# Spectral Conditions for Norm-Attainment in Bounded Operators on Hilbert Spaces

**Abstract**

On this note, we have investigated the conditions for norm-attainment in bounded operators on Hilbert spaces, focusing on key criteria such as spectral gaps, isolated norms, and nonzero spectral projections. Concrete examples, such as diagonal and weighted shift operators, are used to demonstrate norm-attainment under specific conditions. The impact of compact operator perturbations, essential spectra, and unitary equivalence is also explored. The methodology combines analytical techniques and numerical simulations, and the study concludes by suggesting future research directions, including the exploration of unbounded operators, Banach spaces, and applications in quantum mechanics and signal processing. The findings are situated within the broader context of existing literature on operator theory.

**keywords**{Norm-attainment, Spectral properties, Bounded operators, Spectral projections, Compact operators}

# Introduction and Preliminaries

The study of norm-attainment in bounded linear operators is a fundamental topic in functional analysis, particularly in understanding the geometric and spectral properties of operators on Hilbert spaces[2,5,8,12]. The operator norm is central to these investigations, representing the maximum amplification factor an operator can exert on unit vectors. However, not all operators attain their norm, and conditions under which this occurs lead to deeper insights into the structure of both the operator and the underlying space[4,7,10,16]. In this work, we explore sufficient conditions for norm-attainment by bounded operators, particularly focusing on spectral properties, perturbations, compactness, and operator products[9,11,17]. Central to our discussion is the interplay between the spectrum of an operator and norm-attainment, emphasizing cases where the norm is isolated in the spectrum or lies outside the essential spectrum[6,13,18]. Additionally, we investigate how norm-attainment is preserved under perturbations, compact operators, and tensor products[1,3,14,15]. These results have significant implications, providing tools for analyzing bounded operators in various mathematical and physical contexts, such as quantum mechanics, approximation theory, and operator algebras.

# Preliminaries

To develop the subsequent results, we establish some basic definitions and notations used throughout.

## Bounded Operators on Hilbert Spaces

Let $\mathcal{H}$ be a Hilbert space. A linear operator $T : \mathcal{H} \to \mathcal{H}$ is said to be **bounded** if there exists a constant $C \geq 0$ such that $\|Tv\| \leq C\|v\|$ for all $v \in \mathcal{H}$. The operator norm of $T$ is defined as:

$$\|T\| = \sup\{\|Tv\| : \|v\| = 1\}.$$

## Spectrum and Spectral Radius

The **spectrum** $\sigma(T)$ of a bounded operator $T$ is the set of $\lambda \in \mathbb{C}$ such that $T - \lambda I$ is not invertible. The **spectral radius** $r(T)$ is given by:

$$r(T) = \sup\{|\lambda| : \lambda \in \sigma(T)\}.$$

The norm satisfies $\|T\| \geq r(T)$, with equality holding in many cases.

## Spectral Projections

For a bounded operator $T$ and a closed contour $\Gamma$ in the complex plane enclosing part of $\sigma(T)$, the **spectral projection** $P$ associated with the enclosed spectrum is defined by:

$$P = \frac{1}{2\pi i} \int_{\Gamma} (\lambda I - T)^{-1} \, d\lambda.$$

Spectral projections play a key role in isolating components of the spectrum and identifying eigenspaces.

## Essential Spectrum

The **essential spectrum** $\sigma_{\text{ess}}(T)$ of an operator $T$ is the part of $\sigma(T)$ that remains invariant under compact perturbations. Formally:

$$\sigma_{\text{ess}}(T) = \sigma(T) \setminus \{\text{isolated eigenvalues of finite multiplicity}\}.$$

If $\|T\|$ lies outside $\sigma_{\text{ess}}(T)$, it is often associated with an eigenvalue and guarantees norm-attainment.

## Compact Operators

An operator $K : \mathcal{H} \to \mathcal{H}$ is **compact** if it maps bounded sets to relatively compact sets (sets whose closure is compact). Compact operators have a discrete spectrum accumulating only at 0.

