Generalized Jacobsthal-Narayana Numbers and Generalized co-Jacobsthal-Narayana Numbers

Abstract. This paper introduces two innovative third-order recurrence sequences: the generalized Jacobsthal-Narayana sequence and the co-Jacobsthal-Narayana sequence. It examines their interrelated properties, including Binet's formulas, generating functions, Simson's formulas, and matrix representations, as well as their special subsequences. The study highlights unique relationships between recurrence equations and roots of characteristic equations, uncovering novel properties. These sequences hold promising potential for applications in various fields, such as number theory, combinatorics, cryptography, and modeling phenomena in physics, biology, and economics. For instance, the matrix representation of co-Jacobsthal-Narayana-Lucas numbers has implications for cryptographic systems.

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1. Introduction: Generalized Tribonacci Numbers

The generalized Tribonacci numbers

$$\{W_n(W_0, W_1, W_2; r, s, t)\}_{n\geq 0}$$

(or $\{W_n\}_{n\geq 0}$ or shortly $\{W_n\}_{n\geq 0}$) is defined as follows:

$$W_n = rW_{n-1} + sW_{n-2} + tW_{n-3}, \qquad W_0 = a, W_1 = b, W_2 = c, \quad n \geq 3 \tag{1.1}$$

where $W_0, W_1, W_2$are arbitrary complex (or real) numbers and r, s and t are real numbers with $t \neq 0$.

The sequence $\{W_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$W_{-n} = -\frac{s}{t}W_{-(n-1)} - \frac{r}{t}W_{-(n-2)} + \frac{1}{t}W_{-(n-3)}$$

for $n = 1, 2, 3, ...$ when $t \neq 0$. Therefore, recurrence (1.1) holds for all integers n.

For r, s, t satisfying Eq. (1.1), the generalized co-Tribonacci numbers

$$\{Y_n(Y_0, Y_1, Y_2; -s, -rt, t^2)\}_{n\geq 0}$$

(or shortly $\{Y_n\}_{n\geq 0}$) is defined as follows:

$$Y_n = -sY_{n-1} - rtY_{n-2} + t^2Y_{n-3}, \qquad Y_0 = d, Y_1 = e, Y_2 = f, \quad n \geq 3 \tag{1.2}$$

i.e.,

$$Y_n = r_1Y_{n-1} + s_1Y_{n-2} + t_1Y_{n-3}, \qquad Y_0 = d, Y_1 = e, Y_2 = f, \quad n \geq 3$$

where $Y_0, Y_1, Y_2$are arbitrary complex (or real) numbers and $r_1 = -s$, $s_1 = -rt$, $t_1 = t^2$.

The sequence $\{Y_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$\begin{aligned} Y_{-n} &= -\frac{-rt}{t^2}Y_{-(n-1)} - \frac{-s}{t^2}Y_{-(n-2)} + \frac{1}{t^2}Y_{-(n-3)} \\ &= -\frac{s_1}{t_1}Y_{-(n-1)} - \frac{r_1}{t_1}Y_{-(n-2)} + \frac{1}{t_1}Y_{-(n-3)} \end{aligned}$$

for $n = 1, 2, 3, ...$ when $t \neq 0$. Therefore, recurrence (1.2) holds for all integer n. For more information on generalized Tribonacci and co-Tribonacci numbers, see [39].

Note that we can easily use and modify the results given for r, s, t in [39] by substituting r_1, s_1, t_1 for r, s, t and we will do this in this paper.

There are close interrelations between roots of characteristic equations of generalized Tribonacci and generalized co-Tribonacci numbers, see [39, Lemma 17.]: If α, β, γ are the roots of characteristic equation of $\{W_n\}$ which is given as

$$z^3 - rz^2 - sz - t = 0,$$

and if $\theta_1, \theta_2, \theta_3$ are the roots of characteristic equation of $\{Y_n\}$ which is given as

$$y^3 - r_1y^2 - s_1y - t_1 = y^3 + sy^2 + rty - t^2 = 0,$$

then we get

$$\theta_1 = \beta\gamma,$$

$$\theta_2 = \alpha\beta,$$

$$\theta_3 = \alpha\gamma.$$

There are also close connections and relations between recurrence equations of generalized Tribonacci and generalized co-Tribonacci numbers, see, for example, Lemma 32 in [39].

2. Generalized Jacobsthal-Narayana Numbers

In this section, we consider the case $r=1,\ s=0,\ t=2$.The generalized Jacobsthal-Narayana numbers

$$\{W_n(W_0,W_1,W_2;1,0,2)\}_{n\geq 0}$$

(or $\{W_n\}_{n\geq 0}$ or shortly $\{W_n\}_{n\geq 0}$) is defined as follows:

$$W_n = W_{n-1} + 2W_{n-3}, \qquad W_0 = a, W_1 = b, W_2 = c, \quad n \geq 3 \tag{2.1}$$

where W_0, W_1, W_2 are arbitrary complex (or real) numbers.

The sequence $\{W_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$W_{-n} = -\frac{1}{2}W_{-(n-2)} + \frac{1}{2}W_{-(n-3)}$$

for $n = 1, 2, 3, ...$ when $t \neq 0$. Therefore, recurrence (2.1) holds for all integers n.

The first few generalized Jacobsthal-Narayana numbers with positive subscript and negative subscript are given in the following Table 1.

Table 1. A few generalized Jacobsthal-Narayana numbers

n	W_n	W_{-n}
0	W_0	W_0
1	W_1	$\frac{1}{2}W_2 - \frac{1}{2}W_1$
2	W_2	$\frac{1}{2}W_1 - \frac{1}{2}W_0$
3	$2W_0 + W_2$	$\frac{1}{2}W_0 + \frac{1}{4}W_1 - \frac{1}{4}W_2$
4	$2W_0 + 2W_1 + W_2$	$\frac{1}{4}W_0 - \frac{1}{2}W_1 + \frac{1}{4}W_2$
5	$2W_0 + 2W_1 + 3W_2$	$\frac{1}{8}W_1 - \frac{1}{2}W_0 + \frac{1}{8}W_2$
6	$6W_0 + 2W_1 + 5W_2$	$\frac{1}{8}W_0 + \frac{3}{8}W_1 - \frac{1}{4}W_2$
7	$10W_0 + 6W_1 + 7W_2$	$\frac{3}{8}W_0 - \frac{5}{16}W_1 + \frac{1}{16}W_2$
8	$14W_0 + 10W_1 + 13W_2$	$\frac{3}{16}W_2 - \frac{1}{8}W_1 - \frac{5}{16}W_0$
9	$26W_0 + 14W_1 + 23W_2$	$\frac{11}{32}W_1 - \frac{1}{8}W_0 - \frac{5}{32}W_2$
10	$46W_0 + 26W_1 + 37W_2$	$\frac{11}{32}W_0 - \frac{3}{32}W_1 - \frac{1}{16}W_2$
11	$74W_0 + 46W_1 + 63W_2$	$\frac{11}{64}W_2 - \frac{15}{64}W_1 - \frac{3}{32}W_0$
12	$126W_0 + 74W_1 + 109W_2$	$\frac{7}{32}W_1 - \frac{15}{64}W_0 - \frac{3}{64}W_2$
13	$218W_0 + 126W_1 + 183W_2$	$\frac{7}{32}W_0 + \frac{9}{128}W_1 - \frac{15}{128}W_2$

As $\{W_n\}$ is a third-order recurrence sequence (difference equation), it's characteristic equation (cubic equation) is

$$z^3 - z^2 - 2 = (z-\alpha)(z-\beta)(z-\gamma) = 0.$$

The roots α,β,γ of characteristic equation of $\{W_n\}$ are given as

$$\begin{aligned}
\alpha &= \frac{1}{3}+\left(\frac{28}{27}+\sqrt{\frac{29}{27}}\right)^{1/3}+\left(\frac{28}{27}-\sqrt{\frac{29}{27}}\right)^{1/3}, \\
\beta &= \frac{1}{3}+\omega\left(\frac{28}{27}+\sqrt{\frac{29}{27}}\right)^{1/3}+\omega^2\left(\frac{28}{27}-\sqrt{\frac{29}{27}}\right)^{1/3}, \\
\gamma &= \frac{1}{3}+\omega^2\left(\frac{28}{27}+\sqrt{\frac{29}{27}}\right)^{1/3}+\omega\left(\frac{28}{27}-\sqrt{\frac{29}{27}}\right)^{1/3},
\end{aligned}$$

where

$$\omega=\frac{-1+i\sqrt{3}}{2}=\exp(2\pi i/3).$$

There are the following relations between the roots of characteristic equation:

$$\left\{\begin{array}{c}
\alpha+\beta+\gamma=1 \\
\alpha\beta+\alpha\gamma+\beta\gamma=0 \\
\alpha\beta\gamma=2
\end{array}\right.$$

The sequence $\{W_n\}$ can be expressed with Binet's formula. Using the roots of characteristic equation and the recurrence relation of W_n, Binet's formula of W_n can be given as follows:

Theorem 1. *For all integers n, Binet's formula of generalized Jacobsthal-Narayana numbers is given as follows.*

$$\begin{aligned}
W_n &= \frac{p_1\alpha^n}{(\alpha-\beta)(\alpha-\gamma)}+\frac{p_2\beta^n}{(\beta-\alpha)(\beta-\gamma)}+\frac{p_3\gamma^n}{(\gamma-\alpha)(\gamma-\beta)} \\
&= A_1\alpha^n+A_2\beta^n+A_3\gamma^n,
\end{aligned}$$

where

$$\begin{aligned}
p_1 &= W_2-(\beta+\gamma)W_1+\beta\gamma W_0, \\
p_2 &= W_2-(\alpha+\gamma)W_1+\alpha\gamma W_0, \\
p_3 &= W_2-(\alpha+\beta)W_1+\alpha\beta W_0.
\end{aligned}$$

and

$$\begin{aligned}
A_1 &= \frac{p_1}{(\alpha-\beta)(\alpha-\gamma)} = \frac{W_2-(\beta+\gamma)W_1+\beta\gamma W_0}{(\alpha-\beta)(\alpha-\gamma)} \\
&= \frac{(\alpha W_2+\alpha(-1+\alpha)W_1+2W_0)}{\alpha^2+6}, \\
A_2 &= \frac{p_2}{(\beta-\alpha)(\beta-\gamma)} = \frac{W_2-(\alpha+\gamma)W_1+\alpha\gamma W_0}{(\beta-\alpha)(\beta-\gamma)} \\
&= \frac{(\beta W_2+\beta(-1+\beta)W_1+2W_0)}{\beta^2+6}, \\
A_3 &= \frac{p_3}{(\gamma-\alpha)(\gamma-\beta)} = \frac{W_2-(\alpha+\beta)W_1+\alpha\beta W_0}{(\gamma-\alpha)(\gamma-\beta)} \\
&= \frac{(\gamma W_2+\gamma(-1+\gamma)W_1+2W_0)}{\gamma^2+6}
\end{aligned}$$

Proof. Set $r=1$, $s=0$, $t=2$ in [39, Theorem 3 (a)]. □

Next, we give the ordinary generating function $\sum_{n=0}^{\infty} W_n z^n$ of the sequence W_n.

