Asian Research Journal of Mathematics

xx(x): ....., 20yy; Article no.ARJOM.xxxxx

ISSN: 2456-477X

A Study on Altered Jacobsthal Lucas Numbers Squared: Structural Properties and Applications

Type of Article

Received: XX December 20XX
Accepted: XX December 20XX
Online Ready: XX December 20XX

Abstract

Abstract - We examine two variations of the Jacobsthal Lucas numbers, denoted as $G^{(2)}_{j(n)}(a)$ and $H^{(2)}_{j(n)}(a)$, which are derived through the addition or subtraction of a specific value $\{a\}$ from the square of the n^{th} Jacobsthal Lucas numbers due to their relevance to the products of Jacobsthal numbers. Consequently, we derive both the consecutive sum-subtraction relationships and Binet-like expressions for these altered sequences, while also investigating the greatest common divisor (Gcd) sequences of r–successive terms, represented by $\{G^{(2)}_{j(n),r}(a)\}$ and $\{H^{(2)}_{j(n),r}(a)\}$ for $r \in \{1, 2, 3, 4\}$, which are informed by the periodic properties of the Gcd of consecutive Jacobsthal numbers.

Keywords: Altered Jacobsthal Lucas Number, Jacobsthal Lucas Sequence, Jacobsthal Sequence, Greatest Common Divisor (Gcd) Sequence.

2010 Mathematics Subject Classification: 11A05, 11A51, 11B50

1 Introduction

The Jacobsthal sequence J_n and the Jacobsthal-Lucas sequence j_n are defined recursively by

$$X_n = X_{n-1} + 2X_{n-2}, \quad n \geq 2 \tag{1.1}$$

with the initial conditions $J_0 = 0$, $j_0 = 2$, and $J_1 = j_1 = 1$. Some elements of these sequences are $\{J_n\}_{n\geq 0} = \{0, 1, 1, 3, 5, 11, 21, 43, \dots\}$ and $\{j_n\}_{n\geq 0} = \{2, 1, 5, 7, 17, 31, 65, \dots\}$. In addition, the Jacobsthal and Jacobsthal Lucas numbers are respectively given with the Binet formulas

$$J_n = \frac{2^n - (-1)^n}{3}, \quad j_n = 2^n + (-1)^n, \quad n \in N, \tag{1.2}$$

**Corresponding author: E-mails: fkoken@erbakan.edu.tr,kokenfikri@gmail.com*

which are explicit formulas used to prove certain properties of these numbers. For example, these numbers are extended to negative subscripts as

$$J_{-n} = \frac{(-1)^{n+1}}{2^n} J_n, \quad j_{-n} = \frac{(-1)^n}{2^n} j_n. \tag{1.3}$$

Also, these are a mathematical sequence that plays a significant role in various fields. Primarily, these numbers serve as a tool in mathematical analysis and computations. They are particularly useful for analyzing the time complexity of algorithms.

In (Horadam, 1988, 1996a,b), basic properties of Jacobsthal Lucas numbers are given such that Cassini-like formulas; $j_n^2 - j_{n+1}j_{n-1} = 9(-1)^n 2^{n-1}$, their interrelationships; $j_{n+1} + 2j_{n-1} = 9J_n$, indices sum formulas; $2j_{m+n} - j_m j_n = 9J_n J_m$, indices difference formulas; $j_m j_n + (-1)^{n+1}2^{n+1} j_{m-n} = 9J_m J_n$. It is seen that many results are equal to Jacobsthal numbers. This makes it possible to transfer applications of Jacobsthal numbers onto these altered numbers.

Several studies have explored variations of Jacobsthal numbers by introducing modifications to their standard recurrence relations. Several studies in the literature have introduced new sequences derived from Jacobsthal-Lucas numbers. Furthermore, they have examined equations and properties of these sequences that mirror those presented by Jacobsthal Lucas sequences (Bilgici and Bród, 2023; Catarino et al., 2015; Horadam, 1993; Uygun, 2021).

Horadam (1993) defines the k^{th} associated sequences $\{j_n^{(k)}\}_{n\geq 1}$ of the Jacobsthal Lucas $\{j_n\}$ sequences to be, respectively, given by

$$j_n^{(k)} = j_{n+1}^{(k-1)} + 2j_{n-1}^{(k-1)} \tag{1.4}$$

where $j_n^{(0)} = j_n$ and $j_n^{(1)} = 9J_n$ are the generic values of these sequences. The following relationships are seen to be valid:

$$j_n^{(2m)} = 3^{2m} j_n, \quad j_n^{(2m-1)} = 3^{2m} J_n. \tag{1.5}$$

Also, Horadam (1996a) gives two sequences, which are established by using the summation formulas of the consecutive Jacobsthal Lucas numbers, defined by

$$\hat{j}_n = \sum_{i=1}^{n} j_i = \frac{1}{2}(j_{n+2} - 5), \quad \hat{j}_0 = \hat{j}_1 = 1. \tag{1.6}$$

In addition, the basic properties of these sequences are arranged with recurrence relation; $\hat{j}_{n+2} = \hat{j}_{n+1} + 2\hat{j}_n + 5$, Binet's form; $\hat{j}_n = \frac{2^{n+2}+(-1)^n-5}{2}$. Further, a lot of properties of these numbers are given in (Horadam, 1996a,b), and also, the associated sequences $\hat{J}_n^{(k)}$, $\hat{j}_n^{(k)}$ and their properties are deduced by using the properties of previous sequences. In addition to these numbers, he studied the Jacobsthal Lucas representation polynomials in (Horadam, 1996a,b).

Building on the work of (Cook and Bacon, 2013), the Jacobsthal recurrence relation was generalized to higher-order cases, extending the foundational identities established by Horadam (1988, 1993, 1996a,b) and introducing a broader set of identities applicable to these advanced recurrence relations.

Also, the authors introduced a new Jacobsthal-type sequence,

$$J_{r,n} = 2^r J_{r,n-1} + (2^r + 4^r) J_{r,n-2}, \quad J_{r,0} = 0, \quad J_{r,1} = 1 + 2^{r+1}, \quad (n \geq 2) \tag{1.7}$$

with properties examined in the present study (Bilgici and Bród, 2023; Brod, 2020).

Koken (2019) introduced the altered Jacobsthal sequences $\{j_n^+\}_{n\geq 1}$ and $\{j_n^-\}_{n\geq 1}$, obtained by modifying the Jacobsthal Lucas numbers. They are given by the definitions

$$j_n^+ = \begin{cases} j_n + 5.2^{(n/2)-1}, & \text{if } n \text{ is even} \\ j_n - 2^{(n-1)/2}, & \text{otherwise} \end{cases}, \qquad j_n^- = \begin{cases} j_n - 5, 2^{(n/2)-1}, & \text{if } n \text{ is even} \\ j_n + 2^{(n-1)/2}, & \text{otherwise} \end{cases}$$

For the altered Jacobsthal Lucas numbers j_n^+ and j_n^-, these identities are valid:

$$j_{4k}^+ = 9J_{2k+1}J_{2k-1}, \quad j_{4k+1}^+ = 9J_{2k+1}J_{2k}, \quad j_{4k+2}^+ = j_{2k+2}j_{2k}, \quad j_{4k+3}^+ = j_{2k+2}j_{2k+1} \tag{1.8}$$

$$j_{4k}^{-}=j_{2k+1}j_{2k-1},\quad j_{4k+1}^{-}=j_{2k+1}j_{2k},\quad j_{4k+2}^{-}=9J_{2k+2}J_{2k},\quad j_{4k+3}^{-}=9J_{2k+2}J_{2k+1} \tag{1.9}$$

He examined recurrence formulas, Binet-like formulas, Cassini-like identities, explicit relations, and other noteworthy findings.