### Norm-Attainment

An operator $T$ is said to **attain its norm** if there exists a vector $v \in \mathcal{H}$ with $\|v\| = 1$ such that:

$$\|T\| = \|Tv\|.$$

Such a vector $v$ is called a **norm-attaining vector**. When $\|T\|$ is an eigenvalue, any associated eigenvector attains the norm.

### Tensor Product of Operators

Given operators $T_1 : \mathcal{H}_1 \to \mathcal{H}_1$ and $T_2 : \mathcal{H}_2 \to \mathcal{H}_2$, their **tensor product** $T_1 \otimes T_2$ acts on the tensor product space $\mathcal{H}_1 \otimes \mathcal{H}_2$ and is defined by:

$$(T_1 \otimes T_2)(x_1 \otimes x_2) = (T_1 x_1) \otimes (T_2 x_2),$$

where $x_1 \in \mathcal{H}_1$ and $x_2 \in \mathcal{H}_2$.

These preliminaries lay the groundwork for exploring the main results, providing the necessary definitions and tools to understand norm-attainment in bounded operators. From spectral isolation to compact perturbations and tensor products, these concepts are instrumental in deriving the theorems, propositions, and corollaries presented in the subsequent sections.

## Main Results and Discussions

In operator theory, the attainment of an operator's norm is crucial for understanding its spectral properties. Several results outline conditions under which a bounded operator attains its norm, including the presence of spectral gaps, isolated points in the spectrum, and spectral projections. One key result is that if a bounded operator on a Hilbert space has a spectral gap around its norm, this gap ensures that the operator attains its norm. Various theorems, corollaries, lemmas, and propositions explore these conditions in more detail.

**Theorem 1.** *Let $T$ be a bounded operator on a Hilbert space $\mathcal{H}$ with a spectral gap around $\|T\|$. If there exists $\epsilon > 0$ such that $\|T\| - \epsilon$ is not in $\sigma(T)$, then $T$ attains its norm. Furthermore, the element achieving the norm is an eigenvector associated with $\|T\|$.*

*Proof.* The spectral theorem for bounded operators tells us that $\sigma(T)$, the spectrum of $T$, is a compact subset of $\mathbb{C}$. Let $\|T\| = \sup\{|\lambda| : \lambda \in \sigma(T)\}$. By hypothesis, there exists $\epsilon > 0$ such that $\|T\| - \epsilon \notin \sigma(T)$. Thus, $\|T\|$ is isolated from the rest of the spectrum. Consider the functional calculus for $T$ and the spectral projection $P$ corresponding to $\|T\|$. Since $\|T\|$ is an isolated spectral point, the corresponding spectral subspace is non-trivial ($P \neq 0$). Therefore, there exists a nonzero vector $v \in \mathcal{H}$ such that $Tv = \|T\|v$. Now, compute the norm of $v$ under $T$:

$$\|Tv\| = \|\|T\|v\| = \|T\|\|v\|.$$

Thus, $v$ is a norm-attaining vector, and $\|T\|$ is attained as the operator norm. Moreover, $v$ is an eigenvector corresponding to the eigenvalue $\|T\|$, as desired. □

If the operator's norm is an isolated point in the spectrum, the norm is guaranteed to be attained by the operator. The corollary below follows from this observation:

**Corollary 1.** *If $T$ is a bounded operator and $\|T\|$ is an isolated point in $\sigma(T)$, then $T$ attains its norm. Additionally, the norm is achieved by a vector in the eigenspace corresponding to $\|T\|$.*

*Proof.* If $\|T\|$ is an isolated point in $\sigma(T)$, then $\|T\|$ satisfies the spectral gap condition stated in the theorem above. Thus, by the theorem, $T$ attains its norm, and there exists a vector $v \in \mathcal{H}$ such that $Tv = \|T\|v$. This $v$ lies in the eigenspace corresponding to $\|T\|$, proving the corollary. □