Lemma 2. *Suppose that* $f_{W_n}(z) = \sum_{n=0}^{\infty} W_n z^n$ *is the ordinary generating function of the generalized Jacobsthal-Narayana numbers* $\{W_n\}_{n\geq 0}$. *Then,* $\sum_{n=0}^{\infty} W_n z^n$ *is given by*

$$\sum_{n=0}^{\infty} W_n z^n = \frac{W_0+(W_1-W_0)z+(W_2-W_1)z^2}{1-z-2z^3}.$$

Proof. Set $r=1$, $s=0$, $t=2$ in [39, Lemma 9.]. □

Two special cases of the sequence $\{W_n\}$ are the well known Jacobsthal-Narayana sequence $\{B_n\}_{n\geq 0}$ and Jacobsthal-Narayana-Lucas sequence $\{C_n\}_{n\geq 0}$. Jacobsthal-Narayana sequence$\{B_n\}_{n\geq 0}$, Jacobsthal-Narayana-Lucas sequence $\{C_n\}_{n\geq 0}$ are defined, respectively, by the third-order recurrence relations

$$B_n = B_{n-1}+2B_{n-3}, \qquad B_0=0, B_1=1, B_2=1, \tag{2.2}$$

$$C_n = C_{n-1}+2C_{n-3}, \qquad C_0=3, C_1=1, C_2=1. \tag{2.3}$$

The sequences $\{B_n\}_{n\geq 0}$, $\{C_n\}_{n\geq 0}$, can be extended to negative subscripts by defining

$$\begin{aligned}
B_{-n} &= -\frac{1}{2}B_{-(n-2)}+\frac{1}{2}B_{-(n-3)}, \\
C_{-n} &= -\frac{1}{2}B_{-(n-2)}+\frac{1}{2}B_{-(n-3)},
\end{aligned}$$

for $n=1,2,3,...$ respectively. Therefore, recurrences (2.2)-(2.3) hold for all integer n.

Next, we present the first few values of the Jacobsthal-Narayana and Jacobsthal-Narayana-Lucas numbers with positive and negative subscripts:

Table 2. The first few values of the special third-order numbers with positive and negative subscripts.

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13
B_n	0	1	1	1	3	5	7	13	23	37	63	109	183	309
B_{-n}	0	0	$\frac{1}{2}$	0	$-\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{8}$	$-\frac{1}{4}$	$\frac{1}{16}$	$\frac{3}{16}$	$-\frac{5}{32}$	$-\frac{1}{16}$	$\frac{11}{64}$	$-\frac{3}{64}$
C_n	3	1	1	7	9	11	25	43	65	115	201	331	561	963
C_{-n}	3	0	-1	$\frac{3}{2}$	$\frac{1}{2}$	$-\frac{5}{4}$	$\frac{1}{2}$	$\frac{7}{8}$	$-\frac{7}{8}$	$-\frac{3}{16}$	$\frac{7}{8}$	$-\frac{11}{32}$	$-\frac{17}{32}$	$\frac{39}{64}$

The sequence $\{B_n\}$ is labelled in [15] as A077949 with the expansion of $\dfrac{1}{1-z-2z^3}$. In [13], authors defined Jacobsthal-Narayana numbers and then investigated the Binet's formula, generating functions and some identities of the sequence $\{B_n\}$.

For all integers n, Binet's formula of Jacobsthal-Narayana and Jacobsthal-Narayana-Lucas numbers (using initial conditions ((2.2) and (2.3)) in Theorem 1) can be expressed as follows:

Theorem 3. *For all integers n, Binet's formulas of Jacobsthal-Narayana and Jacobsthal-Narayana-Lucas numbers are*

$$\begin{aligned} B_n &= \frac{\alpha^{n+1}}{(\alpha-\beta)(\alpha-\gamma)}+\frac{\beta^{n+1}}{(\beta-\alpha)(\beta-\gamma)}+\frac{\gamma^{n+1}}{(\gamma-\alpha)(\gamma-\beta)} \\ &= \frac{\alpha^{n+2}}{\alpha^2+6}+\frac{\beta^{n+2}}{\beta^2+6}+\frac{\gamma^{n+2}}{\gamma^2+6}, \\ C_n &= \alpha^n+\beta^n+\gamma^n, \end{aligned}$$

respectively.

Lemma 2 gives the following results as particular examples (generating functions of Jacobsthal-Narayana and Jacobsthal-Narayana-Lucas numbers).

Corollary 4. *Generating functions of Jacobsthal-Narayana and Jacobsthal-Narayana-Lucas numbers are*

$$\begin{aligned} \sum_{n=0}^{\infty} B_n z^n &= \frac{z}{1-z-2z^3}, \\ \sum_{n=0}^{\infty} C_n z^n &= \frac{3-2z}{1-z-2z^3}, \end{aligned}$$

respectively.

2.1. Some Identities of Generalized Jacobsthal-Narayana Numbers. Now, we present some identities of generalized Jacobsthal-Narayana, Jacobsthal-Narayana and Jacobsthal-Narayana-Lucas numbers. First, we can give a few basic relations between $\{B_n\}$ and $\{C_n\}$.

Lemma 5. *The following equalities are true:*

- **(a):** $2C_n = 3B_{n+4} - 5B_{n+3} + 2B_{n+2}$.
- **(b):** $C_n = -B_{n+3} + B_{n+2} + 3B_{n+1}$.
- **(c):** $2C_n = 6B_{n+1} - 4B_n$.

(d): $C_n = 3B_{n+1} - 2B_n$.

(e): $C_n = B_n + 6B_{n-2}$.

(f): $116B_n = -3C_{n+4} + C_{n+3} + 20C_{n+2}$.

(g): $58B_n = -C_{n+3} + 10C_{n+2} - 3C_{n+1}$.

(h): $58B_n = 9C_{n+2} - 3C_{n+1} - 2C_n$.

(i): $58B_n = 6C_{n+1} - 2C_n + 18C_{n-1}$.

(j): $29B_n = 2C_n + 9C_{n-1} + 6C_{n-2}$.

Proof. Set $G_n = B_n$, $H_n = C_n$ and $r = 1$, $s = 0$, $t = 2$ in [39, Lemma 36.]. □

Note that all the identities in the above lemma can be proved by induction as well.

Next, we give a few basic relations between $\{B_n\}$ and $\{W_n\}$.

Lemma 6. *The following equalities are true:*

(a): $(W_2^3 + 2W_1^3 + 4W_0^3 + W_1^2W_2 - 2W_1W_2^2 + 2W_0^2W_2 + 2W_0W_1^2 - 6W_0W_1W_2)\ B_n = (W_1^2 + 2W_0^2 - W_1W_2)W_{n+2} + (W_2^2 - W_1W_2 - 2W_0W_1)W_{n+1} + (2W_1^2 - 2W_0W_2)W_n$

(b): $(W_2^3 + 2W_1^3 + 4W_0^3 + W_1^2W_2 - 2W_1W_2^2 + 2W_0^2W_2 + 2W_0W_1^2 - 6W_0W_1W_2)\ B_n = (W_2^2 + W_1^2 + 2W_0^2 - 2W_1W_2 - 2W_0W_1)W_{n+1} + (2W_1^2 - 2W_0W_2)W_n + (2W_1^2 + 4W_0^2 - 2W_1W_2)W_{n-1}$

(c): $(W_2^3 + 2W_1^3 + 4W_0^3 + W_1^2W_2 - 2W_1W_2^2 + 2W_0^2W_2 + 2W_0W_1^2 - 6W_0W_1W_2)\ B_n = (W_2^2 + 3W_1^2 + 2W_0^2 - 2W_1W_2 - 2W_0W_2 - 2W_0W_1)W_n + (2W_1^2 + 4W_0^2 - 2W_1W_2)W_{n-1} + (2W_2^2 + 2W_1^2 + 4W_0^2 - 4W_1W_2 - 4W_0W_1)W_{n-2}$

(d): $2W_n = (W_2 - W_1)B_{n+2} + (-W_2 + W_1 + 2W_0)B_{n+1} + (2W_1 - 2W_0)B_n$.

(e): $W_n = W_0B_{n+1} + (W_1 - W_0)B_n + (W_2 - W_1)B_{n-1}$.

(f): $W_n = W_1B_n + (W_2 - W_1)B_{n-1} + 2W_0B_{n-2}$.

Proof. Set $G_n = B_n$ and $r = 1$, $s = 0$, $t = 2$ in [39, Lemma 37.]. □

Now, we present a few basic relations between $\{C_n\}$ and $\{W_n\}$.

Lemma 7. *The following equalities are true:*

(a): $(W_2^3 + 2W_1^3 + 4W_0^3 + W_1^2W_2 - 2W_1W_2^2 + 2W_0^2W_2 + 2W_0W_1^2 - 6W_0W_1W_2)C_n = (3W_2^2 + W_1^2 + 2W_0^2 - 4W_1W_2 - 6W_0W_1)W_{n+2} + (-2W_2^2 + 6W_1^2 - 6W_0W_2 + 2W_1W_2 + 4W_0W_1)W_{n+1} + (2W_1^2 + 12W_0^2 - 6W_1W_2 + 4W_0W_2)W_n$.

(b): $(W_2^3 + 2W_1^3 + 4W_0^3 + W_1^2W_2 - 2W_1W_2^2 + 2W_0^2W_2 + 2W_0W_1^2 - 6W_0W_1W_2)C_n = (W_1^2 + W_2^2 + 6W_1^2 + 2W_0^2 - 6W_0W_2 - 2W_1W_2 - 2W_0W_1)W_{n+1} + (12W_0^2 - 6W_1W_2 + 2W_1^2 + 4W_0W_2)W_n + (6W_2^2 + 2W_1^2 + 4W_0^2 - 8W_1W_2 - 12W_0W_1)W_{n-1}$.

(c): $(W_2^3 + 2W_1^3 + 4W_0^3 + W_1^2W_2 - 2W_1W_2^2 + 2W_0^2W_2 + 2W_0W_1^2 - 6W_0W_1W_2)C_n = (W_2^2 + 9W_1^2 + 14W_0^2 - 8W_1W_2 - 2W_0W_2 - 2W_0W_1)W_n + (6W_2^2 + 2W_1^2 + 4W_0^2 - 8W_1W_2 - 12W_0W_1)W_{n-1} + (2W_2^2 + 14W_1^2 + 4W_0^2 - 4W_1W_2 - 12W_0W_2 - 4W_0W_1)W_{n-2}$.

(d): $116W_n = (-2W_2 + 20W_1 - 6W_0)C_{n+2} + (20W_2 - 26W_1 + 2W_0)C_{n+1} + (-6W_2 + 2W_1 + 40W_0)C_n$.

(e): $116W_n = (18W_2 - 6W_1 - 4W_0)C_{n+1} + (-6W_2 + 2W_1 + 40W_0)C_n + (-4W_2 + 40W_1 - 12W_0)C_{n-1}$.

(f): $116W_n = (12W_2 - 4W_1 + 36W_0)C_n + (-4W_2 + 40W_1 - 12W_0)C_{n-1} + (36W_2 - 12W_1 - 8W_0)C_{n-2}$.

Proof. Set $H_n = C_n$, and $r = 1$, $s = 0$, $t = 2$ in [39, Lemma 38.]. □

Now, we present some identities of generalized Jacobsthal-Narayana numbers and its special cases.

Lemma 8. *Suppose that* $\{Z_n\}_{n\geq 0} = \{Z_n(Z_0, Z_1, Z_2)\}_{n\geq 0}$ *is also defined by the third-order recurrence relations*

$$Z_n = Z_{n-1} + 2Z_{n-3} \tag{2.4}$$

i.e.,

$$Z_{n+3} = Z_{n+2} + 2Z_n$$

with the initial values Z_0, Z_1, Z_2 *not all being zero and*

$$Z_{-n} = -\frac{1}{2}Z_{-(n-2)} + \frac{1}{2}Z_{-(n-3)}$$

so that (2.4) is true for all integer n.

