is not constant function. Therefore, the coefficients of andin equation (3.74) should be zero. In this case, we obtain three algebraic relationships.

Rewriting (3.75) and (3.76) in terms ofand, we obtain:

These relationships are considered conditions for the solution of (2.16).

**3.6. General Expression for and**

Now, we have the general expressions of and, whereis constant. The function and its derivative take the form

whereandare real constant.

Therefore, the expressions of andare:

As a result, we obtain the following Lie point symmetries:

The linear combination of (3.85) and (3.86) is

That is

**4. PARTICULAR SOLUTIONS OF THE FOURTH-ORDER NONLINEAR ODE**

For the invariant solution, the characteristic equation of the main group is

**Case 1: whenand**

Using, (4.0) simplifies to:

Integrating both sides, we obtain:

whereis a real constant and we define .

Thus, the invariant solution is

**4.1. Determination of the Constant**

Substituting (4.3) into the main fourth-order nonlinear ODE (2.16), we obtain:

By identifying (4.4), we obtain the following system of equations

 For simplification, we have :

Solving (4.9), we have two roots nonzero:

To ensure real solutions forand, we impose the condition:

or equivalently:

The other equations verify the algebraic relations (3.78) (3.79) and (3.80).

**Case 2: when and**

Using, (4.0) becomes:

We obtain the trivial solution of (2.7)

whereis constant.

**4.2. Determination of the Constant**.

From equation (4.18), this leads to:

Substituting in (2.16), and using (4.18), the equation reduces to:

In equation (4.20), let us substituteandwith their corresponding expressions (3.78), (3.79) and (3.80). Upon solving, we obtain:

In hydrodynamics, a constant solution to the equation indicates either a rise in ocean level or a fall in ocean level.

**Case 3: whenand**

Using the general symmetry generator, equation (4.0) takes the form:

Integrating (4.22), we obtain

whereis a constant.

Thus

We rewrite it in a simplified form:

where

*,*

**4.3. Determination of the Constant**.

Substituting (4.26) into the main fourth-order nonlinear ODE (2.16), we obtain:

By identifying (4.28), we obtain the following system of equations

Here we have the same system as in the first case, so we obtain the same constants, that is:

**Remark 2**

We see that the first case is identical to the third case. The profile of their solutions is therefore the same.

Then, the solution of the nonlinear ordinary equation (2.16) is:

**4.4. Final Solution Expression**

We return to the initial form of equation (1.0) and using (2.17), obtain the solution according to their variables and their coefficients.

We call back that

and

Using (2.17), the solution of (1.0) is

and with the relations (3.74), (3.75) and (3.76), we obtain:

**Remarks 3**

* To obtain the real solution, we assume thatand that.
* The relationships (4.36), (4.37) and (4.38) are considered as conditions of the solution for (1.0).
* For, so , leading to a constant solution.
* For, we require and, ensuring that andare nonzero.
* To keep the fifth order in equation (1.0)should not be zero.

In summary, to maintain the fifth-order nature of equation (1.0), andshould all be different from zero.

We have another possibility to determine the solution of (1.0). We take (1.0) according to (2.18).

The equation (1.0) becomes

Therefore, the corresponding solution is

**5. ILLUSTRATIONS BY SOME EXAMPLES**

For the following examples, we take

**Example 4**

We directly determine the solution of the equation (1.0) by choosing the values ofand.

We chooseand. Using (4.37) and (4.38), we obtainand*.*

The expression (1.0) becomes

The solutions of (5.0) are



Fig 1: the graph of (5.1) in 3D, the range of and



Fig 2: the graph of (5.2) in 3D, the range of and

**Exemple 5**

We determine the solution of (1.0) in passing by the nonlinear EDO (2.16).

For the nonlinear EDO, we choose

and

Thus, we obtain the other values

and

The expression (2.16) becomes

The solutions of (5.5) are:

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Fig 3: the graph of (5.6) in 2D,

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Fig 4: the graph of (5.7) in 2D,

With the coefficients (5.3) and (5.4), we obtain the coefficient (1.0) via the relationship (2.17).

We have:

and

We see that these coefficients are dependent on. We choose

We obtain:

and

The expression of (1.0) becomes

Using (4.35), the solutions of (5.11) are

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Fig 5: the graph of (5.12) in 3D, the range of and

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Fig 6: the graph of (5.13) in 3D, the range of and

Example 5 clearly demonstrates the importance of the derivative solution. When a variable is changed in an equation, this can indeed lead to changes in the coefficients. These new equations maintain specific relationships with the original coefficients. This makes it possible to link the solutions of the transformed equations to those of the initial equations, thus facilitating the resolution of complex problems. By respecting these relationships, we can obtain solutions that are directly applicable to the initial situation.

**6. CONCLUSION**

In this work, we derived two nontrivial invariant solutions for the fifth-order Korteweg–de Vries (fKdV) equation. Our analysis revealed that the existence of solutions for the associated fourth-order nonlinear ordinary differential equation (ODE) depends on specific algebraic relationships between the coefficients. In particular, we established that the coefficients andmust be nonzero to ensure a valid solution.

For the fKdV equation, we identified necessary conditions on its coefficients, demonstrating that most of them must be nonzero, except for. Furthermore, we showed that three key coefficients are interdependent, and their relationships serve as constraints that allow the existence of invariant solutions.

This study highlights the effectiveness of Lie symmetry analysis as a powerful method for solving nonlinear differential equations. By leveraging symmetry-based approaches, we obtained exact solutions that can contribute to a deeper understanding of the underlying physical phenomena modeled by the fKdV equation.

For future research, we will use the same method applied to the same equation but using experimental values, taking into account appropriate initial and boundary conditions and studding additional symmetry reductions in order to obtain a soliton solution that models a solitary wave propagation like the tsunami.

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