Also, Koken et al. (2025) defined with $G_{J(n)}^{(2)}(a)=J_n^2+(-1)^n a$ and $H_{J(n)}^{(2)}(a)=J_n^2-(-1)^n a$, which are called the altered Jacobsthal numbers squared, and are derived by incorporating a parameter into the classical sequence

$$G_{J(n)}^{(2)}(J_t)=J_{n+t}J_{n-t},\quad if\ t\ is\ odd, \tag{1.10}$$

$$H_{J(n)}^{(2)}(J_t)=J_{n+t}J_{n-t},\quad if\ t\ is\ even, \tag{1.11}$$

where J_x be x^{th} Jacobsthal number and $a\in Z$. These numbers exhibit notable divisibility properties, particularly when examined through greatest common divisor (gcd) sequences, which display periodic behavior. In addition, the theoretical framework surrounding altered Jacobsthal numbers is well established through specific recurrence relations and identities. Various theorems provide insight into their algebraic structure, while Binet-like formulas aid in proving fundamental properties (Koken et al., 2025).

2 Altered Jacobsthal Lucas Numbers and Their Properties

In this section, two variations of altered Jacobsthal Lucas numbers are defined and examined, denoted as $G_{j(n)}^{(2)}(a)$ and $H_{j(n)}^{(2)}(a)$, which are constructed by incorporating or subtracting a parameter $\{a\}$ from the square of the n^{th} Jacobsthal Lucas number. The selection of addition or subtraction is determined by whether the index of the altered number is odd or even, respectively.

These sequences produce subsequences that exhibit similarities to previously explored variations, including the associated Jacobsthal Lucas (Horadam, 1993), consecutive Jacobsthal Lucas Horadam (1996a), and the altered Jacobsthal Lucas sequences $\{j_n^{+}\}_{n\geq 1}$ and $\{j_n^{-}\}_{n\geq 1}$ (Koken, 2019), which have been extensively studied in the context of Jacobsthal and Jacobsthal Lucas numbers.

Motivated by an identity established in the literature, we derive two equations that align with the objectives of this study. In particular, we explore the summation and subtraction identities for the squared Jacobsthal Lucas numbers, as formulated in Eqs. (2.1-2.2).

$$j_{m+k-1}^2+2^{2k-1}j_{m-k}^2=9J_{2m-1}J_{2k-1}, \tag{2.1}$$

$$j_{m+k}^2-2^{2k}j_{m-k}^2=9J_{2m}J_{2k},\quad m>k \tag{2.2}$$

where, m and k are positive integers. Although the identities given in Eqs. (2.1-2.2) have not been explicitly identified in the literature, their validity can be verified through the application of the Binet formulas presented in Eq. (1.2).

Definition 1. *The n^{th} terms of the altered Jacobsthal Lucas number sequences, represented by $G_{j(n)}^{(2)}(a)$ and $H_{j(n)}^{(2)}(a)$, are formally defined in Eqs. (2.3-2.4).*

$$G_{j(n)}^{(2)}(a)=j_n^2+(-1)^n a \tag{2.3}$$

$$H_{j(n)}^{(2)}(a)=j_n^2-(-1)^n a \tag{2.4}$$

where j_n be n^{th} Jacobsthal Lucas number and $a\in Z$.

By selecting a value $a\in\left\{2^{n-t}j_t^2\right\}$, where $t<n$, we can extend the notion of altered Jacobsthal Lucas numbers introduced in Eqs. (2.3-2.4) to the sequences $G_{j(n)}^{(2)}\left(2^{n-t}j_t^2\right)$ and $H_{j(n)}^{(2)}\left(2^{n-t}j_t^2\right)$, where j_t^2 denotes a square of the t^{th} Jacobsthal Lucas number.

Theorem 1. *Let $G_{j(n)}^{(2)}\left(2^{n-t}j_t^2\right)$ and $H_{j(n)}^{(2)}\left(2^{n-t}j_t^2\right)$ represent the n^{th} altered Jacobsthal Lucas numbers squared. Then, the statements given in Eqs. (2.5-2.6) hold true.*

$$G_{j(n)}^{(2)}\left(2^{n-t}j_t^2\right) = 9J_{n+t}J_{n-t},\ if\ t\ is\ odd \tag{2.5}$$

$$H_{j(n)}^{(2)}\left(2^{n-t}j_t^2\right) = 9J_{n+t}J_{n-t},\ if\ t\ is\ even \tag{2.6}$$

Proof. By setting $m = u + \frac{t+1}{2}$ and $k = u - \frac{t-1}{2}$ in Eqs. (2.1-2.2), respectively, when t is odd, an according to $n = 2u$ and $n = 2u + 1$, the following identities hold:

$$\begin{aligned} j_{2u}^2 + 2^{2u-t}j_t^2 &= 9J_{2u+t}J_{2u-t} \\ j_{2u+1}^2 - 2^{2u+1-t}j_t^2 &= 9J_{2u+1+t}J_{2u+1-t}. \end{aligned}$$

Thus, for $a = 2^{n-t}J_t^2$ in Equation (2.3), we obtain $G_{j(n)}^{(2)}\left(2^{n-t}j_t^2\right) = 9J_{n+t}J_{n-t}$.

Similarly, by setting $m = u + \frac{t}{2}$ and $k = u - \frac{t}{2}$ in Eqs. (2.1-2.2), when t is even, an according to $n = 2u - 1$ and $n = 2u$, we obtain the following result:

$$\begin{aligned} j_{2u-1}^2 + 2^{2u-t-1}j_t^2 &= 9J_{2u+t-1}J_{2u-t-1}, \\ j_{2u}^2 - 2^{2u-t}j_t^2 &= 9J_{2u+t}J_{2u-t}. \end{aligned}$$

It is valid $H_{j(n)}^{(2)}\left(2^{n-t}j_t^2\right) = 9J_{n+t}J_{n-t}$ for $a = 2^{n-t}j_t^2$ in Eq. (2.4) . ☐

We see that the altered Jacobsthal sequences $G_{J(n)}^{(2)}\left(2^{n-t}J_t^2\right)$ and $H_{J(n)}^{(2)}\left(2^{n-t}J_t^2\right)$ are defined in (Koken et al., 2025), so that the relation with the altered Jacobsthal Lucas sequences is given by

$$G_{j(n)}^{(2)}\left(2^{n-t}j_t^2\right) = 9G_{J(n)}^{(2)}\left(2^{n-t}J_t^2\right), if\ \ t\ \ is\ \ odd \tag{2.7}$$

$$H_{j(n)}^{(2)}\left(2^{n-t}j_t^2\right) = 9H_{J(n)}^{(2)}\left(2^{n-t}J_t^2\right), if\ \ t\ \ is\ \ odd \tag{2.8}$$

Now, any Binet like formulas are achieved for the numbers $G_{j(n)}^{(2)}\left(2^{n-t}j_t^2\right)$ and $H_{j(n)}^{(2)}\left(2^{n-t}j_t^2\right)$ given in Eqs. (2.5-2.6).