Then the following equalities are true:

(a): $(Z_0Z_3^2 + Z_1^2Z_4 + Z_2^3 - Z_0Z_2Z_4 - 2Z_1Z_2Z_3)W_n = ((Z_1^2 - Z_0Z_2)W_2 + (Z_0Z_3 - Z_1Z_2)W_1 + (Z_2^2 - Z_1Z_3)W_0)Z_{n+2} + ((Z_0Z_3 - Z_1Z_2)W_2 + (Z_2^2 - Z_0Z_4)W_1 + (Z_1Z_4 - Z_2Z_3)W_0)Z_{n+1} + ((Z_2^2 - Z_1Z_3)W_2 + (Z_1Z_4 - Z_2Z_3)W_1 + (Z_3^2 - Z_2Z_4)W_0)Z_n$.

(b): $(W_0W_3^2 + W_1^2W_4 + W_2^3 - W_0W_2W_4 - 2W_1W_2W_3)B_n = (W_1^2 + 2W_0^2 - W_1W_2)W_{n+2} + (W_2^2 - W_1W_2 - 2W_0W_1)W_{n+1} + 2(W_1^2 - W_0W_2)W_n$.

(c): $2W_n = (W_2 - W_1)B_{n+2} + (-W_2 + W_1 + 2W_0)B_{n+1} + 2(W_1 - W_0)B_n$.

(d): $(W_0W_3^2 + W_1^2W_4 + W_2^3 - W_0W_2W_4 - 2W_1W_2W_3)C_n = (3W_2^2 + W_1^2 + 2W_0^2 - 4W_1W_2 - 6W_0 W_1)W_{n+2} + (-2W_2^2 + 6W_1^2 + 2W_1W_2 - 6W_0W_2 + 2(s^2 + tr)W_0W_1)W_{n+1} + 2(W_1^2 + 6W_0^2 - 3W_1W_2 + 2W_0W_2)W_n$.

(e): $116W_n = 2(-W_2 + 10W_1 - 3W_0)C_{n+2} + 2(10W_2 - 13W_1 + W_0)C_{n+1} + 2(-3W_2 + W_1 + 20W_0)C_n$.

Proof.

(a): Writing

$$W_n = p_1 \times Z_{n+2} + p_2 \times Z_{n+1} + p_3 \times Z_n$$

and solving the system of equations

$$\begin{aligned} W_0 &= p_1 \times Z_2 + p_2 \times Z_1 + p_3 \times Z_0 \\ W_1 &= p_1 \times Z_3 + p_2 \times Z_2 + p_3 \times Z_1 \\ W_2 &= p_1 \times Z_4 + p_2 \times Z_3 + p_3 \times Z_2 \end{aligned}$$

we find the required identity.

(b): Replace W_n and Z_n with B_n and W_n, respectively in (a).

(c): Replace Z_n with B_n in (a).

(d): Replace W_n and Z_n with C_n and W_n, respectively in (a).

(e): Replace Z_n with C_n in (a). □

2.2. Simson's Formulas of Generalized Jacobsthal-Narayana Numbers. The following theorem gives Simson's formula of the generalized Jacobsthal-Narayana numbers $\{W_n\}$.

THEOREM 9 (Simson's Formula of Generalized Jacobsthal-Narayana Numbers). *For all integers n, we have*

$$\begin{vmatrix} W_{n+2} & W_{n+1} & W_n \\ W_{n+1} & W_n & W_{n-1} \\ W_n & W_{n-1} & W_{n-2} \end{vmatrix} = 2^n \begin{vmatrix} W_2 & W_1 & W_0 \\ W_1 & W_0 & W_{-1} \\ W_0 & W_{-1} & W_{-2} \end{vmatrix}$$
$$= 2^n \begin{vmatrix} W_2 & W_1 & W_0 \\ W_1 & W_0 & \frac{1}{2}(W_2 - W_1) \\ W_0 & \frac{1}{2}(W_2 - W_1) & \frac{1}{2}(W_1 - W_0) \end{vmatrix}.$$

Proof. Set $r = 1$, $s = 0$, $t = 2$ in [39, Theorem 33.]. □

The previous theorem gives the following results as particular examples.

COROLLARY 10. *For all integers n, Simson's formula of Jacobsthal-Narayana and Jacobsthal-Narayana-Lucas numbers are given as*

$$\begin{vmatrix} B_{n+2} & B_{n+1} & B_n \\ B_{n+1} & B_n & B_{n-1} \\ B_n & B_{n-1} & B_{n-2} \end{vmatrix} = -2^{n-1},$$

$$\begin{vmatrix} C_{n+2} & C_{n+1} & C_n \\ C_{n+1} & C_n & C_{n-1} \\ C_n & C_{n-1} & C_{n-2} \end{vmatrix} = -29 \times 2^n,$$

respectively.

Proof. Set $W_n = B_n$ and $W_n = C_n$ in Theorem 9, respectively. □

2.3. Recurrence Properties of Generalized Jacobsthal-Narayana Numbers. The generalized Jacobsthal-Narayana numbers W_n at negative indices can be expressed by the sequence itself at positive indices.

THEOREM 11. *For $n \in \mathbb{Z}$, we have*

$$W_{-n} = 2^{-n}(W_{2n} - C_n W_n + \frac{1}{2}(C_n^2 - C_{2n})W_0).$$

Proof. Set $r=1$, $s=0$, $t=2$ and $H_n=C_n$ in [39, Theorem 39.]. □

As special cases of Theorem 11, we have the following corollary.

Corollary 12. *For $n\in\mathbb{Z}$, we have*

(a): $B_{-n}=\frac{1}{2^n}(2B_n^2+B_{2n}-3B_nB_{n+1})$

(b): $C_{-n}=\frac{1}{2^{n+1}}(C_n^2-C_{2n})$.

Proof. Set $r=1$, $s=0$, $t=2$ and $G_n=B_n$ and $H_n=C_n$,respectively, in [39, Corollary 42.] or take $W_n=B_n$ and $W_n=C_n$,respectively, in Theorem 11. □

2.4. Sum Formulas $\sum\limits_{k=0}^{n}W_k$, $\sum\limits_{k=0}^{n}W_{2k}$, $\sum\limits_{k=0}^{n}W_{2k+1}$, $\sum\limits_{k=0}^{n}W_{-k}$, $\sum\limits_{k=0}^{n}W_{-2k}$, $\sum\limits_{k=0}^{n}W_{-2k+1}$ and Generating Functions $\sum_{n=0}^{\infty}W_nz^n$, $\sum_{n=0}^{\infty}W_{2n}z^n$, $\sum_{n=0}^{\infty}W_{2n+1}z^n$, $\sum_{n=0}^{\infty}W_{-n}z^n$, $\sum_{n=0}^{\infty}W_{-2n}z^n$, $\sum_{n=0}^{\infty}W_{-2n+1}z^n$ of Generalized Jacobsthal-Narayana Numbers. Next, we present sum formulas of generalized Jacobsthal-Narayana numbers

Theorem 13. *For $n\geq 0$, we have the following sum formulas for generalized Jacobsthal-Narayana numbers:*

(a): $\sum\limits_{k=0}^{n}W_k=\frac{1}{2}(W_{n+2}+2W_n-W_2)$.

(b): $\sum\limits_{k=0}^{n}W_{2k}=\frac{1}{8}(W_{2n+2}+2W_{2n+1}+6W_{2n}-W_2-2W_1+2W_0)$.

(c): $\sum\limits_{k=0}^{n}W_{2k+1}=\frac{1}{8}(3W_{2n+2}+6W_{2n+1}+2W_{2n}-3W_2+2W_1-2W_0)$.

(d): $\sum\limits_{k=0}^{n}W_{-k}=\frac{1}{2}(-W_{-n+2}+W_2+2W_0)$.

(e): $\sum\limits_{k=0}^{n}W_{-2k}=\frac{1}{8}(-W_{-2n}-2W_{-2n-1}-6W_{-2n-2}+W_2+2W_1+6W_0)$.

(f): $\sum\limits_{k=0}^{n}W_{-2k+1}=\frac{1}{8}(-3W_{-2n}-6W_{-2n-1}-2W_{-2n-2}+3W_2+6W_1+2W_0)$.

Proof.

(a): Set $r=1,s=0,t=2$ and $z=1$ in [39, Theorem 62 (a) (i)].

(b): Set $r=1,s=0,t=2$ and $z=1$ in [39, Theorem 62 (b) (i)].

(c): Set $r=1,s=0,t=2$ and $z=1$ in [39, Theorem 62 (c) (i)].

(d): Set $r=1,s=0,t=2$ and $z=1$ in [39, Theorem 62 (d) (i)].

(e): Set $r=1,s=0,t=2$ and $z=1$ in [39, Theorem 62 (e) (i)].

(f): Set $r=1,s=0,t=2$ and $z=1$ in [39, Theorem 62 (f) (i)]. □

From the last Theorem, we have the following Corollary which gives sum formulas of Jacobsthal-Narayana numbers (take $W_n=B_n$ with $B_0=0,B_1=1,B_2=1$).

Corollary 14. *For $n\geq 0$, Jacobsthal-Narayana numbers have the following properties.*

(a): $\sum\limits_{k=0}^{n}B_k=\frac{1}{2}(B_{n+2}+2B_n-1)$.

(b): $\sum\limits_{k=0}^{n} B_{2k} = \frac{1}{8}(B_{2n+2} + 2B_{2n+1} + 6B_{2n} - 3)$.
(c): $\sum\limits_{k=0}^{n} B_{2k+1} = \frac{1}{8}(3B_{2n+2} + 6B_{2n+1} + 2B_{2n} - 1)$.
(d): $\sum\limits_{k=0}^{n} B_{-k} = \frac{1}{2}(-B_{-n+2} + 1)$.
(e): $\sum\limits_{k=0}^{n} B_{-2k} = \frac{1}{8}(-B_{-2n} - 2B_{-2n-1} - 6B_{-2n-2} + 3)$.
(f): $\sum\limits_{k=0}^{n} B_{-2k+1} = \frac{1}{8}(-3B_{-2n} - 6B_{-2n-1} - 2B_{-2n-2} + 9)$.

Taking $W_n = C_n$ with $C_0 = 3, C_1 = 1, C_2 = 1$ in the last Theorem, we have the following Corollary which gives sum formulas of Jacobsthal-Narayana-Lucas numbers.

Corollary 15. *For $n \geq 0$, Jacobsthal-Narayana-Lucas numbers have the following properties:*
(a): $\sum\limits_{k=0}^{n} C_k = \frac{1}{2}(C_{n+2} + 2C_n - 1)$.
(b): $\sum\limits_{k=0}^{n} C_{2k} = \frac{1}{8}(C_{2n+2} + 2C_{2n+1} + 6C_{2n} + 3)$.
(c): $\sum\limits_{k=0}^{n} C_{2k+1} = \frac{1}{8}(3C_{2n+2} + 6C_{2n+1} + 2C_{2n} - 7)$.
(d): $\sum\limits_{k=0}^{n} C_{-k} = \frac{1}{2}(-C_{-n+2} + 7)$.
(e): $\sum\limits_{k=0}^{n} C_{-2k} = \frac{1}{8}(-C_{-2n} - 2C_{-2n-1} - 6C_{-2n-2} + 21)$.
(f): $\sum\limits_{k=0}^{n} C_{-2k+1} = \frac{1}{8}(-3C_{-2n} - 6C_{-2n-1} - 2C_{-2n-2} + 15)$.