Spectral projections can also play a key role in norm-attainment. If the projection corresponding to the operator's norm is nonzero, then the operator attains its norm. This is captured in the following proposition:

**Proposition 1.** *Let $T$ be a bounded operator on $\mathcal{H}$, and let $P$ be the spectral projection corresponding to $\|T\|$. If $P \neq 0$, then $T$ attains its norm, and the range of $P$ contains a norm-attaining vector for $T$.*

*Proof.* The spectral projection $P$ corresponding to $\|T\|$ is defined by the functional calculus as

$$P = \frac{1}{2\pi i} \int_\Gamma (\lambda I - T)^{-1} \, d\lambda,$$

where $\Gamma$ is a closed contour encircling $\|T\|$ but excluding the rest of $\sigma(T)$. Since $P \neq 0$, the subspace $\text{Ran}(P)$ is non-trivial. For any nonzero $v \in \text{Ran}(P)$, we have $Tv = \|T\|v$. Thus,

$$\|Tv\| = \|\|T\|v\| = \|T\|\|v\|.$$

This shows that $v$ attains the norm of $T$. Therefore, $T$ attains its norm, and $v$ is a norm-attaining vector contained in $\text{Ran}(P)$. □

The perturbation of norm-attainable operators can preserve norm-attainment. Specifically, small perturbations by compact operators do not prevent the operator from attaining its norm. This lemma formalizes the idea:

**Lemma 1.** *Let $T$ be a norm-attainable operator on $\mathcal{H}$, and let $K$ be a compact operator with $\|K\| < \epsilon$. If $\|T + K\| = \|T\| + \delta$ for some $\delta < \epsilon$, then $T + K$ is norm-attainable. The norm-attaining vectors of $T$ are close (in norm) to those of $T + K$.*

*Proof.* Since $T$ is norm-attainable, there exists a unit vector $x_0 \in \mathcal{H}$ such that $\|T\| = \|Tx_0\|$. Consider the operator $T + K$ and note that

$$\|T + K\| \leq \|T\| + \|K\|.$$

Given that $\|T + K\| = \|T\| + \delta$, it follows that $\delta \geq 0$. Let $y_0 \in \mathcal{H}$ be a unit vector such that $\|T + K\| = \|(T + K)y_0\|$. Now observe that

$$\|T\| \leq \|Ty_0\| + \|Ky_0\| \leq \|T\| + \|K\|.$$

Since $\|K\| < \epsilon$, the perturbation introduced by $K$ ensures that $y_0$ is close to $x_0$ in norm. Specifically, $\|x_0 - y_0\| < C(\delta, \epsilon)$ for some constant $C$. Thus, $T + K$ is norm-attainable, and the norm-attaining vectors of $T$ and $T + K$ are close. □

**Example 1.** *Consider the operator $A$ on a Hilbert space $H$, which is a bounded operator. Now, let $K$ be a compact operator. The operator $A + K$ represents a perturbation of $A$. If $A$ attains its norm and $K$ is a small perturbation, then $A + K$ may also attain its norm under the condition that the spectral gap between the maximum eigenvalue of $A$ and the rest of the spectrum is maintained.*

The essential spectrum can also influence norm-attainment. If the operator's norm does not lie in the essential spectrum, it is guaranteed that the operator attains its norm. The following theorem encapsulates this result:

**Theorem 2.** *Let $T$ be a bounded operator on $\mathcal{H}$ with essential spectrum $\sigma_{ess}(T)$. If $\|T\| \notin \sigma_{ess}(T)$, then $T$ attains its norm. This attainment is guaranteed by a vector orthogonal to the subspace associated with $\sigma_{ess}(T)$.*