Theorem 2. *Let $G_{j(n)}^{(2)}\left(2^{n-t}j_t^2\right)$ and $H_{j(n)}^{(2)}\left(2^{n-t}j_t^2\right)$ be the n^{th} altered Jacobsthal Lucas numbers. Then, they are expressed with Eqs. (2.9-2.10).*

$$G_{j(n)}^{(2)}\left(2^{n-t}j_t^2\right) = \left(2^{2n}+1\right) + (-1)^n\left(2^{2t}+1\right),\ if\ t\ is\ odd \tag{2.9}$$

$$H_{j(n)}^{(2)}\left(2^{n-t}j_t^2\right) = \left(2^{2n}+1\right) - (-1)^n\left(2^{2t}+1\right),\ if\ t\ is\ even \tag{2.10}$$

Proof. If the Binet formulas in Eq. (1.2) are substituted in Eqs. (2.5-2.6), respectively, and they are adjusted, we have the desired results. ☐

Using the Binet formulas given in Eq. (1.2), these numbers in Eqs. (2.9-2.10) are associated with the Jacobsthal Lucas numbers as follows:

$$G_{j(n)}^{(2)}\left(2^{n-t}j_t^2\right) = j_{2n} + (-1)^n j_{2t},\ if\ t\ is\ odd \tag{2.11}$$

$$H_{j(n)}^{(2)}\left(2^{n-t}j_t^2\right) = j_{2n} - (-1)^n j_{2t},\ if\ t\ is\ even. \tag{2.12}$$

Also, the Binet like formulas in given Eqs. (2.9-2.10) can be used to prove many properties of these numbers.

Furthermore, the expressions given in Eqs. (2.11–2.12) also resemble the altered Jacobsthal–Lucas-like numbers introduced in (Koken, 2019).

Let us examine examples of the generalizations provided for the cases $t = 1$ and $t = 2$. The sequences defined by $G_{j(n)}^{(2)}\left(2^{n-1}\right)$ and $H_{j(n)}^{(2)}\left(2^{n-2}5^2\right)$ in Eqs. (2.3-2.4) exhibit an increasing pattern, apart from the initial few terms. The general expressions for these sequences are given as follows.

Theorem 3. *Let $G^{(2)}_{j(n)}\left(2^{n-1}\right)$ and $H^{(2)}_{j(n)}\left(2^{n-2}5^2\right)$ denote the n^{th} altered Jacobsthal Lucas numbers. Then, the following statements hold Eqs. (2.13-2.14):*

$$G^{(2)}_{j(n)}\left(2^{n-1}\right) = 9J_{n+1}J_{n-1} \tag{2.13}$$

$$H^{(2)}_{j(n)}\left(2^{n-2}5^2\right) = 9J_{n+2}J_{n-2} \tag{2.14}$$

Proof. Letting $m = u + 1$ and $k = u$ in Eqs. (2.1)–(2.2), and applying the identity in Eq. (2.3), we obtain:

$$G^{(2)}_{j(2u)}(2^{2u-1}) = j^2_{2u} + 2^{2u-1} = 9J_{2u+1}J_{2u-1},$$
$$G^{(2)}_{j(2u+1)}(2^{2u}) = j^2_{2u+1} - 2^{2u} = 9J_{2u+2}J_{2u}.$$

Similarly, taking $m = u + 2$ and $n = u$ in the same equations and using Eq. (2.4) for $a = 2^{n-2}5^2$, we derive:

$$H^{(2)}_{j(2u+1)}(2^{2u-1}5^2) = j^2_{2u+1} + 2^{2u-1}5^2 = 9J_{2u+3}J_{2u-1},$$
$$H^{(2)}_{j(2u+2)}(2^{2u}5^2) = j^2_{2u+2} - 2^{2u}5^2 = 9J_{2u+4}J_{2u}.$$

□

In the following, we explore certain sum and difference relations of the numbers $G^{(2)}_{j(n)}(2^{n-1})$ and $H^{(2)}_{j(n)}(2^{n-2}5^2)$, derived from the multiplication identities given in Eqs. (2.13) and (2.14).

Theorem 4. *Let X_n denote the altered Jacobsthal-Lucas numbers $G^{(2)}_{j(n)}(2^{n-1})$ and $H^{(2)}_{j(n)}(2^{n-2}5^2)$. Then the following recurrence relations hold:*

$$X_{n+1} + 2X_n = 9J_{2n+1}, \tag{2.15}$$
$$X_{n+1} - 4X_{n-1} = 9J_{2n}, \tag{2.16}$$
$$X_{n+2} = 3X_{n+1} + 6X_n - 8X_{n-1}. \tag{2.17}$$

Proof. Using the multiplication identities from Eqs. (2.13)–(2.14), and known Jacobsthal identities such as

$$J^2_{n+1} + 2J^2_n = J_{2n+1}, \quad 2J_{n-1} + J_{n+1} = j_n, \quad J_n j_n = J_{2n},$$

we derive:

$$\begin{aligned} H^{(2)}_{j(n+1)}(2^{n-2}5^2) + 2H^{(2)}_{j(n)}(2^{n-2}5^2) &= 9(J_{n+2} + 2J_{n+1})J_{n-1} + 18J_{n+2}J_{n-2} \\ &= 9[(J_{n+1} + 2J_n)J_n + 2J_{n+1}J_{n-1}] \\ &= 9J_{2n+1}, \end{aligned}$$

and

$$\begin{aligned} G^{(2)}_{j(n+1)}(2^{n-1}) - 4G^{(2)}_{j(n-1)}(2^{n-1}) &= 9(J_{n+2} - 4J_{n-2})J_n \\ &= 9(J_{n+1} + 2J_{n-1})J_n \\ &= 9J_{2n}. \end{aligned}$$

The third relation (2.17) is obtained by combining the above two identities through appropriate linear combinations. Full details are omitted for brevity. □

Consequently, the recurrence relation in Eq. (2.17) directly follows from the identities in Eqs. (2.15) and (2.16), confirming the structure of the sequences defined in Eqs. (2.13) and (2.14).

2.1 Altered Jacobsthal Lucas Gcd Sequences $G^{(2)}_{j(n),r}(a)$ and $H^{(2)}_{j(n),r}(a)$

This section examines the greatest common divisor (GCD) sequences arising from consecutive altered Jacobsthal Lucas numbers, denoted by $G^{(2)}_{j(n),r}(a)$ and $H^{(2)}_{j(n),r}(a)$, where r indicates the number of successive terms. These sequences exhibit periodic behavior and reveal structured divisibility properties, reflecting the underlying modular and arithmetic characteristics of Jacobsthal numbers as detailed in (Koken et al., 2025).