Next, we give the ordinary generating function of special cases of the generalized Jacobsthal-Narayana numbers $\{W_{mn+j}\}$.

Corollary 16. *The ordinary generating functions of the sequences W_n, W_{2n}, W_{2n+1}, W_{-n}, W_{-2n}, W_{-2n+1} are given as follows:*

(a): $(|z| < \min\{|\alpha|^{-1}, |\beta|^{-1}, |\gamma|^{-1}\} = |\alpha|^{-1} \simeq 0.589754)$.
$$\sum_{n=0}^{\infty} W_n z^n = \frac{-W_0 + (W_0 - W_1)z + (W_1 - W_2)z^2}{2z^3 + z - 1}.$$
(b): $(|z| < \min\{|\alpha|^{-2}, |\beta|^{-2}, |\gamma|^{-2}\} = |\alpha|^{-2} \simeq 0.34781)$.
$$\sum_{n=0}^{\infty} W_{2n} z^n = \frac{-W_0 + (W_0 - W_2)z + (2W_0 - 2W_1)z^2}{4z^3 + 4z^2 + z - 1}.$$
(c): $(|z| < \min\{|\alpha|^{-2}, |\beta|^{-2}, |\gamma|^{-2}\} = |\alpha|^{-2} \simeq 0.34781)$.
$$\sum_{n=0}^{\infty} W_{2n+1} z^n = \frac{-W_1 - (2W_0 - W_1 + W_2)z + 2(W_1 - W_2)z^2}{4z^3 + 4z^2 + z - 1}.$$
(d): $(|z| < \min\{|\alpha|, |\beta|, |\gamma|\} = |\beta| = |\gamma| \simeq 1.086052)$.
$$\sum_{n=0}^{\infty} W_{-n} z^n = \frac{2W_0 + (W_2 - W_1)z + W_1 z^2}{-z^3 + z^2 + 2}.$$

(e): $(|z|<\min\{|\alpha|^{2},|\beta|^{2},|\gamma|^{2}\}=|\beta|^{2}=|\gamma|^{2}\simeq 1.179509)$.

$$\sum_{n=0}^{\infty}W_{-2n}z^{n}=\frac{4W_{0}+(2W_{0}+2W_{1})z+W_{2}z^{2}}{-z^{3}+z^{2}+4z+4}.$$

(f): $(|z|<\min\{|\alpha|^{2},|\beta|^{2},|\gamma|^{2}\}=|\beta|^{2}=|\gamma|^{2}\simeq 1.179509)$.

$$\sum_{n=0}^{\infty}W_{-2n+1}z^{n}=\frac{4W_{1}+(2W_{1}+2W_{2})z+(2W_{0}+W_{2})z^{2}}{-z^{3}+z^{2}+4z+4}.$$

Proof. Take $r=1,s=0,t=2$ in [39, Corollary 67.]. □

Now, we consider special cases of the last corollary.

Corollary 17. *The ordinary generating functions of special cases of the generalized Jacobsthal-Narayana numbers are given as follows:*

(a): $(|z|<\min\{|\alpha|^{-1},|\beta|^{-1},|\gamma|^{-1}\}=|\alpha|^{-1}\simeq 0.589754)$.

$$\begin{aligned}\sum_{n=0}^{\infty}B_{n}z^{n}&=\frac{-z}{2z^{3}+z-1},\\ \sum_{n=0}^{\infty}C_{n}z^{n}&=\frac{2z-3}{2z^{3}+z-1}.\end{aligned}$$

(b): $(|z|<\min\{|\alpha|^{-2},|\beta|^{-2},|\gamma|^{-2}\}=|\alpha|^{-2}\simeq 0.34781)$.

$$\begin{aligned}\sum_{n=0}^{\infty}B_{2n}z^{n}&=\frac{-2z^{2}-z}{4z^{3}+4z^{2}+z-1},\\ \sum_{n=0}^{\infty}C_{2n}z^{n}&=\frac{4z^{2}+2z-3}{4z^{3}+4z^{2}+z-1}.\end{aligned}$$

(c): $(|z|<\min\{|\alpha|^{-2},|\beta|^{-2},|\gamma|^{-2}\}=|\alpha|^{-2}\simeq 0.34781)$.

$$\begin{aligned}\sum_{n=0}^{\infty}B_{2n+1}z^{n}&=\frac{-1}{4z^{3}+4z^{2}+z-1},\\ \sum_{n=0}^{\infty}C_{2n+1}z^{n}&=\frac{-6z-1}{4z^{3}+4z^{2}+z-1}.\end{aligned}$$

(d): $(|z|<\min\{|\alpha|,|\beta|,|\gamma|\}=|\beta|=|\gamma|\simeq 1.086052)$.

$$\begin{aligned}\sum_{n=0}^{\infty}B_{-n}z^{n}&=\frac{z^{2}}{-z^{3}+z^{2}+2},\\ \sum_{n=0}^{\infty}C_{-n}z^{n}&=\frac{z^{2}+6}{-z^{3}+z^{2}+2}.\end{aligned}$$

(e): $(|z|<\min\{|\alpha|^{2},|\beta|^{2},|\gamma|^{2}\}=|\beta|^{2}=|\gamma|^{2}\simeq 1.179509)$.

$$\begin{aligned}\sum_{n=0}^{\infty}B_{-2n}z^{n}&=\frac{z^{2}+2z}{-z^{3}+z^{2}+4z+4},\\ \sum_{n=0}^{\infty}C_{-2n}z^{n}&=\frac{z^{2}+8z+12}{-z^{3}+z^{2}+4z+4}.\end{aligned}$$

(f): $(|z| < \min\{|\alpha|^2, |\beta|^2, |\gamma|^2\} = |\beta|^2 = |\gamma|^2 \simeq 1.179509)$.

$$\begin{aligned}\sum_{n=0}^{\infty} B_{-2n+1} z^n &= \frac{z^2+4z+4}{-z^3+z^2+4z+4},\\ \sum_{n=0}^{\infty} C_{-2n+1} z^n &= \frac{7z^2+4z+4}{-z^3+z^2+4z+4}.\end{aligned}$$

From the last corollary, we obtain the following results for special cases of z.

Corollary 18. *We have the following infinite sums .*

(a): $z = \frac{1}{2}$

$$\begin{aligned}\sum_{n=0}^{\infty} \frac{B_n}{2^n} &= 2,\\ \sum_{n=0}^{\infty} \frac{C_n}{2^n} &= 8.\end{aligned}$$

(b): $z = \frac{1}{4}$

$$\begin{aligned}\sum_{n=0}^{\infty} \frac{B_{2n}}{4^n} &= \frac{6}{7},\\ \sum_{n=0}^{\infty} \frac{C_{2n}}{4^n} &= \frac{36}{7}.\end{aligned}$$

(c): $z = \frac{1}{4}$

$$\begin{aligned}\sum_{n=0}^{\infty} \frac{B_{2n+1}}{4^n} &= \frac{16}{7},\\ \sum_{n=0}^{\infty} \frac{C_{2n+1}}{4^n} &= \frac{40}{7}.\end{aligned}$$

(d): $z = 1$

$$\begin{aligned}\sum_{n=0}^{\infty} B_{-n} &= \frac{1}{2},\\ \sum_{n=0}^{\infty} C_{-n} &= \frac{7}{2}.\end{aligned}$$

(e): $z = 1$

$$\begin{aligned}\sum_{n=0}^{\infty} B_{-2n} &= \frac{3}{8},\\ \sum_{n=0}^{\infty} C_{-2n} &= \frac{21}{8}.\end{aligned}$$

(f): $z=1$

$$\sum_{n=0}^{\infty} B_{-2n+1} = \frac{9}{8},$$
$$\sum_{n=0}^{\infty} C_{-2n+1} = \frac{15}{8}.$$

2.5. Sum Formulas $\sum_{k=0}^{n} z^k W_k^2$, $\sum_{k=0}^{n} z^k W_{k+1}W_k$, $\sum_{k=0}^{n} z^k W_{k+2}W_k$ and Generating Functions $\sum_{n=0}^{\infty} W_n^2 z^n$, $\sum_{n=0}^{\infty} W_{n+1}W_n z^n$, $\sum_{n=0}^{\infty} W_{n+2}W_n z^n$ of Generalized Jacobsthal-Narayana Numbers. Next, we present sum formulas of generalized Jacobsthal-Narayana numbers.

THEOREM 19. *For $n \geq 0$, we have the following sum formulas for generalized Jacobsthal-Narayana numbers:*

(a): $\sum_{k=0}^{n} W_k^2 = \frac{1}{8}(5W_{n+3}^2 + 12W_{n+2}^2 + 12W_{n+1}^2 - 12W_{n+2}W_{n+3} - 4W_{n+1}W_{n+3} - 8W_{n+1}W_{n+2} - 5W_2^2 - 12W_1^2 - 12W_0^2 + 12W_1W_2 + 4W_0W_2 + 8W_0W_1)$.

(b): $\sum_{k=0}^{n} W_{k+1}W_k = \frac{1}{8}(-W_{n+3}^2 - 4W_{n+2}^2 - 4W_{n+1}^2 + 4W_{n+2}W_{n+3} + 4W_{n+1}W_{n+3} + W_2^2 + 4W_1^2 + 4W_0^2 - 4W_1W_2 - 4W_0W_2)$.

(c): $\sum_{k=0}^{n} W_{k+2}W_k = \frac{1}{8}(-3W_{n+3}^2 - 12W_{n+2}^2 - 12W_{n+1}^2 + 12W_{n+2}W_{n+3} + 4W_{n+1}W_{n+3} + 8W_{n+1}W_{n+2} + 3W_2^2 + 12W_1^2 + 12W_0^2 - 12W_1W_2 - 4W_0W_2 - 8W_0W_1)$.

Proof.

(a): Set $r=1, s=0, t=2,$ and $z=1$ in [42, Theorem 2.1 (a) (i)].

(b): Set $r=1, s=0, t=2,$ and $z=1$ in [42, Theorem 2.1 (b) (i)].

(c): Set $r=1, s=0, t=2,$ and $z=1$ in [42, Theorem 2.1 (c) (i)]. □

From the last Theorem, we have the following Corollary which gives sum formulas of Jacobsthal-Narayana numbers (take $W_n = B_n$ with $B_0 = 0, B_1 = 1, B_2 = 1$).

COROLLARY 20. *For $n \geq 0$, Jacobsthal-Narayana numbers have the following properties.*

(a): $\sum_{k=0}^{n} B_k^2 = \frac{1}{8}(5B_{n+3}^2 + 12B_{n+2}^2 + 12B_{n+1}^2 - 12B_{n+2}B_{n+3} - 4B_{n+1}B_{n+3} - 8B_{n+1}B_{n+2} - 5)$.

(b): $\sum_{k=0}^{n} B_{k+1}B_k = \frac{1}{8}(-B_{n+3}^2 - 4B_{n+2}^2 - 4B_{n+1}^2 + 4B_{n+2}B_{n+3} + 4B_{n+1}B_{n+3} + 1)$.

(c): $\sum_{k=0}^{n} B_{k+2}B_k = \frac{1}{8}(-3B_{n+3}^2 - 12B_{n+2}^2 - 12B_{n+1}^2 + 12B_{n+2}B_{n+3} + 4B_{n+1}B_{n+3} + 8B_{n+1}B_{n+2} + 3)$.