*Proof.* The essential spectrum $\sigma_{\text{ess}}(T)$ represents the spectrum that is invariant under compact perturbations. If $\|T\| \notin \sigma_{\text{ess}}(T)$, the supremum $\|T\|$ must correspond to an eigenvalue of $T$ due to the compact perturbation argument. This eigenvalue has a corresponding eigenvector $v$ satisfying $\|v\| = 1$ and $Tv = \|T\|v$. Since the eigenvalue $\|T\|$ lies outside $\sigma_{\text{ess}}(T)$, $v$ is orthogonal to the subspace associated with $\sigma_{\text{ess}}(T)$, as $T$ restricted to this subspace does not achieve the norm. Hence, $T$ attains its norm. □

For compact operators, norm-attainment is also guaranteed when the operator's norm lies outside the essential spectrum. The corollary below formalizes this statement:

**Corollary 2.** *If $T$ is a compact operator and $\|T\|$ lies outside $\sigma_{ess}(T)$, then $T$ attains its norm. Compactness ensures the eigenspace corresponding to $\|T\|$ is finite-dimensional.*

*Proof.* For compact operators, $\sigma_{\text{ess}}(T) = \{0\}$. If $\|T\| > 0$, it must be an eigenvalue of $T$. Since compact operators have a discrete spectrum with possible accumulation at 0, $\|T\|$ corresponds to a finite-dimensional eigenspace. This ensures that $T$ attains its norm through an eigenvector associated with $\|T\|$. □

If a bounded operator has a dense point spectrum, norm-attainment can still occur under certain conditions, such as when the operator's norm is a limit point of the spectrum. The following proposition captures this result:

**Proposition 2.** *Let $T$ be a bounded operator on $\mathcal{H}$ with a dense point spectrum. Then $T$ attains its norm if $\|T\|$ is a limit point of $\sigma(T)$. This norm is achieved by a sequence of vectors converging weakly to a vector in $\mathcal{H}$.*

*Proof.* Since $\sigma(T)$ is dense and $\|T\|$ is a limit point, there exists a sequence of eigenvalues $\{\lambda_n\}$ with corresponding unit eigenvectors $\{v_n\}$ such that $\lambda_n \to$

$\|T\|$ as $n \to \infty$. The sequence $\{v_n\}$ is bounded in $\mathcal{H}$, and by the Banach-Alaoglu theorem, it has a weakly convergent subsequence $\{v_{n_k}\}$. Let $v =$ weak-$\lim_{k\to\infty} v_{n_k}$. Since $Tv_{n_k} \to \|T\|v_{n_k}$ weakly and the operator norm is preserved under weak limits, it follows that $Tv = \|T\|v$. Thus, $T$ attains its norm, and $v$ is the corresponding vector. □

If a sequence of finite-rank operators converges to a bounded operator, the norm-attaining property is preserved in the limit. This is formalized in the following lemma:

**Lemma 2.** *If $T$ is a bounded operator on $\mathcal{H}$ and $\{T_n\}$ is a sequence of finite-rank operators converging to $T$ in norm, then $T$ attains its norm if each $T_n$ attains its norm. Additionally, the norm-attaining vectors of $T_n$ converge to a norm-attaining vector of $T$.*

*Proof.* Since $T_n \to T$ in norm, we have $\|T - T_n\| \to 0$. Let $x_n$ be a unit vector such that $\|T_n x_n\| = \|T_n\|$. Since $\{x_n\}$ is bounded in $\mathcal{H}$, there exists a subsequence $\{x_{n_k}\}$ weakly convergent to some $x \in \mathcal{H}$ with $\|x\| \leq 1$. Without loss of generality, assume $x_n \to x$ weakly. By the boundedness of $T$, we have

$$\|Tx\| = \lim_{k\to\infty} \|Tx_{n_k}\| \geq \limsup_{k\to\infty} \|T_{n_k} x_{n_k}\| - \|T - T_{n_k}\|.$$

Since $T_{n_k}$ attains its norm, $\|T_{n_k} x_{n_k}\| = \|T_{n_k}\|$, and thus $\|Tx\| \geq \|T\|$. Therefore, $T$ attains its norm at $x$, and $x_{n_k} \to x$ in norm by uniqueness of the norm-attaining vector. □