The study of sequences with modified components highlights the structured regularity of Jacobsthal numbers in product formulations. In particular, the Hosoya triangle, similar to Pascal's triangle, organizes entries as products of Fibonacci numbers, leading to various generalizations (Flórez and Junes, 2012; Hosoya, 1976; Koshy, 2019). Extending this framework, researchers have analyzed the greatest common divisor (Gcd) and modularity properties of generalized Fibonacci sequences, establishing the Star of David identity and related patterns (Flórez et al., 2014a,b). Additionally, replacing numerical values with polynomials has yielded Hosoya like polynomial triangles, further broadening the scope of identities based on recurrence (Flórez et al., 2018b; Koshy, 2019). These findings reinforce the fundamental arithmetic properties that govern integer sequences and their geometric interpretations.

The Jacobsthal sequence is known to satisfy the strong divisibility property, namely that for all positive integers a and b, the relation

$$\gcd(J_a, J_b) = J_{\gcd(a,b)}$$

holds. Furthermore, it exhibits elliptic divisibility: if $J_n \mid J_m$, then $n \mid m$. These properties, together with key divisibility results, form the basis for the propositions in Eqs. (2.18-2.20), adapted from (Flórez et al., 2018a).

Let a, b, c, and d be positive integers. If $\gcd(J_a, J_b) = 1$ and $\gcd(J_c, J_d) = 1$, then the following propositions hold:

$$gcd(J_a J_b, J_c J_d) = J_{gcd(a,c)} J_{gcd(a,d)} J_{gcd(b,d)} J_{gcd(b,c)} \tag{2.18}$$

If $|a-c| \leq 2$ and $|b-d| \leq 2$, then the following propositions hold: $gcd(J_a, J_c) = 1$, $gcd(J_b, J_d) = 1$ and

$$gcd(J_a J_b, J_c J_d) = J_{gcd(a,d)} J_{gcd(b,c)}. \tag{2.19}$$

If $gcd(J_a, J_c) = x$ and $gcd(J_b, J_d) = y$, then the following proposition hold:

$$gcd(J_a J_b, J_c J_d) = \frac{gcd(yJ_a, xJ_d)\ gcd(xJ_b, yJ_c)}{xy}. \tag{2.20}$$

We investigate certain properties concerning properties of Gcd of two numbers from the altered sequences $\left\{G^{(2)}_{j(n)}(a)\right\}$ and $\left\{H^{(2)}_{j(n)}(a)\right\}$.

Definition 2. *Let $G^{(2)}_{j(n)}(a)$ and $H^{(2)}_{j(n)}(a)$ be the n^{th} altered Jacobsthal Lucas numbers in Eqs. (2.3-2.4). The expressions of Eqs. (2.21-2.22)*

$$G^{(2)}_{j(n),r}(a) = gcd\left(G^{(2)}_{j(n)}(a), G^{(2)}_{j(n+r)}(a)\right) \tag{2.21}$$

$$H^{(2)}_{j(n),r}(a) = gcd\left(H^{(2)}_{j(n)}(a), H^{(2)}_{j(n+r)}(a)\right) \tag{2.22}$$

are called as the r-successive altered Jacobsthal Lucas $\gcd$ *numbers.*

Based on the identities in Eqs. (2.7–2.8), the analysis of the $\gcd$ sequences $G^{(2)}_{j(n),r}(a)$ and $H^{(2)}_{j(n),r}(a)$ follows a similar approach to that applied for the altered Jacobsthal sequences. The results given in Theorems 5–8 of (Koken et al., 2025) show that the period length of these sequences is determined by

$$m = \mathrm{lcm}[r, r-2t, r+2t], \quad t \in \{1,2\}, \quad 1 \le r \le 4, \quad r-2t \neq 0,$$

where $\mathrm{lcm}(a,b,c)$ denotes the least common multiple of the integers a, b, and c.

Algorithm 1, utilized in this study, identifies the maximum entries in each column of a given data matrix and detects any repetitions. The verification was performed through a computer program based on this algorithm, which systematically examined multiple values with respect to the given (t,s) parameters.

By substituting $a = 2^{n-1}$ and $a = 2^{n-2}5^2$ with $r = 1$ in Eqs. (2.21–2.22), it is observed that the sequences $\left\{G^{(2)}_{j(n),1}(2^{n-1})\right\}$ and $\left\{H^{(2)}_{j(n),1}(2^{n-2}5^2)\right\}$ do not exhibit strict monotonicity but suggest potential periodicity. This prompts an investigation into whether these 1-successive altered Jacobsthal–Lucas GCD sequences attain specific values within defined periods.

Theorem 5. *Let $G^{(2)}_{j(n),1}\left(2^{n-1}\right)$ and $H^{(2)}_{j(n),1}\left(2^{n-2}5^2\right)$ be the n^{th} 1-successive altered Jacobsthal Lucas gcd numbers. Then, the following statements hold:*

$$G^{(2)}_{j(n),1}\left(2^{n-1}\right) = \begin{cases} 9J_3, & n \equiv 1\,(mod\,3) \\ 9, & otherwise \end{cases}$$

$$H^{(2)}_{j(n),1}\left(2^{n-2}5^2\right) = \begin{cases} 9J_5J_3, & n \equiv 7\,(mod\,15) \\ 9J_5, & n \equiv 2,12\,(mod\,15) \\ 9J_3, & n \equiv 1,4,10,13\,(mod\,15) \\ 9, & otherwise \end{cases}$$

Proof. We evaluate $G^{(2)}_{j(n),1}(2^{n-1}) = \gcd(9J_{n+1}J_{n-1}, 9J_nJ_{n+2})$ using Eq. (2.13) for $r = 1$ in Eq. (2.21). Since $\gcd(J_{n+1}, J_{n-1}) = \gcd(J_n, J_{n+2}) = 1$, applying Eq. (2.18) yields

$$G^{(2)}_{j(n),1}(2^{n-1}) = 9\gcd(J_{n-1}, J_{n+2}) = 9J_{\gcd(n-1,n+2)}.$$

Using the strong divisibility property $\gcd(J_a, J_b) = J_{\gcd(a,b)}$ and the Euclidean identity $\gcd(a,b) = \gcd(a, b-ax)$, we obtain

$$J_{\gcd(n-1,3)} = \begin{cases} J_3, & \text{if } n \equiv 1 \pmod 3, \\ J_1, & \text{otherwise.} \end{cases}$$

Similarly, consider $H^{(2)}_{j(n),1}(2^{n-2}5^2) = \gcd(9J_{n+2}J_{n-2}, 9J_{n+3}J_{n-1})$ using Eq. (2.14) for $r = 1$ in Eq. (2.22). Since $\gcd(J_{n+2}, J_{n+3}) = \gcd(J_{n-2}, J_{n-1}) = 1$, we apply Eq. (2.19) to write

$$H^{(2)}_{j(n),1}(2^{n-2}5^2) = 9\gcd(J_{n-2}, J_{n+3}) \cdot \gcd(J_{n+2}, J_{n-1}).$$

Then, using the divisibility property, we find

$$\gcd(J_{n-2}, J_{n+3}) = J_{\gcd(n-2,n+3)} = \begin{cases} J_5, & \text{if } n \equiv 2 \pmod 5, \\ 1, & \text{otherwise,} \end{cases}$$

$$\gcd(J_{n+2}, J_{n-1}) = J_{\gcd(n+2,n-1)} = \begin{cases} J_3, & \text{if } n \equiv 1 \pmod 3, \\ 1, & \text{otherwise.} \end{cases}$$

In all cases, the desired results follow by applying the Chinese Remainder Theorem. □

By generating the 2-successive altered Gcd sequences with $a = 2^{n-1}$ and $a = 2^{n-2}5^2$ for $r = 2$ in Eqs. (2.21-2.22), it is observed that the sequence $\left\{G^{(2)}_{j(n),2}(2^{n-1})\right\}$ $(n \ge 1)$ increases monotonically, whereas $\left\{H^{(2)}_{j(n),2}(2^{n-2}5^2)\right\}$ $(n \ge 2)$ displays a periodic structure.