Taking $W_n = C_n$ with $C_0 = 3, C_1 = 1, C_2 = 1$ in the last Theorem, we have the following Corollary which gives sum formulas of Jacobsthal-Narayana-Lucas numbers.

COROLLARY 21. *For $n \geq 0$, Jacobsthal-Narayana-Lucas numbers have the following properties:*

(a): $\sum_{k=0}^{n} C_k^2 = \frac{1}{8}(5C_{n+3}^2 + 12C_{n+2}^2 + 12C_{n+1}^2 - 12C_{n+2}C_{n+3} - 4C_{n+1}C_{n+3} - 8C_{n+1}C_{n+2} - 77)$.

(b): $\sum_{k=0}^{n} C_{k+1}C_k = \frac{1}{8}(-C_{n+3}^2 - 4C_{n+2}^2 - 4C_{n+1}^2 + 4C_{n+2}C_{n+3} + 4C_{n+1}C_{n+3} + 25)$.

(c): $\sum\limits_{k=0}^{n} C_{k+2}C_k = \frac{1}{8}(-3C_{n+3}^2 - 12C_{n+2}^2 - 12C_{n+1}^2 + 12C_{n+2}C_{n+3} + 4C_{n+1}C_{n+3} + 8C_{n+1}C_{n+2} + 75).$

Next, we give the ordinary generating functions $\sum\limits_{n=0}^{\infty} W_n^2 z^n$, $\sum\limits_{n=0}^{\infty} W_{n+1}W_n z^n$, $\sum\limits_{n=0}^{\infty} W_{n+2}W_n z^n$ of the sequences $\{W_n^2\}$, $\{W_{n+1}W_n\}$, $\{W_{n+2}W_n\}$.

THEOREM 22. *Assume that* $|z| < \min\{|\alpha|^{-2}, |\beta|^{-2}, |\gamma|^{-2}, |\alpha\beta|^{-1}, |\alpha\gamma|^{-1}, |\beta\gamma|^{-1}\} = |\alpha|^{-2} \simeq 0.34781$. *Then the ordinary generating functions of the sequences* $\{W_n^2\}$, $\{W_{n+1}W_n\}$, $\{W_{n+2}W_n\}$ *are given as follows:*

(a): $\sum\limits_{n=0}^{\infty} W_n^2 z^n = \frac{1}{-16z^6 - 8z^5 + 4z^4 + 10z^3 + 2z^2 + z - 1}(-W_0^2 + (W_0^2 - W_1^2)z + (2W_0^2 + W_1^2 - W_2^2)z^2 + (6W_0^2 - 4W_2W_0 + 2W_1^2)z^3 + 2(2W_0^2 - 4W_0W_1 + 3W_1^2 - 2W_1W_2 + W_2^2)z^4 + 4(W_1 - W_2)^2 z^5).$

(b): $\sum\limits_{n=0}^{\infty} W_{n+1}W_n z^n = \frac{1}{-16z^6 - 8z^5 + 4z^4 + 10z^3 + 2z^2 + z - 1}(-W_0W_1 + W_1(W_0 - W_2)z + (2W_0W_1 - W_2^2 - 2W_0W_2 + W_1W_2)z^2 + (6W_0W_1 - 4W_0^2 - 2W_0W_2)z^3 + 4W_1(W_2 - W_1)z^4 + 8W_0(W_2 - W_1)z^5).$

(c): $\sum\limits_{n=0}^{\infty} W_{n+2}W_n z^n = \frac{1}{-16z^6 - 8z^5 + 4z^4 + 10z^3 + 2z^2 + z - 1}(-W_0W_2 - (2W_0W_1 - W_0W_2 + W_1W_2)z - (W_2^2 + W_1W_2 - 2W_0W_1)z^2 + (-4W_0^2 + 4W_0W_2 - 2W_2^2 + 2W_1W_2)z^3 + 2(-4W_0^2 + 4W_0W_1 - 2W_1^2 + 2W_2W_1)z^4 + 8W_1(W_2 - W_1)z^5).$

Proof. Take $r = 1, s = 0, t = 2$ in [42, Theorem 3.1]. □

Now, we consider special cases of the last Theorem.

COROLLARY 23. *Assume that* $|z| < \min\{|\alpha|^{-2}, |\beta|^{-2}, |\gamma|^{-2}, |\alpha\beta|^{-1}, |\alpha\gamma|^{-1}, |\beta\gamma|^{-1}\} = |\alpha|^{-2} \simeq 0.34781$. *The ordinary generating functions of the sequences* $\{B_n^2\}$, $\{B_{n+1}B_n\}$, $\{B_{n+2}B_n\}$ *and* $\{C_n^2\}$, $\{C_{n+1}C_n\}$, $\{C_{n+2}C_n\}$ *are given as follows:*

(a):

$$\sum_{n=0}^{\infty} B_n^2 z^n = \frac{4z^4 + 2z^3 - z}{-16z^6 - 8z^5 + 4z^4 + 10z^3 + 2z^2 + z - 1},$$
$$\sum_{n=0}^{\infty} C_n^2 z^n = \frac{16z^4 + 44z^3 + 18z^2 + 8z - 9}{-16z^6 - 8z^5 + 4z^4 + 10z^3 + 2z^2 + z - 1}.$$

(b):

$$\sum_{n=0}^{\infty} B_{n+1}B_n z^n = \frac{-z}{-16z^6 - 8z^5 + 4z^4 + 10z^3 + 2z^2 + z - 1},$$
$$\sum_{n=0}^{\infty} C_{n+1}C_n z^n = \frac{-24z^3 + 2z - 3}{-16z^6 - 8z^5 + 4z^4 + 10z^3 + 2z^2 + z - 1}.$$

(c):

$$\sum_{n=0}^{\infty} B_{n+2}B_n z^n = \frac{-2z^2 - z}{-16z^6 - 8z^5 + 4z^4 + 10z^3 + 2z^2 + z - 1},$$
$$\sum_{n=0}^{\infty} C_{n+2}C_n z^n = \frac{-48z^4 - 24z^3 + 4z^2 - 4z - 3}{-16z^6 - 8z^5 + 4z^4 + 10z^3 + 2z^2 + z - 1}.$$

From the last corollary, we obtain the following results for special cases of generalized Jacobsthal-Narayana numbers.

Corollary 24. *Some infinite sums of* $\{B_n^2\}$, $\{B_{n+1}B_n\}$, $\{B_{n+2}B_n\}$ *and* $\{C_n^2\}$, $\{C_{n+1}C_n\}$, $\{C_{n+2}C_n\}$ *are given as follows:*

(a): $z = \frac{1}{4}$.

$$\sum_{n=0}^{\infty} \frac{B_n^2}{4^n} = \frac{52}{119},$$
$$\sum_{n=0}^{\infty} \frac{C_n^2}{4^n} = \frac{1312}{119}.$$

(b): $z = \frac{1}{4}$.

$$\sum_{n=0}^{\infty} \frac{B_{n+1}B_n}{4^n} = \frac{64}{119},$$
$$\sum_{n=0}^{\infty} \frac{C_{n+1}C_n}{4^n} = \frac{736}{119}.$$

(c): $z = \frac{1}{4}$.

$$\sum_{n=0}^{\infty} \frac{B_{n+2}B_n}{4^n} = \frac{96}{119},$$
$$\sum_{n=0}^{\infty} \frac{C_{n+2}C_n}{4^n} = \frac{1104}{119}.$$

2.6. Generalized Jacobsthal-Narayana Numbers by Matrix Methods. In this section, we present matrix representations of the sequences W_n, B_n and C_n. We also introduce Simson matrix and investigate its properties.

2.6.1. *Matrix Representations of the Sequences* W_n, B_n *and* C_n. We define the square matrix A of order 3 as:

$$A = \begin{pmatrix} 1 & 0 & 2 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

and such that $\det A = 2$. Some properties of matrix A^n can be given as

$$A^n = A^{n-1} + 2A^{n-3},$$
$$A^{n+m} = A^n A^m = A^m A^n,$$

for all integers m and n. Note that we have the following formulas:

$$\begin{pmatrix} W_{n+2} \\ W_{n+1} \\ W_n \end{pmatrix} = \begin{pmatrix} 1 & 0 & 2 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} W_{n+1} \\ W_n \\ W_{n-1} \end{pmatrix},$$

and

$$\begin{pmatrix} W_{n+2} \\ W_{n+1} \\ W_n \end{pmatrix} = \begin{pmatrix} 1 & 0 & 2 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^n \begin{pmatrix} W_2 \\ W_1 \\ W_0 \end{pmatrix},$$

and

$$\begin{pmatrix} B_{n+2} \\ B_{n+1} \\ B_n \end{pmatrix} = \begin{pmatrix} 1 & 0 & 2 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} B_{n+1} \\ B_n \\ B_{n-1} \end{pmatrix}.$$

We also define

$$E_n = \begin{pmatrix} B_{n+1} & 2B_{n-1} & 2B_n \\ B_n & 2B_{n-2} & 2B_{n-1} \\ B_{n-1} & 2B_{n-3} & 2B_{n-2} \end{pmatrix}$$

and

$$D_n = \begin{pmatrix} W_{n+1} & 2W_{n-1} & 2W_n \\ W_n & 2W_{n-2} & 2W_{n-1} \\ W_{n-1} & 2W_{n-3} & 2W_{n-2} \end{pmatrix}.$$

THEOREM 25. *For all integers m, n, we have the following properties:*

(a): $E_n = A^n$, *i.e.,*

$$\begin{pmatrix} 1 & 0 & 2 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^n = \begin{pmatrix} B_{n+1} & 2B_{n-1} & 2B_n \\ B_n & 2B_{n-2} & 2B_{n-1} \\ B_{n-1} & 2B_{n-3} & 2B_{n-2} \end{pmatrix}.$$

(b): $D_1 A^n = A^n D_1$.

(c): $D_{n+m}=D_nE_m=E_mD_n$, *i.e.,*

$$\begin{pmatrix} W_{n+m+1} & 2W_{n+m-1} & 2W_{n+m} \\ W_{n+m} & 2W_{n+m-2} & 2W_{n+m-1} \\ W_{n+m-1} & 2W_{n+m-3} & 2W_{n+m-2} \end{pmatrix}$$

$$= \begin{pmatrix} W_{n+1} & 2W_{n-1} & 2W_{n} \\ W_{n} & 2W_{n-2} & 2W_{n-1} \\ W_{n-1} & 2W_{n-3} & 2W_{n-2} \end{pmatrix} \begin{pmatrix} B_{m+1} & 2B_{m-1} & 2B_{m} \\ B_{m} & 2B_{m-2} & 2B_{m-1} \\ B_{m-1} & 2B_{m-3} & 2B_{m-2} \end{pmatrix}$$

$$= \begin{pmatrix} B_{m+1} & 2B_{m-1} & 2B_{m} \\ B_{m} & 2B_{m-2} & 2B_{m-1} \\ B_{m-1} & 2B_{m-3} & 2B_{m-2} \end{pmatrix} \begin{pmatrix} W_{n+1} & 2W_{n-1} & 2W_{n} \\ W_{n} & 2W_{n-2} & 2W_{n-1} \\ W_{n-1} & 2W_{n-3} & 2W_{n-2} \end{pmatrix}.$$

(d):

$$A^n = B_{n-1}A^2 + 2B_{n-3}A + 2B_{n-2}I$$

i.e.,

$$A^n = \frac{1}{2}((B_{n+2}-B_{n+1})A^2 + (-B_{n+2}+B_{n+1}+2B_n)A + (2B_{n+1}-2B_n)I)$$

that is,

$$A^n = \frac{1}{2}(B_{n+2}(A^2-A) + B_{n+1}(-A^2+A+2I) + B_n(2A-2I))$$

where

$$I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Proof. Set $G_n = B_n$, and $r=1$, $s=0$, $t=2$ in [39, Theorem 51.]. □

Next, we present matrix formulas for the generalized Jacobsthal-Narayana and Jacobsthal-Narayana-Lucas numbers numbers.