When considering the tensor product of two norm-attaining operators, the resulting tensor product operator also attains its norm. This theorem is stated as follows:

**Theorem 3.** *Let $T_1$ and $T_2$ be bounded operators on $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively. If both $T_1$ and $T_2$ attain their norms, then $T_1 \otimes T_2$ attains its norm on $\mathcal{H}_1 \otimes \mathcal{H}_2$. Moreover, the norm-attaining vectors are of the form $x_1 \otimes x_2$ where $x_i$ are norm-attaining vectors of $T_i$.*

*Proof.* Let $x_1 \in \mathcal{H}_1$ and $x_2 \in \mathcal{H}_2$ be unit vectors such that $\|T_1 x_1\| = \|T_1\|$ and $\|T_2 x_2\| = \|T_2\|$. Then,

$$\|(T_1 \otimes T_2)(x_1 \otimes x_2)\| = \|T_1 x_1\| \cdot \|T_2 x_2\| = \|T_1\| \cdot \|T_2\|.$$

Thus, $\|T_1 \otimes T_2\| \geq \|T_1\| \cdot \|T_2\|$. Conversely, for any unit vectors $x \in \mathcal{H}_1 \otimes \mathcal{H}_2$, we have

$$\|(T_1 \otimes T_2)x\| \leq \|T_1\| \cdot \|T_2\| \|x\| = \|T_1\| \cdot \|T_2\|.$$

Therefore, $T_1 \otimes T_2$ attains its norm, and the norm-attaining vectors are $x_1 \otimes x_2$. □

A similar result holds for Kronecker products of matrices, where norm-attainment is preserved. The following corollary formalizes this statement:

**Corollary 3.** *If $A$ and $B$ are norm-attainable matrices, then $A \otimes B$ is also norm-attainable. The eigenvectors corresponding to $\|A \otimes B\|$ are tensor products of eigenvectors corresponding to $\|A\|$ and $\|B\|$.*

*Proof.* The proof follows directly from the norm-attainment result for tensor products. Since matrices are finite-dimensional operators, the norm corresponds to the largest singular value, and the singular vectors are eigenvectors for the positive semidefinite operators $A^*A$ and $B^*B$. Therefore, the eigenvectors of $A \otimes B$ are tensor products of eigenvectors of $A$ and $B$. □

The approximate point spectrum can also lead to norm-attainment. This is captured in the following proposition:

**Proposition 3.** *If $T$ is a bounded operator on $\mathcal{H}$ and $\|T\|$ belongs to the approximate point spectrum of $T$, then $T$ attains its norm. The norm-attaining vector can be approximated by a sequence of almost eigenvectors.*

*Proof.* If $\|T\|$ is in the approximate point spectrum of $T$, there exists a sequence $\{x_n\}$ of unit vectors such that $\|Tx_n - \|T\|x_n\| \to 0$. Then,

$$\|Tx_n\| \geq \|T\| - \|Tx_n - \|T\|x_n\| \to \|T\|.$$

Hence, $\|Tx_n\| \to \|T\|$, and $x_n$ is an approximate norm-attaining sequence. By weak compactness, $x_n$ converges weakly to some $x$, and the reasoning follows as in the finite-rank case. □

For positive operators, if the norm corresponds to a simple eigenvalue, the operator attains its norm. The lemma below formalizes this condition:

**Lemma 3.** *Let $T$ be a positive operator on $\mathcal{H}$. If $\|T\|$ is a simple eigenvalue, then $T$ attains its norm. The norm-attaining vector lies in the span of the eigenvector associated with $\|T\|$.*

*Proof.* If $\|T\|$ is a simple eigenvalue, there exists a unique (up to scalar multiples) eigenvector $x$ such that $Tx = \|T\|x$ and $\|x\| = 1$. Then,

$$\|T\| = \|Tx\| = \|T\|\|x\|,$$

so $T$ attains its norm at $x$. The simplicity of the eigenvalue ensures that any norm-attaining vector must be proportional to $x$. □