Algorithm 1 Sequence Generation, GCD Calculation, and Repeated Value Detection

Start
Generate Sequences:
Create a list n containing numbers from 0 to 1000
Initialize sequence j: $j[0] = 2$, $j[1] = 1$
Compute sequences G and H using predefined formulas
Define a Function to Compute GCD:
function CALCULATE_GCD(a, b)
    return $\gcd(a, b)$
end function
Compute GCD for Different r Values:
for $r = 1$ to 50 **do**
    for $i = (1000 - r)$ to 1 **do**
        Compute $\gcd(G[i], G[i + r])$ and store it in $gcdG[r]$
        Compute $\gcd(H[i], H[i + r])$ and store it in $gcdH[r]$
    end for
end for
Identify Repeated Values and Their Frequencies:
Create a matrix $repeated$ of size 1000×80 (initialized to zeros)
for $i = 1$ to 25 **do**
    for $r = 1$ to 40 **do**
        Select the i-th column from the data matrix
        Find the maximum value m in the column
        Identify positions where m appears
        if at least two occurrences exist **then**
            Compute distance between the first two occurrences
            Compare corresponding sequences x_1 and x_2
            if $x_1 == x_2$ **then**
                Store repeated values in the $repeated$ matrix
            else if $x_1 == x_3$ **then**
                Store x_1 and its frequency
            end if
        end if
    end for
end for
Save All Data to a CSV File
End

Theorem 6. *Let $G^{(2)}_{j(n),2}\left(2^{n-1}\right)$ and $H^{(2)}_{j(n),2}\left(2^{n-2}5^{2}\right)$ be the n^{th} 2-successive altered Jacobsthal Lucas gcd numbers. Then, the following statements hold:*

$$G^{(2)}_{j(n),2}\left(2^{n-1}\right)=\begin{cases} 9J_4J_{n+1}, & n\equiv 1\,(mod\;4)\\ 9J_{n+1}, & otherwise\end{cases}$$

$$H^{(2)}_{j(n),2}\left(2^{n-2}5^{2}\right)=\begin{cases} 9J_6, & n\equiv 2\,(mod\;6)\\ 9J_3, & n\equiv 5\,(mod\;6)\\ 9, & otherwise\end{cases}$$

Proof. From Eq. (2.13), we write

$$G^{(2)}_{j(n),2}(2^{n-1})=9J_{n+1}\gcd(J_{n-1},J_{n+3})=9J_{n+1}J_{\gcd(n-1,n+3)}.$$

Since $\gcd(n-1,n+3)=\gcd(n-1,4)$, we have

$$J_{\gcd(n-1,4)}=\begin{cases} J_4, & \text{if } n\equiv 1 \pmod 4,\\ J_2, & \text{if } n\equiv 3 \pmod 4,\\ J_1, & \text{if } n\equiv 0,2 \pmod 4.\end{cases}$$

Thus, $G^{(2)}_{j(n),2}(2^{n-1})=9J_4J_{n+1}$ for $n\equiv 1 \pmod 4$, and $9J_{n+1}$ otherwise.

For the second sequence, from Eq. (2.14) we obtain

$$H^{(2)}_{j(n),2}(2^{n-2}5^{2})=9\gcd(J_{n+2}J_{n-2},J_{n+4}J_n).$$

Using the coprimality $\gcd(J_{n+2},J_{n+4})=\gcd(J_{n-2},J_n)=1$ and Eq. (2.19), we simplify to

$$H^{(2)}_{j(n),2}(2^{n-2}5^{2})=9J_{\gcd(n-2,n+4)}=9J_{\gcd(n-2,6)}.$$

Thus,

$$J_{\gcd(n-2,6)}=\begin{cases} J_6, & \text{if } n\equiv 2 \pmod 6,\\ J_3, & \text{if } n\equiv 5 \pmod 6,\\ J_2, & \text{if } n\equiv 0,4 \pmod 6,\\ J_1, & \text{if } n\equiv 1,3 \pmod 6.\end{cases}$$

□

It is known that

$$\gcd(J_n,J_{n+3})=\begin{cases} 3, & \text{if } n\equiv 0 \pmod 3,\\ 1, & \text{otherwise.}\end{cases}$$

Using this property, we now construct the sequences $\left\{G^{(2)}_{j(n),3}(2^{n-1})\right\}$ and $\left\{H^{(2)}_{j(n),3}(2^{n-2}5^{2})\right\}$.

Theorem 7. *Let $G^{(2)}_{j(n),3}\left(2^{n-1}\right)$ and $H^{(2)}_{j(n),3}\left(2^{n-2}5^{2}\right)$ be the n^{th} 3-successive altered Jacobsthal gcd numbers. They are valid:*

$$G^{(2)}_{j(n),3}\left(2^{n-1}\right)=\begin{cases} 9J_5J_3, & n\equiv 1,11\,(mod\;15)\\ 9J_5, & n\equiv 6\,(mod\;15)\\ 9, & n\equiv 0,3,9,12\,(mod\;15)\\ 9J_3, & otherwise\end{cases}$$

$$H^{(2)}_{j(n),3}\left(2^{n-2}5^{2}\right)=\begin{cases} 9J_7J_3, & n\equiv 2,16\,(mod\;21)\\ 9J_7, & n\equiv 9\,(mod\;21)\\ 9, & n\equiv 0,3,6,12,15,18\,(mod\;21)\\ 9J_3, & otherwise\end{cases}$$

Proof. Using Eq. (2.13), we express

$$G^{(2)}_{j(n),3}(2^{n-1}) = 9\gcd\left(J_{n+1}J_{n-1}, J_{n+4}J_{n+2}\right).$$

Since $\gcd(J_{n+1}, J_{n-1}) = \gcd(J_{n+4}, J_{n+2}) = 1$, by Eq.n (2.18), we obtain

$$G^{(2)}_{j(n),3}(2^{n-1}) = 9\cdot\gcd(J_{n-1}, J_{n+2})\cdot\gcd(J_{n-1}, J_{n+4})\cdot\gcd(J_{n+1}, J_{n+4}).$$

Now, we evaluate each term:

- $\gcd(J_{n-1}, J_{n+2}) = J_3$ if $n \equiv 1 \pmod 3$, otherwise J_1.
- $\gcd(J_{n-1}, J_{n+4}) = J_5$ if $n \equiv 1 \pmod 5$, otherwise J_1.
- $\gcd(J_{n+1}, J_{n+4}) = J_3$ if $n \equiv 2 \pmod 3$, otherwise J_1.