Corollary 26. *For all integers n, we have the following formulas for generalized Jacobsthal-Narayana and Jacobsthal-Narayana-Lucas numbers.*

(a): *Generalized Jacobsthal-Narayana numbers.*

$$\begin{pmatrix} 1 & 0 & 2 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^n = \frac{1}{\Lambda_W(0)} \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

where

$$a_{11} = (W_1^2 + 2W_0^2 - W_1W_2)W_{n+3} + (W_2^2 - W_1W_2 - 2W_0W_1)W_{n+2} + (2W_1^2 - 2W_0W_2)W_{n+1}$$
$$a_{21} = (W_1^2 + 2W_0^2 - W_1W_2)W_{n+2} + (W_2^2 - W_1W_2 - 2W_0W_1)W_{n+1} + (2W_1^2 - 2W_0W_2)W_{n}$$
$$a_{31} = (W_1^2 + 2W_0^2 - W_1W_2)W_{n+1} + (W_2^2 - W_1W_2 - 2W_0W_1)W_{n} + (2W_1^2 - 2W_0W_2)W_{n-1}$$

$$a_{12} = 2((W_1^2 + 2W_0^2 - W_1W_2)W_{n+1} + (W_2^2 - W_1W_2 - 2W_0W_1)W_n + 2(W_1^2 - W_0W_2)W_{n-1})$$
$$a_{22} = 2((W_1^2 + 2W_0^2 - W_1W_2)W_n + (W_2^2 - W_1W_2 - 2W_0W_1)W_{n-1} + 2(W_1^2 - W_0W_2)W_{n-2})$$
$$a_{32} = 2((W_1^2 + 2W_0^2 - W_1W_2)W_{n-1} + (W_2^2 - W_1W_2 - 2W_0W_1)W_{n-2} + 2(W_1^2 - W_0W_2)W_{n-3})$$
$$a_{13} = 2((W_1^2 + 2W_0^2 - W_1W_2)W_{n+2} + (W_2^2 - W_1W_2 - 2W_0W_1)W_{n+1} + 2(W_1^2 - W_0W_2)W_n)$$
$$a_{23} = 2((W_1^2 + 2W_0^2 - W_1W_2)W_{n+1} + (W_2^2 - W_1W_2 - 2W_0W_1)W_n + 2(W_1^2 - W_0W_2)W_{n-1})$$
$$a_{33} = 2((W_1^2 + 2W_0^2 - W_1W_2)W_n + (W_2^2 - W_1W_2 - 2W_0W_1)W_{n-1} + 2(W_1^2 - W_0W_2)W_{n-2})$$

and

$$\Lambda_W(0) = W_2^3 + 2W_1^3 + 4W_0^3 - 2W_1W_2^2 + W_2W_1^2 + 2W_0W_1^2 + 2W_0^2W_2 - 6W_2W_1W_0.$$

(b): *Jacobsthal-Narayana-Lucas numbers.*

$$\begin{pmatrix} 1 & 0 & 2 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^n = \frac{1}{116}\begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix}$$

where

$$b_{11} = 18C_{n+3} - 6C_{n+2} - 4C_{n+1}$$
$$b_{21} = 18C_{n+2} - 6C_{n+1} - 4C_n$$
$$b_{31} = 18C_{n+1} - 4C_{n-1} - 6C_n$$
$$b_{12} = 36C_{n+1} - 12C_n - 8C_{n-1}$$
$$b_{22} = 36C_n - 12C_{n-1} - 8C_{n-2}$$
$$b_{32} = 36C_{n-1} - 12C_{n-2} - 8C_{n-3}$$
$$b_{13} = 36C_{n+2} - 12C_{n+1} - 8C_n$$
$$b_{23} = 36C_{n+1} - 12C_n - 8C_{n-1}$$
$$b_{33} = 36C_n - 12C_{n-1} - 8C_{n-2}$$

Proof. Set $r = 1, s = 0, t = 2$ and then take $W_n = C_n$ respectively, in [39, Corollary 52.]. □

Now, we present an identity for W_{n+m}.

THEOREM 27. *(Honsberger's Identity) For all integers m and n, we have*

$$W_{n+m} = W_nB_{m+1} + 2W_{n-1}B_{m-1} + 2W_{n-2}B_m$$

Proof. Set $G_n = B_n$ and $r = 1, s = 0, t = 2$ in [39, Theorem 53.]. □

As special cases of the last Theorem, we have the following corollary.

COROLLARY 28. *For all integers m, n, we have the following properties:*

$$B_{n+m} = B_nB_{m+1} + 2B_{n-1}B_{m-1} + 2B_{n-2}B_m$$
$$C_{n+m} = C_nB_{m+1} + 2C_{n-1}B_{m-1} + 2C_{n-2}B_m$$

Next, we present identies for W_{mn+j} and its special cases.

COROLLARY 29. *For all integers m, n, j, we have the following properties:*

$$\begin{array}{rcl} W_{mn+j} & = & B_{mn-1}W_{j+2}+2B_{mn-3}W_{j+1}+2B_{mn-2}W_{j} \\ B_{mn+j} & = & B_{mn-1}B_{j+2}+2B_{mn-3}B_{j+1}+2B_{mn-2}B_{j} \\ C_{mn+j} & = & B_{mn-1}C_{j+2}+2B_{mn-3}C_{j+1}+2B_{mn-2}C_{j} \end{array}$$

is true

Proof. Set $r=1, s=0, t=2$ and then take $W_n=B_n$, $W_n=C_n$, respectively, in [39, Corollary 55.]. □

2.6.2. *Simson Matrix and its Properties.* For $n\in\mathbb{Z}$, we define

$$f_W(n)=\begin{pmatrix} W_{n+2} & W_{n+1} & W_{n} \\ W_{n+1} & W_{n} & W_{n-1} \\ W_{n} & W_{n-1} & W_{n-2} \end{pmatrix}.$$

We call this matrix as Simson matrix of the sequence W_n. Similarly, as special cases of W_n, Simson matrices of the sequences B_n and C_n are

$$f_B(n)=\begin{pmatrix} B_{n+2} & B_{n+1} & B_{n} \\ B_{n+1} & B_{n} & B_{n-1} \\ B_{n} & B_{n-1} & B_{n-2} \end{pmatrix} \text{ and } f_C(n)=\begin{pmatrix} C_{n+2} & C_{n+1} & C_{n} \\ C_{n+1} & C_{n} & C_{n-1} \\ C_{n} & C_{n-1} & C_{n-2} \end{pmatrix}$$

respectively.

LEMMA 30. *For all integers n, m and j, the followings hold.*

(a): $f_W(n)=f_W(n-1)+2f_W(n-3)$.

(b): $f_W(n)=Af_W(n-1)$ *and* $f_W(n)=A^nf_W(0)$, *i.e.,*

$$\begin{pmatrix} W_{n+2} & W_{n+1} & W_{n} \\ W_{n+1} & W_{n} & W_{n-1} \\ W_{n} & W_{n-1} & W_{n-2} \end{pmatrix}=\begin{pmatrix} 1 & 0 & 2 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} W_{n+1} & W_{n} & W_{n-1} \\ W_{n} & W_{n-1} & W_{n-2} \\ W_{n-1} & W_{n-2} & W_{n-3} \end{pmatrix}$$

and

$$\begin{pmatrix} W_{n+2} & W_{n+1} & W_{n} \\ W_{n+1} & W_{n} & W_{n-1} \\ W_{n} & W_{n-1} & W_{n-2} \end{pmatrix}=\begin{pmatrix} 1 & 0 & 2 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^n\begin{pmatrix} W_{2} & W_{1} & W_{0} \\ W_{1} & W_{0} & W_{-1} \\ W_{0} & W_{-1} & W_{-2} \end{pmatrix}.$$

(c): $f_W(n+m)=A^nf_W(m)$ *and* $f_W(n+m)=A^mf_W(n)$ *i.e.,*

$$\begin{pmatrix} W_{n+m+2} & W_{n+m+1} & W_{n+m} \\ W_{n+m+1} & W_{n+m} & W_{n+m-1} \\ W_{n+m} & W_{n+m-1} & W_{n+m-2} \end{pmatrix}=\begin{pmatrix} 1 & 0 & 2 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^n\begin{pmatrix} W_{m+2} & W_{m+1} & W_{m} \\ W_{m+1} & W_{m} & W_{m-1} \\ W_{m} & W_{m-1} & W_{m-2} \end{pmatrix},$$

and

$$\begin{pmatrix} W_{m+n+2} & W_{m+n+1} & W_{m+n} \\ W_{m+n+1} & W_{m+n} & W_{m+n-1} \\ W_{m+n} & W_{m+n-1} & W_{m+n-2} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 2 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^m \begin{pmatrix} W_{n+2} & W_{n+1} & W_n \\ W_{n+1} & W_n & W_{n-1} \\ W_n & W_{n-1} & W_{n-2} \end{pmatrix},$$

and $f_W(n) = A^m f_W(n-m)$, *i.e.,*

$$\begin{pmatrix} W_{n+2} & W_{n+1} & W_n \\ W_{n+1} & W_n & W_{n-1} \\ W_n & W_{n-1} & W_{n-2} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 2 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^m \begin{pmatrix} W_{n-m+2} & W_{n-m+1} & W_{n-m} \\ W_{n-m+1} & W_{n-m} & W_{n-m-1} \\ W_{n-m} & W_{n-m-1} & W_{n-m-2} \end{pmatrix}.$$

Proof. Set $r = 1$, $s = 0$, $t = 2$ in [39, Lemma 56.]. □

Taking the determinant of both sides of the identities given in the last Lemma, we obtain the following Theorem.

Theorem 31. *For all integers n and m, the following identities hold.*

(a): *Catalan's Identity:*

$$\det(f_W(n+m)) = 2^n \det(f_W(m)) \quad \text{and} \quad \det(f_W(n)) = 2^m \det(f_W(n-m)),$$

i.e.,

$$\begin{vmatrix} W_{n+m+2} & W_{n+m+1} & W_{n+m} \\ W_{n+m+1} & W_{n+m} & W_{n+m-1} \\ W_{n+m} & W_{n+m-1} & W_{n+m-2} \end{vmatrix} = 2^n \begin{vmatrix} W_{m+2} & W_{m+1} & W_m \\ W_{m+1} & W_m & W_{m-1} \\ W_m & W_{m-1} & W_{m-2} \end{vmatrix},$$

and

$$\begin{vmatrix} W_{n+2} & W_{n+1} & W_n \\ W_{n+1} & W_n & W_{n-1} \\ W_n & W_{n-1} & W_{n-2} \end{vmatrix} = 2^m \begin{vmatrix} W_{n-m+2} & W_{n-m+1} & W_{n-m} \\ W_{n-m+1} & W_{n-m} & W_{n-m-1} \\ W_{n-m} & W_{n-m-1} & W_{n-m-2} \end{vmatrix}.$$

(b): *(see Theorem 9) Simson's (or Cassini's) Identity:*

$$\det(f_W(n)) = 2^n \det(f_W(0)),$$

i.e.,

$$\begin{vmatrix} W_{n+2} & W_{n+1} & W_n \\ W_{n+1} & W_n & W_{n-1} \\ W_n & W_{n-1} & W_{n-2} \end{vmatrix} = 2^n \begin{vmatrix} W_2 & W_1 & W_0 \\ W_1 & W_0 & W_{-1} \\ W_0 & W_{-1} & W_{-2} \end{vmatrix}.$$

Proof. Set $r = 1$, $s = 0$, $t = 2$ in [39, Theorem 57.]. □

From the last Theorem, we have the following Corollary which gives determinantal formulas of Jacobsthal-Narayana numbers (take $W_n = B_n$ with $B_0 = 0, B_1 = 1, B_2 = 1$).