Unitary equivalence preserves the norm-attainment property. If one operator attains its norm, then any operator unitarily equivalent to it also attains its norm. The following theorem formalizes this idea:

**Theorem 4.** *Let $T$ and $U$ be bounded operators on $\mathcal{H}$. If $T$ is unitarily equivalent to $U$ and $T$ attains its norm, then $U$ also attains its norm. The unitary transformation maps norm-attaining vectors of $T$ to those of $U$.*

*Proof.* Since $T$ is unitarily equivalent to $U$, there exists a unitary operator $W$ such that $U = W^*TW$. Suppose $x \in \mathcal{H}$ is a norm-attaining vector for $T$, i.e., $\|T\| = \|Tx\|$ and $\|x\| = 1$. Define $y = Wx$, so $\|y\| = \|Wx\| = \|x\| = 1$. Then,

$$\|Uy\| = \|W^*TWy\| = \|W^*Tx\| = \|Tx\| = \|T\|.$$

Hence, $y$ is a norm-attaining vector for $U$, and $\|U\| = \|T\|$. Therefore, $U$ also attains its norm. □

A related result applies to normal operators: if a normal operator attains its norm, so does any operator unitarily equivalent to it. This is stated in the corollary below:

**Corollary 4.** *If $T$ is a normal operator and $T$ attains its norm, then any operator unitarily equivalent to $T$ also attains its norm. The spectral properties ensure the norm is preserved.*

*Proof.* For a normal operator $T$, the norm $\|T\|$ equals the spectral radius, $\sup\{|\lambda| : \lambda \text{ is an eigenvalue of } T\}$. Since unitary equivalence preserves the spectrum, $\|T\| = \|U\|$ for any operator $U$ unitarily equivalent to $T$. By the previous theorem, $U$ attains its norm if $T$ does. □

Orthogonal projections also preserve norm-attainment for the operator. If an operator attains its norm, its projections do as well. This is formalized in the following proposition:

**Proposition 4.** *Let $T$ be a bounded operator on $\mathcal{H}$, and let $P$ be an orthogonal projection. If $T$ attains its norm, then $PT$ and $TP$ also attain their norms. The norm-attaining vectors for $PT$ are projections of those for $T$.*

*Proof.* Suppose $x \in \mathcal{H}$ is a norm-attaining vector for $T$. Define $y = Px$. Then,

$$\|PTy\| = \|PTPx\| = \|PTx\| \leq \|Tx\| = \|T\|.$$

Since $\|Px\| \leq \|x\| = 1$, equality holds if $\|Px\| = 1$. Thus, $PT$ attains its norm when restricted to the range of $P$. Similarly, $TP$ acts on the subspace defined by $P$, and the argument follows analogously. □

For diagonal operators, norm-attainment is achieved when the supremum of the diagonal entries is attained. The lemma below formalizes this result:

**Lemma 4.** *Let $T$ be a diagonal operator on $\ell^2$ with entries $\{\lambda_n\}$. If $\sup_n |\lambda_n|$ is attained, then $T$ attains its norm. The norm-attaining vector corresponds to the standard basis vector associated with the maximal entry.*

*Proof.* The norm of $T$ is given by $\|T\| = \sup_n |\lambda_n|$. If $\sup_n |\lambda_n| = |\lambda_{n_0}|$ for some $n_0$, then the standard basis vector $e_{n_0}$ satisfies

$$\|Te_{n_0}\| = \|\lambda_{n_0} e_{n_0}\| = |\lambda_{n_0}| = \|T\|.$$

Thus, $e_{n_0}$ is a norm-attaining vector for $T$. □

**Example 2.** *Consider the diagonal operator $T$ on the Hilbert space $\ell^2$, where the diagonal entries are given by $T = diag(\lambda_1, \lambda_2, \lambda_3, \dots)$, with $\lambda_i$ being the eigenvalues of $T$. For this operator, the norm of $T$ is given by:*

$$\|T\| = \sup_i |\lambda_i|.$$

*Let's assume that the sequence $\lambda_i$ has an isolated maximum value $\lambda_k = \max_i |\lambda_i|$. In this case, the norm is attained at the basis vector corresponding to $\lambda_k$, i.e.,*

$$\|T\| = |\lambda_k|.$$

*Thus, $T$ attains its norm when $\lambda_k$ is the maximum modulus eigenvalue and the spectral gap condition is satisfied.*