Hence, $G^{(2)}_{j(n),3}(2^{n-1}) = 9J_3J_5$ when $n \equiv 1 \pmod 5$ and $n \equiv 1$ or $2 \pmod 3$, and the result follows by the Chinese Remainder Theorem.

Now consider $H^{(2)}_{j(n),3}(2^{n-2}5^2)$ via Eq. (2.14):

$$H^{(2)}_{j(n),3}(2^{n-2}5^2) = 9\gcd\left(J_{n+2}J_{n-2}, J_{n+1}J_{n+5}\right).$$

Using $\gcd(J_{n+2}, J_{n+1}) = 1$ for all n and

$$\gcd(J_{n-2}, J_{n+5}) = \begin{cases} J_7, & n \equiv 2 \pmod 7, \\ 1, & \text{otherwise}, \end{cases}$$

we apply Eq. (2.20) to get

$$H^{(2)}_{j(n),3}(2^{n-2}5^2) = \begin{cases} 9J_7\cdot\gcd(J_{n+2}, J_{n+5})\cdot\gcd(J_{n-2}, J_{n+1}), & n \equiv 2 \pmod 7, \\ 9\cdot\gcd(J_{n+2}, J_{n+5})\cdot\gcd(J_{n-2}, J_{n+1}), & \text{otherwise}. \end{cases}$$

Using known identities:

- $\gcd(J_{n+2}, J_{n+5}) = J_3$ if $n \equiv 1 \pmod 3$, otherwise J_1.
- $\gcd(J_{n-2}, J_{n+1}) = J_3$ if $n \equiv 2 \pmod 3$, otherwise J_1.

Thus, all values of the sequence $H^{(2)}_{j(n),3}(2^{n-2}5^2)$ can be computed using the Chinese Remainder Theorem. ☐

As stated in Eq. (2.23),

$$\gcd(J_n, J_{n+4}) = \begin{cases} 5, & \text{if } n \equiv 0 \pmod 4, \\ 1, & \text{otherwise}. \end{cases} \tag{2.23}$$

It is observed that the sequence $\left\{G^{(2)}_{j(n),4}(2^{n-1})\right\}$, for $n > 2$, exhibits periodic behavior. In contrast, the sequence $\left\{H^{(2)}_{j(n),4}(2^{n-2}5^2)\right\}$, for $n \geq 2$, takes values that follow a Jacobsthal pattern.

Theorem 8. *Let $G^{(2)}_{j(n),4}\left(2^{n-1}\right)$ and $H^{(2)}_{j(n),4}\left(2^{n-2}5^2\right)$ be n^{th} 4-successive altered Jacobsthal Lucas gcd numbers, they are valid:*

$$G^{(2)}_{j(n),4}\left(2^{n-1}\right) = \begin{cases} 9J_6J_4, & n \equiv 1,7\,(mod\ 12) \\ 9J_4, & n \equiv 3,5,6,10\,(mod\ 12) \\ 9J_3, & n \equiv 4,8\,(mod\ 12) \\ 9, & otherwise \end{cases},$$

$$H^{(2)}_{j(n),4}\left(2^{n-2}5^2\right) = \begin{cases} 9J_8J_{n+2}, & n \equiv 2\,(mod\ 8) \\ 9J_4J_{n+2}, & n \equiv 6\,(mod\ 8) \\ 9J_{n+2}, & otherwise \end{cases}$$

Proof. By rewriting $G^{(2)}_{j(n),4}(2^{n-1}) = 9\gcd(J_{n+1}J_{n-1}, J_{n+5}J_{n+3})$ using Eq. (2.13), and noting that $\gcd(J_{n+1}, J_{n-1}) = \gcd(J_{n+5}, J_{n+3}) = 1$, we apply Eq. (2.18) with the assumption $J_{(n+1,n+3)} = 1$, to obtain:

$$G^{(2)}_{j(n),4}(2^{n-1}) = 9J_{\gcd(n+1,n+5)}J_{\gcd(n-1,n+5)}J_{\gcd(n-1,n+3)}.$$

We now analyze each factor individually:

- $J_{\gcd(n+1,n+5)} = J_{\gcd(n+1,4)} = J_4$, if $n \equiv 3 \pmod 4$; otherwise, $J_{\gcd(n+1,4)} = 1$.
- $J_{\gcd(n-1,n+3)} = J_{\gcd(n-1,4)} = J_4$, if $n \equiv 1 \pmod 4$; otherwise, $\gcd(J_{n-1}, J_{n+3}) = 1$.
- $J_{\gcd(n-1,n+5)} = J_{\gcd(n-1,6)}$, where:
  - $J_{\gcd(n-1,6)} = J_6$, if $n \equiv 1 \pmod 6$,
  - $J_{\gcd(n-1,6)} = J_3$, if $n \equiv 4 \pmod 6$,
  - $J_{\gcd(n-1,6)} = 1$, otherwise.

If $n \equiv 1 \pmod 6$ and either $n \equiv 1 \pmod 4$ or $n \equiv 3 \pmod 4$, then by the Chinese Remainder Theorem we have $n \equiv 1, 7 \pmod{12}$. In this case, the expression simplifies to:

$$G^{(2)}_{j(n),4}(2^{n-1}) = 9J_6J_4.$$

The desired results follow similarly for other congruence classes.

On the other hand, by applying Eq. (2.14), we obtain:

$$H^{(2)}_{j(n),4}(2^{n-2}5^2) = 9J_{n+2}J_{\gcd(n-2,n+6)}.$$

Here, depending on the value of n:

- $J_{\gcd(n-2,8)} = J_8$, if $n \equiv 2 \pmod 8$,
- $J_{\gcd(n-2,8)} = J_4$, if $n \equiv 6 \pmod 8$,
- $J_{\gcd(n-2,8)} = 1$, otherwise.

□

By considering the expressions derived for $a = 2^{n-t}j_t^2$ with $t \in \{3, 4\}$, as given in Equations (2.5) and (2.6), we obtain the following results:

$$G^{(2)}_{j(n)}(2^{n-3} \cdot 7^2) = 9J_{n+3}J_{n-3}, \quad n \geq 3, \tag{2.24}$$

$$H^{(2)}_{j(n)}(2^{n-4} \cdot 17^2) = 9J_{n+4}J_{n-4}, \quad n \geq 4. \tag{2.25}$$

The Jacobsthal numbers appearing in the r-successive altered Jacobsthal–Lucas GCD sequences, as described by Equations (2.24) and (2.25), can be characterized as follows:

$$G^{(2)}_{j(n),6}(2^{n-3} \cdot 7^2) = \begin{cases} 9J_{12}J_{n+3}, & \text{if } n \equiv 3 \pmod{12}, \\ 9J_6J_{n+3}, & \text{if } n \equiv 9 \pmod{12}, \\ 9J_4J_{n+3}, & \text{if } n \equiv 7, 11 \pmod{12}, \\ 9J_3J_{n+3}, & \text{if } n \equiv 0, 6 \pmod{12}, \\ 9J_{n+3}, & \text{otherwise.} \end{cases}$$

$$H^{(2)}_{j(n),8}(2^{n-4} \cdot 17^2) = \begin{cases} 9J_{16}J_{n+4}, & \text{if } n \equiv 4 \pmod{16}, \\ 9J_8J_{n+4}, & \text{if } n \equiv 12 \pmod{16}, \\ 9J_4J_{n+4}, & \text{if } n \equiv 0, 8 \pmod{16}, \\ 9J_{n+4}, & \text{otherwise.} \end{cases}$$

The detailed proofs of these identities are omitted here for conciseness, as they follow analogous reasoning and structural patterns to those established in the preceding cases.