Corollary 32. *For all integers n and m, the following identities hold.*

(a): *Catalan's Identity:*

$$\det(f_B(n+m)) = 2^n \det(f_B(m)) \quad \textit{and} \quad \det(f_B(n)) = 2^m \det(f_B(n-m)),$$

i.e.,

$$\begin{vmatrix} B_{n+m+2} & B_{n+m+1} & B_{n+m} \\ B_{n+m+1} & B_{n+m} & B_{n+m-1} \\ B_{n+m} & B_{n+m-1} & B_{n+m-2} \end{vmatrix} = 2^n \begin{vmatrix} B_{m+2} & B_{m+1} & B_m \\ B_{m+1} & B_m & B_{m-1} \\ B_m & B_{m-1} & B_{m-2} \end{vmatrix},$$

and

$$\begin{vmatrix} B_{n+2} & B_{n+1} & B_n \\ B_{n+1} & B_n & B_{n-1} \\ B_n & B_{n-1} & B_{n-2} \end{vmatrix} = 2^m \begin{vmatrix} B_{n-m+2} & B_{n-m+1} & B_{n-m} \\ B_{n-m+1} & B_{n-m} & B_{n-m-1} \\ B_{n-m} & B_{n-m-1} & B_{n-m-2} \end{vmatrix}.$$

(b): *Simson's (or Cassini's) Identity:*

$$\det(f_B(n)) = 2^n \det(f_B(0)),$$

i.e.,

$$\begin{vmatrix} B_{n+2} & B_{n+1} & B_n \\ B_{n+1} & B_n & B_{n-1} \\ B_n & B_{n-1} & B_{n-2} \end{vmatrix} = -2^{n-1}.$$

Taking $W_n = C_n$ with $C_0 = 3, C_1 = 1, C_2 = 1$ in the last Theorem, we have the following Corollary which gives determinantal formulas of Jacobsthal-Narayana-Lucas numbers.

Corollary 33. *For all integers n and m, the following identities hold.*

(a): *Catalan's Identity:*

$$\det(f_C(n+m)) = 2^n \det(f_C(m)) \quad \textit{and} \quad \det(f_C(n)) = 2^m \det(f_C(n-m))$$

i.e.,

$$\begin{vmatrix} C_{n+m+2} & C_{n+m+1} & C_{n+m} \\ C_{n+m+1} & C_{n+m} & C_{n+m-1} \\ C_{n+m} & C_{n+m-1} & C_{n+m-2} \end{vmatrix} = 2^n \begin{vmatrix} C_{m+2} & C_{m+1} & C_m \\ C_{m+1} & C_m & C_{m-1} \\ C_m & C_{m-1} & C_{m-2} \end{vmatrix},$$

and

$$\begin{vmatrix} C_{n+2} & C_{n+1} & C_n \\ C_{n+1} & C_n & C_{n-1} \\ C_n & C_{n-1} & C_{n-2} \end{vmatrix} = 2^m \begin{vmatrix} C_{n-m+2} & C_{n-m+1} & C_{n-m} \\ C_{n-m+1} & C_{n-m} & C_{n-m-1} \\ C_{n-m} & C_{n-m-1} & C_{n-m-2} \end{vmatrix}.$$

(b): *Simson's (or Cassini's) Identity:*

$$\det(f_C(n)) = 2^n \det(f_C(0)),$$

i.e.,

$$\begin{vmatrix} C_{n+2} & C_{n+1} & C_n \\ C_{n+1} & C_n & C_{n-1} \\ C_n & C_{n-1} & C_{n-2} \end{vmatrix} = -29 \times 2^n.$$

3. Generalized co-Jacobsthal-Narayana Numbers

If $r = 1$, $s = 0$, $t = 2$, then we get $r_1 = -s = 0$, $s_1 = -rt = -2$, $t_1 = t^2 = 4$. From now on, throughout the paper the chapter, we also use the notation $r = 0$, $s = -2$, $t = 4$ for $r_1 = 0$, $s_1 = -2$, $t_1 = 4$ and we consider the case $r = 0$, $s = -2$, $t = 4$ to use results in the paper [39].

In this section, we define and investigate a new sequence and its two special cases, namely the generalized co-Jacobsthal-Narayana, co-Jacobsthal-Narayana and co-Jacobsthal-Narayana-Lucas numbers. The generalized co-Jacobsthal-Narayana numbers

$$\{Y_n(Y_0, Y_1, Y_2; 0, -2, 4)\}_{n\geq 0}$$

(or shortly $\{Y_n\}_{n\geq 0}$) is defined as follows:

$$Y_n = -2Y_{n-2} + 4Y_{n-3}, \qquad Y_0 = d, Y_1 = e, Y_2 = f, \quad n \geq 3 \tag{3.1}$$

where Y_0, Y_1, Y_2 are arbitrary complex (or real) numbers with real coefficients.

The sequence $\{Y_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$Y_{-n} = \frac{1}{2}Y_{-(n-1)} + \frac{1}{4}Y_{-(n-3)}$$

for $n = 1, 2, 3, \ldots$ when $t \neq 0$. Therefore, recurrence (3.1) holds for all integer n.

The first few generalized co-Jacobsthal-Narayana numbers with positive subscript and negative subscript are given in the following Table 3.

Table 3. A few generalized co-Jacobsthal-Narayana numbers

n	Y_n	Y_{-n}
0	Y_0	Y_0
1	Y_1	$\frac{1}{2}Y_0+\frac{1}{4}Y_2$
2	Y_2	$\frac{1}{4}Y_0+\frac{1}{4}Y_1+\frac{1}{8}Y_2$
3	$4Y_0-2Y_1$	$\frac{3}{8}Y_0+\frac{1}{8}Y_1+\frac{1}{16}Y_2$
4	$4Y_1-2Y_2$	$\frac{5}{16}Y_0+\frac{1}{16}Y_1+\frac{3}{32}Y_2$
5	$4Y_1-8Y_0+4Y_2$	$\frac{7}{32}Y_0+\frac{3}{32}Y_1+\frac{5}{64}Y_2$
6	$16Y_0-16Y_1+4Y_2$	$\frac{13}{64}Y_0+\frac{5}{64}Y_1+\frac{7}{128}Y_2$
7	$16Y_0+8Y_1-16Y_2$	$\frac{23}{128}Y_0+\frac{7}{128}Y_1+\frac{13}{256}Y_2$
8	$48Y_1-64Y_0+8Y_2$	$\frac{37}{256}Y_0+\frac{13}{256}Y_1+\frac{23}{512}Y_2$
9	$32Y_0-80Y_1+48Y_2$	$\frac{63}{512}Y_0+\frac{23}{512}Y_1+\frac{37}{1024}Y_2$
10	$192Y_0-64Y_1-80Y_2$	$\frac{109}{1024}Y_0+\frac{37}{1024}Y_1+\frac{63}{2048}Y_2$
11	$352Y_1-320Y_0-64Y_2$	$\frac{183}{2048}Y_0+\frac{63}{2048}Y_1+\frac{109}{4096}Y_2$
12	$352Y_2-192Y_1-256Y_0$	$\frac{309}{4096}Y_0+\frac{109}{4096}Y_1+\frac{183}{8192}Y_2$
13	$1408Y_0-960Y_1-192Y_2$	$\frac{527}{8192}Y_0+\frac{183}{8192}Y_1+\frac{309}{16\,384}Y_2$

Remark 34. *In this paper we will extensively use the paper [39]. Note that in the notation of [39], here we have* $r=1,\ s=0,\ t=2$ *and* $r_1=0,\ s_1=-2,\ t_1=4$. *For simplicity, we can use the result of [39] by taking and replacing* $r=0,\ s=-2,\ t=4$.

As $\{Y_n\}$ is a third-order recurrence sequence (difference equation), it's characteristic equation (cubic equation) is

$$y^3+2y-4=0.$$

The roots $\theta_1,\theta_2,\theta_3$ of characteristic equation of $\{Y_n\}$ are given as

$$\begin{aligned}
\theta_1 &= \left(2+\sqrt{\frac{116}{27}}\right)^{1/3}-\left(-2+\sqrt{\frac{116}{27}}\right)^{1/3}, \\
\theta_2 &= \omega\left(2+\sqrt{\frac{116}{27}}\right)^{1/3}-\omega^2\left(-2+\sqrt{\frac{116}{27}}\right)^{1/3}, \\
\theta_3 &= \omega^2\left(2+\sqrt{\frac{116}{27}}\right)^{1/3}-\omega\left(-2+\sqrt{\frac{116}{27}}\right)^{1/3},
\end{aligned}$$

where

$$\omega=\frac{-1+i\sqrt{3}}{2}=\exp(2\pi i/3).$$

There are the following relations between the roots of characteristic equation:

$$\begin{cases}
\theta_1+\theta_2+\theta_3=0, \\
\theta_1\theta_2+\theta_1\theta_3+\theta_2\theta_3=2, \\
\theta_1\theta_2\theta_3=4.
\end{cases}$$

Note that there are an important relation between $\theta_1, \theta_2, \theta_3$ and α, β, γ:

$$\theta_1 = \beta\gamma,$$
$$\theta_2 = \alpha\beta,$$
$$\theta_3 = \alpha\gamma.$$

The sequence $\{Y_n\}$ can be expressed with Binet's formula. Using the roots of characteristic equation and the recurrence relation of Y_n, Binet's formula of Y_n can be given as follows:

THEOREM 35. *For all integers n, Binet's formula of generalized co-Jacobsthal-Narayana numbers is given as follows.*

$$\begin{aligned} Y_n &= \frac{p_1\theta_1^n}{(\theta_1-\theta_2)(\theta_1-\theta_3)} + \frac{p_2\theta_2^n}{(\theta_2-\theta_1)(\theta_2-\theta_3)} + \frac{p_3\theta_3^n}{(\theta_3-\theta_1)(\theta_3-\theta_2)} \\ &= A_1\theta_1^n + A_2\theta_2^n + A_3\theta_3^n, \end{aligned}$$

where

$$p_1 = Y_2 - (\theta_2+\theta_3)Y_1 + \theta_2\theta_3Y_0,\ p_2 = Y_2 - (\theta_1+\theta_3)Y_1 + \theta_1\theta_3Y_0,\ p_3 = Y_2 - (\theta_1+\theta_2)Y_1 + \theta_1\theta_2Y_0,$$

and

$$\begin{aligned} A_1 &= \frac{p_1}{(\theta_1-\theta_2)(\theta_1-\theta_3)} = \frac{Y_2-(\theta_2+\theta_3)Y_1+\theta_2\theta_3Y_0}{(\theta_1-\theta_2)(\theta_1-\theta_3)} \\ &= \frac{(\theta_1Y_2+\theta_1\theta_1Y_1+tY_0)}{-4\theta_1+12}, \\ A_2 &= \frac{p_2}{(\theta_2-\theta_1)(\theta_2-\theta_3)} = \frac{Y_2-(\theta_1+\theta_3)Y_1+\theta_1\theta_3Y_0}{(\theta_2-\theta_1)(\theta_2-\theta_3)} \\ &= \frac{(\theta_2Y_2+\theta_2\theta_2Y_1+tY_0)}{-4\theta_2+12}, \\ A_3 &= \frac{p_3}{(\theta_3-\theta_1)(\theta_3-\theta_2)} = \frac{Y_2-(\theta_1+\theta_2)Y_1+\theta_1\theta_2Y_0}{(\theta_3-\theta_1)(\theta_3-\theta_2)} \\ &= \frac{(\theta_3Y_2+\theta_3\theta_3Y_1+tY_0)}{-4\theta_3+12}. \end{aligned}$$

Proof. For the proof, take $r = 0$, $s = -2$, $t = 4$ in [39, Theorem 3 (a)] or $r = 0$, $s = -2$, $t = 4$ in [39, Theorem 19 (a)]. □

Next, we give the ordinary generating function $\sum_{n=0}^{\infty} Y_n z^n$ of the sequence Y_n.