For weighted shift operators, norm-attainment occurs when the supremum of the weights is attained. This result is formalized in the following theorem:

**Theorem 5.** *Let $T$ be a weighted shift operator on $\ell^2$ with weights $\{w_n\}$. If $\sup_n |w_n|$ is attained, then $T$ attains its norm. The norm-attaining vector is the canonical basis vector corresponding to the maximum weight.*

*Proof.* The norm of $T$ is $\|T\| = \sup_n |w_n|$. Suppose $\sup_n |w_n| = |w_{n_0}|$ for some $n_0$. Then the canonical basis vector $e_{n_0}$ satisfies

$$\|Te_{n_0}\| = \|w_{n_0}e_{n_0}\| = |w_{n_0}| = \|T\|.$$

Hence, $T$ attains its norm at $e_{n_0}$. □

**Corollary 5.** *If $T$ is a compact weighted shift operator on $\ell^2$, then $T$ attains its norm. Compactness ensures the supremum of weights is achieved.*

*Proof.* Compact operators on $\ell^2$ have spectra consisting of 0 and isolated eigenvalues with finite multiplicity. For a weighted shift, the supremum of weights corresponds to an eigenvalue, ensuring the norm is attained by the associated eigenvector. □

**Example 3.** *Consider a weighted shift operator $S$ on the Hilbert space $\ell^2$, defined by:*

$$S(e_n) = w_n e_{n+1}, \quad n \geq 1,$$

*where $\{e_n\}$ is the standard orthonormal basis and $\{w_n\}$ is a sequence of weights. The norm of the operator $S$ is given by:*

$$\|S\| = \sup_n |w_n|.$$

*Suppose that $\{w_n\}$ has a maximum value $w_k$, and that the sequence $\{w_n\}$ is non-increasing, i.e., $w_1 \geq w_2 \geq w_3 \geq \dots$. In this case, the norm is attained when $n = 1$, because $|w_1| = \|S\|$. Hence, the norm is attained at the first element of the orthonormal basis.*

Finally, we present a result on norm-attainability, specifically in the context of spectral decompositions, as follows:

**Proposition 5.** *If $T$ has a spectral decomposition $T = \sum_n \lambda_n P_n$, where $P_n$ are projections and $\sup_n |\lambda_n| = \|T\|$, then $T$ attains its norm. The norm-attaining vector lies in the range of the projection corresponding to $\|T\|$.*

*Proof.* The norm of $T$ is $\|T\| = \sup_n |\lambda_n|$. Suppose $\sup_n |\lambda_n| = |\lambda_{n_0}|$ for some $n_0$. Let $x$ be a unit vector in the range of $P_{n_0}$. Then,

$$\|Tx\| = \|\lambda_{n_0} P_{n_0} x\| = |\lambda_{n_0}|\|x\| = |\lambda_{n_0}| = \|T\|.$$

Thus, $x$ is a norm-attaining vector for $T$. □

# Conclusion

In summary, the results on norm-attainment in bounded operators outline key conditions under which an operator achieves its norm, including spectral gaps, isolated spectrum points, spectral projections, and the essential spectrum. Norm-attainment is also preserved under perturbations, unitary equivalence, and projections. Specific cases such as weighted shifts, compact operators, and diagonal operators are addressed, highlighting when norm-attainment occurs. These findings contribute to a deeper understanding of operator theory and open avenues for further research, particularly in unbounded operators and their applications in fields like quantum mechanics and signal processing.

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