Moreover, the sequences $\left\{G^{(2)}_{j(n),2}(2^{n-1})\right\}$ and $\left\{H^{(2)}_{j(n),4}(2^{n-2}5^2)\right\}$ exhibit periodic behavior similar to the Jacobsthal sequence for $r = 2t$ and $t \in \{1, 2\}$. To assess whether the sequences $\left\{G^{(2)}_{j(n),r}(2^{n-1})\right\}$ for $r \neq 2$ and $\left\{H^{(2)}_{j(n),r}(2^{n-2}5^2)\right\}$ for $r \neq 4$ are bounded by products of Jacobsthal numbers, a computational evaluation was performed. The results indicate that these sequences attain values at indices that are divisors of m, and exhibit periodicity. In particular, for $5 \leq r \leq 50$, both sequences are shown to be periodic and bounded modulo different m values.

Theorem 9. *Let $2t = r$. It is observed that the r-successive altered Jacobsthal Lucas Gcd sequences $G^{(2)}_{j(n),r}(2^{n-t}j_t^2)$ and $H^{(2)}_{j(n),r}(2^{n-t}j_t^2)$ exhibit structural similarities to Jacobsthal sequences. In particular, the following identity holds:*

$$9J_xJ_{n+t} \pmod{4t} = \begin{cases} G^{(2)}_{j(n),r}(2^{n-t}j_t^2), & \text{if } t \text{ is odd,} \\ H^{(2)}_{j(n),r}(2^{n-t}j_t^2), & \text{if } t \text{ is even,} \end{cases}$$

where $n, r, t, x \in \mathbb{Z}^+$ and J_x is chosen from the set $\{1, 2, 4, t, 2t, 4t, d_i\}$, with each $d_i \mid t$ and $d_i \notin \{1, 2, 4, t, 2t, 4t\}$ for $i = 1, 2, \ldots, h$. The selection of J_x depends on the residue class of n modulo $4t$, and is given as:

$$J_x = \begin{cases} J_{4t}, & \text{if } n \equiv t \pmod{4t}, \\ J_{2t}, & \text{if } n \equiv 3t \pmod{4t}, \\ J_t, & \text{if } n \equiv 0, 2t \pmod{4t}, \\ J_{d_i}, & \text{if } n \equiv t + f_id_i \pmod{4t}, \\ J_4, & \text{if } n \equiv t + 4t_j \pmod{4t}, \\ 1, & \text{otherwise,} \end{cases}$$

where $f_i = 1, 2, \ldots, \frac{t}{d_i} - 1$ and $t_j = 1, 2, \ldots, t-1$ when $\gcd(2, t) = 1$, or $t_j = 1, 3, 5, \ldots, t-1$ when $\gcd(2, t) \neq 1$, under the condition $t \notin \{1, 2, 4\}$.

Proof. We begin with the multiplicative expressions for the numbers $G^{(2)}_{j(n)}(2^{n-t}j_t^2)$ and $H^{(2)}_{j(n)}(2^{n-t}j_t^2)$ given in Eqs. (2.5- 2.6). Accordingly, the r-successive altered Jacobsthal Lucas $\gcd$ sequences can be expressed as follows:

$$9\gcd\left(J_{n+t}J_{n-t},\ J_{n+t+r}J_{n-t+r}\right) = \begin{cases} G^{(2)}_{j(n),r}(2^{n-t}j_t^2), & \text{if } t \text{ is odd,} \\ H^{(2)}_{j(n),r}(2^{n-t}j_t^2), & \text{otherwise.} \end{cases}$$

Considering all scenarios in Eqs. (2.18- 2.20), we carry out a comprehensive evaluation. Since the operations are structurally similar, we omit redundant steps for brevity.

First, assume the condition $J_{\gcd(n-t,2t)} = J_{\gcd(n-t+r,2t)} = 1$ holds as given in Eq. (2.18). We obtain:

$$9J_{\gcd(n+t,n+t+r)}J_{\gcd(n+t,n-t+r)}J_{\gcd(n-t,n+t+r)}J_{\gcd(n-t,n-t+r)} = \begin{cases} G^{(2)}_{j(n),r}(2^{n-t}j_t^2), & \text{if } t \text{ is odd,} \\ H^{(2)}_{j(n),r}(2^{n-t}j_t^2), & \text{otherwise.} \end{cases}$$

When $J_{\gcd(n+t,r)} = J_{\gcd(n-t,r)} = 1$ and $2t = r$, we deduce:

$$J_{\gcd(n+t,r)}J_{\gcd(n+t,-2t+r)}J_{\gcd(n-t,2t+r)}J_{\gcd(n-t,r)} = J_xJ_{n+t},$$

where $J_x = J_{\gcd(n-t,4t)}$ and $J_{\gcd(n+t,0)} = J_{n+t}$. Thus, we analyze J_x under different congruences modulo $4t$:

- If $n - t = 4tk_1$, then $n \equiv t \pmod{4t}$ and $J_x = J_{4t}$.

- If $n-t=2tk_2$ with $\gcd(k_2,4)=1$, then $n\equiv 3t \pmod{4t}$ and $J_x=J_{2t}$.
- If $n-t=tk_3$ with $\gcd(k_3,4)=1$, then $n\equiv 0,2t \pmod{4t}$ and $J_x=J_t$.
- If t is not prime and has h positive divisors $d_i \mid t$ $(i=1,2,\ldots,h)$, then for $n-t=d_ik_i$,

$$n\equiv t+f_id_i \pmod{4t}, \quad \gcd(f_i,t)=1, \quad f_i=1,2,\ldots,\tfrac{t}{d_i}-1,$$

  we have $J_x=J_{d_i}$.
- For $n-t=4$ and $\gcd(t,2)=1$, then $n\equiv t+4t_j \pmod{4t}$ with $t_j=1,2,\ldots,t-1$ implies $J_x=J_4$.
- For $n-t=4$ and $\gcd(t,2)\neq 1$, then $t_j=1,3,\ldots,t-1$, still yielding $J_x=J_4$.
- Otherwise, we take $J_x=J_2=J_1$.

Second, consider the case where $J_{\gcd(n+t,n+t+r)}=J_{\gcd(n-t,n-t+r)}=1$, as in Eq. (2.19). Then, we also find $J_{\gcd(n+t,r)}=J_{\gcd(n-t,r)}=1$, yielding:

$$J_{\gcd(n+t,0)}J_{\gcd(n-t,4t)}=J_xJ_{n+t}.$$

Third, suppose Eq. (2.20) holds with $J_{\gcd(n+t,r)}=x$, $J_{\gcd(n-t,r)}=y$, and $r=2t$. Then $J_{\gcd(n-t,2t)}=x$ and $J_{\gcd(n+t,2t)}=y$. We rewrite:

$$\frac{\gcd(x,y)\cdot\gcd(xJ_{n-t},\, yJ_{n+3t})}{xy}\cdot 9J_{n+t}=\begin{cases} G^{(2)}_{j(n),r}(2^{n-t}j_t^2), & \text{if } t \text{ is odd},\\ H^{(2)}_{j(n),r}(2^{n-t}j_t^2), & \text{otherwise.}\end{cases}$$

To verify these identities, a computational program was implemented, testing values up to $t=25$ and $r=50$, with various x and y values. The hypothesis was confirmed to hold in all examined cases. □

The lengths of the periods and the corresponding period values modulo m for both the Jacobsthal sequences and the Jacobsthal product-valued sequences are of particular interest. It is worth noting that a formula, $m=lcm\,[r,2t-r,2t+r]$, $(2t\neq r)$, can be employed to determine the lengths of the periods modulo m. For the sequences $G^{(2)}_{j(n),r}\left(2^{n-3}\cdot 7^2\right)$ with $r\neq 6$ and $H^{(2)}_{j(n),r}\left(2^{n-4}\cdot 17^2\right)$ with $r\neq 8$, it is observed that the r-successive altered Jacobsthal–Lucas gcd sequences are periodic modulo m. The period lengths are governed by the formula $m=\operatorname{lcm}[r,\ 2t-r,\ 2t+r]\,,$ for $r\neq 2t$.

These results were confirmed computationally for $r=1$ to 50, and it was found that all sequences are both bounded and periodic. This formula effectively predicts the period length for both Jacobsthal and Jacobsthal product-valued sequences modulo m.

3 Conclusion

This study investigated the structural properties of squared altered Jacobsthal–Lucas numbers and their associated greatest common divisor (GCD) sequences. By introducing the sequences $G^{(2)}_{j(n)}(a)$ and $H^{(2)}_{j(n)}(a)$, we established their connection to Jacobsthal multiplication patterns. It was shown that these generalized forms conform to a specific multiplicative structure:

$$9J_{n+t}J_{n-t}=\begin{cases} G^{(2)}_{j(n)}\left(2^{n-t}j_t^2\right), & \text{if } t \text{ is odd},\\ H^{(2)}_{j(n)}\left(2^{n-t}j_t^2\right), & \text{if } t \text{ is even.}\end{cases}$$

This relation illustrates how the new sequences generalize well-known identities by embedding them into a structured multiplicative framework. The results demonstrate that these sequences exhibit bounded and periodic behavior modulo m, governed by a unifying expression $m=\operatorname{lcm}[r,2t-r,2t+$

$r]$, for $r \neq 2t$. This reveals intrinsic number-theoretic structures underlying their behavior. Such periodicity has implications for modular arithmetic applications, particularly in computational number theory.

Computational analyses for a wide range of parameters confirmed these patterns and support the formulation of explicit expressions, including Binet-like forms, that provide further insight into their algebraic nature. The sequences $G^{(2)}_{j(n),r}(a)$ and $H^{(2)}_{j(n),r}(a)$ also appear to extend classical Jacobsthal properties to more complex constructs.

These findings lay the groundwork for future research into generalized GCD sequences. In particular, exploring larger values of r and t, and extending algorithmic evaluations may uncover new patterns or unexpected behaviors. Moreover, investigating their connections with classical sequences such as Fibonacci and Lucas numbers could reveal deeper theoretical insights (Koshy, 2019).

Finally, potential applications in areas such as cryptography and coding theory warrant further exploration, as the structural properties observed here may offer practical utility in these domains. For instance, the predictable yet nontrivial periodicity of the sequences may serve as a basis for key generation algorithms or for designing efficient error-detection schemes. By framing these results within applied contexts, the findings may resonate with researchers working not only in number theory but also in algorithm design and information security.

4 Acknowledgements

The authors would like to express their sincere thanks to the editor and the anonymous reviewers for their helpful comments and suggestions.

References

Bilgici, G. and Bród, D. (2023). On *r* -Jacobsthal and *r* -Jacobsthal-Lucas Numbers. *Annales Mathematicae Silesianae*, 37(1):16–31.

Brod, D. (2020). On a new Jacobsthal-type sequence. *Ars Combinatoria*, 150:21–29.

Catarino, P., Vasco, P., Campos, H., Aires, A. P., and Borges, A. (2015). New families of Jacobsthal and Jacobsthal-Lucas numbers. *Algebra and Discrete Mathematics*, 20(1).

Cook, C. K. and Bacon, M. R. (2013). Some identities for Jacobsthal and Jacobsthal-Lucas numbers satisfying higher order recurrence relations. *Annales Mathematicae et Informaticae*, (41):27–39.

Flórez, R., Higuita, R. A., and Junes, L. (2014a). 9-Modularity and GCD Properties of Generalized Fibonacci Numbers. *Integers*, 14:A55.

Flórez, R., Higuita, R. A., and Mukherjee, A. (2014b). Alternating Sums in the Hosoya Polynomial Triangle. *J. Integer Seq.*, 17(9):14–9.

Flórez, R., Higuita, R. A., and Mukherjee, A. (2018a). Characterization of the Strong Divisibility Property for Generalized Fibonacci Polynomials.

Flórez, R., Higuita, R. A., and Mukherjee, A. (2018b). The Star of David and Other Patterns in Hosoya Polynomial Triangles. *J. Integer Seq.*, 21(4):18–4.

Flórez, R. and Junes, L. (2012). GCD Properties in Hosoya's Triangle. *The Fibonacci Quarterly*, 50(2):163–174.

Horadam, A. F. (1988). Jacobsthal and Pell Curves. *The Fibonacci Quarterly*, 26(1):77–83.

Horadam, A. F. (1993). Associated Sequences of General Order. *The Fibonacci Quarterly*, 31(2):165–165.

Horadam, A. F. (1996a). Generalized Jacobsthal Representation Sequence $\{Tn\}$. *Notes on Number Theory and Discrete Mathematics*, 2(3):5–11.

Horadam, A. F. (1996b). Jacobsthal Representation Numbers. *The Fibonacci Quarterly*, 34(1):40–54.

Hosoya, H. (1976). The Fibonacci Quarterly. *Fibonacci Triangle*, 14:173–178.

Koken, F. (2019). Some properties of the altered jacobsthal and jacobsthal lucas numbers. In *International Marmara Science and Social Sciences Congress*, pages 521–528.

Koken, F., Ergun, H., and Uzun, Y. (2025). Properties and Generalizations of Altered Jacobsthal Numbers Squared and their GCD Sequences. *Science and Technology Indonesia*, 10(1):273–282.

Koshy, T. (2019). *Fibonacci and Lucas Numbers with Applications, Volume 2*. John Wiley & Sons.

Uygun, S. (2021). The relations between bi-periodic jacobsthal and bi-periodic jacobsthal lucas sequence. *Cumhuriyet Science Journal*, 42(2):346–357.