LEMMA 36. *Suppose that $f_{Y_n}(z) = \sum_{n=0}^{\infty} Y_n z^n$ is the ordinary generating function of the generalized co-Jacobsthal-Narayana numbers $\{Y_n\}_{n\geq 0}$. Then, $\sum_{n=0}^{\infty} Y_n z^n$ is given by*

$$\sum_{n=0}^{\infty} Y_n z^n = \frac{Y_0 + Y_1 z + (Y_2 + 2Y_0)z^2}{1 + 2z^2 - 4z^3}.$$

Proof. Set $r = 0$, $s = -2$, $t = 4$ in [39, Lemma 9.] or $r = 0$, $s = -2$, $t = 4$ in [39, Lemma 24.]. □

In this paper, we define and investigate, in detail, two special cases of the generalized co-,Jacobsthal-Narayana numbers $\{Y_n\}$ which we call them co-,Jacobsthal-Narayana and co-,Jacobsthal-Narayana-Lucas numbers. co-,Jacobsthal-Narayana numbers $\{U_n\}_{n\geq 0}$ and co-Jacobsthal-Narayana-Lucas numbers $\{S_n\}_{n\geq 0}$ are defined, respectively, by the third-order recurrence relations

$$U_{n+3} = -2U_{n+2} + 4U_n, \qquad U_0 = 0, U_1 = 1, U_2 = 0, \tag{3.2}$$

$$S_{n+3} = -2S_{n+2} + 4S_n, \qquad S_0 = 3, S_1 = 0, S_2 = -4. \tag{3.3}$$

i.e.,

$$U_n = -2U_{n-2} + 4U_{n-3}, \qquad U_0 = 0, U_1 = 1, U_2 = 0,$$

$$S_n = -2S_{n-2} + 4S_{n-3}, \qquad S_0 = 3, S_1 = 0, S_2 = -4.$$

The sequences $\{U_n\}_{n\geq 0}$ and $\{S_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$U_{-n} = \frac{1}{2}U_{-(n-1)} + \frac{1}{4}U_{-(n-3)},$$

$$S_{-n} = \frac{1}{2}S_{-(n-1)} + \frac{1}{4}S_{-(n-3)},$$

for $n = 1, 2, 3, \ldots$ respectively. Therefore, recurrences (3.2) and (3.3) hold for all integers n.

Next, we present the first few values of the co-Jacobsthal-Narayana and co-Jacobsthal-Narayana-Lucas numbers with positive and negative subscripts.

Table 4. The first few values of the special third-order numbers with positive and negative subscripts.

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13
U_n	0	1	0	-2	4	4	-16	8	48	-80	-64	352	-192	-960
U_{-n}	0	0	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{16}$	$\frac{3}{32}$	$\frac{5}{64}$	$\frac{7}{128}$	$\frac{13}{256}$	$\frac{23}{512}$	$\frac{37}{1024}$	$\frac{63}{2048}$	$\frac{109}{4096}$	$\frac{183}{8192}$
S_n	3	0	-4	12	8	-40	32	112	-224	-96	896	-704	-2176	4992
S_{-n}	3	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{7}{8}$	$\frac{9}{16}$	$\frac{11}{32}$	$\frac{25}{64}$	$\frac{43}{128}$	$\frac{65}{256}$	$\frac{115}{512}$	$\frac{201}{1024}$	$\frac{331}{2048}$	$\frac{561}{4096}$	$\frac{963}{8192}$

For all integers n, Binet's formula of co-Jacobsthal-Narayana and co-Jacobsthal-Narayana-Lucas numbers (using initial conditions (3.2) and (3.3) in Theorem theo:smeako7) can be expressed as follows:

Theorem 37. *For all integers n, Binet's formulas of co-Jacobsthal-Narayana and co-Jacobsthal-Narayana-Lucas numbers are*

$$\begin{aligned} U_n &= \frac{\theta_1^{n+1}}{(\theta_1-\theta_2)(\theta_1-\theta_3)} + \frac{\theta_2^{n+1}}{(\theta_2-\theta_1)(\theta_2-\theta_3)} + \frac{\theta_3^{n+1}}{(\theta_3-\theta_1)(\theta_3-\theta_2)} \\ &= \frac{\theta_1^{n+2}}{-4\theta_1+12} + \frac{\theta_2^{n+2}}{-4\theta_2+12} + \frac{\theta_3^{n+2}}{-4\theta_3+12}, \end{aligned}$$

and

$$S_n = \theta_1^n + \theta_2^n + \theta_3^n,$$

respectively.

Lemma 36 gives the following results as particular examples (generating functions of co-Jacobsthal-Narayana and co-Jacobsthal-Narayana-Lucas numbers).

COROLLARY 38. *Generating functions of co-Jacobsthal-Narayana and co-Jacobsthal-Narayana-Lucas numbers are*

$$\sum_{n=0}^{\infty} U_n z^n = \frac{z}{1+2z^2-4z^3},$$
$$\sum_{n=0}^{\infty} S_n z^n = \frac{3+2z^2}{1+2z^2-4z^3},$$

respectively.

3.1. Connections between B_n, C_n and U_n, S_n. S_n can be given as follows.

LEMMA 39. *For all integers n, we have the following formula for S_n:*

$$\begin{aligned} S_n &= \theta_1^n + \theta_2^n + \theta_3^n \\ &= \alpha^n\beta^n + \alpha^n\gamma^n + \beta^n\gamma^n. \end{aligned}$$

Proof. Use [39, Lemma 30.]. □

We can present the relations between U_n, S_n and B_n, C_n as follows.

LEMMA 40. *For all integers n, we have the following formulas:*

- **(a):** $S_n = \frac{1}{2}(C_n^2 - C_{2n})$.
- **(b):** $U_n = 2^n B_{-n-1}$ *and* $U_{-n} = 2^{-n} B_{n-1}$.
- **(c):** $S_n = 2^n C_{-n}$ *and* $S_{-n} = 2^{-n} C_n$.

Proof. Use [39, Lemma 32.]. □

3.2. Some Identities of Generalized co-Jacobsthal-Narayana Numbers. In this section, we obtain some identities of generalized co-Jacobsthal-Narayana, co-Jacobsthal-Narayana and co-Jacobsthal-Narayana-Lucas numbers. First, we can give a few basic relations between $\{U_n\}$ and $\{S_n\}$.

LEMMA 41. *The following equalities are true:*

- **(a):** $16S_n = 14U_{n+4} + 4U_{n+3} + 36U_{n+2}$.
- **(b):** $4S_n = U_{n+3} + 2U_{n+2} + 14U_{n+1}$.
- **(c):** $2S_n = U_{n+2} + 6U_{n+1} + 2U_n$.
- **(d):** $S_n = 3U_{n+1} + 2U_{n-1}$.
- **(e):** $S_n = -4U_{n-1} + 12U_{n-2}$.
- **(f):** $116U_n = S_{n+4} + 3S_{n+3} + 11S_{n+2}$.
- **(g):** $116U_n = 3S_{n+3} + 9S_{n+2} + 4S_{n+1}$.
- **(h):** $116U_n = 9S_{n+2} - 2S_{n+1} + 12S_n$.

(i): $116U_n = -2S_{n+1} - 6S_n + 36S_{n-1}$.

(j): $116U_n = -6S_n + 40S_{n-1} - 8S_{n-2}$.

Proof. Set $G_n = U_n$, $H_n = S_n$ and $r = 0$, $s = -2$, $t = 4$ in [39, Lemma 36.]. □

Note that all the identities in the above lemma can be proved by induction as well.

Next, we give a few basic relations between $\{U_n\}$ and $\{Y_n\}$.

Lemma 42. *The following equalities are true:*

(a): $(Y_2^3+4Y_1^3+16Y_0^3+2Y_1^2Y_2+2Y_0Y_2^2+4Y_0Y_1^2-16Y_0^2Y_1-12Y_0Y_1Y_2)U_n = (4Y_0^2-Y_1Y_2-2Y_0Y_1)Y_{n+2}+(Y_2^2 + 2Y_0Y_2 - 4Y_0Y_1)Y_{n+1} + 4(Y_1^2 - Y_0Y_2)Y_n$.

(b): $(Y_2^3+4Y_1^3+16Y_0^3+2Y_1^2Y_2+2Y_0Y_2^2+4Y_0Y_1^2-16Y_0^2Y_1-12Y_0Y_1Y_2)U_n = (Y_2^2+2Y_0Y_2-4Y_0Y_1)Y_{n+1}+(4Y_1^2 - 8Y_0^2 + 2Y_1Y_2 - 4Y_0Y_2 + 4Y_0Y_1)Y_n + (16Y_0^2 - 4Y_1Y_2 - 8Y_0Y_1)Y_{n-1}$.

(c): $(Y_2^3 + 4Y_1^3 + 16Y_0^3 + 2Y_1^2Y_2 + 2Y_0Y_2^2 + 4Y_0Y_1^2 - 16Y_0^2Y_1 - 12Y_0Y_1Y_2)U_n = (4Y_1^2 - 8Y_0^2 + 2Y_1Y_2 - 4Y_0Y_2 + 4Y_0Y_1)Y_n + (-2Y_2^2 + 16Y_0^2 - 4Y_1Y_2 - 4Y_0Y_2)Y_{n-1} + (4Y_2^2 + 8Y_0Y_2 - 16Y_0Y_1)Y_{n-2}$.

(d): $4Y_n = (Y_2 + 2Y_0)U_{n+2} + 4Y_0U_{n+1} + (2Y_2 + 4Y_1 + 4Y_0)U_n$.

(e): $Y_n = Y_0U_{n+1} + Y_1U_n + (Y_2 + 2Y_0)U_{n-1}$.

(f): $Y_n = Y_1U_n + Y_2U_{n-1} + 4Y_0U_{n-2}$.

Proof. Set $W_n = Y_n$, $G_n = U_n$ and $r = 0$, $s = -2$, $t = 4$ in [39, Lemma 37.]. □

Now, we present a few basic relations between $\{S_n\}$ and $\{Y_n\}$.

Lemma 43. *The following equalities are true:*

(a): $(Y_2^3 + 4Y_1^3 + 16Y_0^3 + 2Y_1^2Y_2 + 2Y_0Y_2^2 + 4Y_0Y_1^2 - 16Y_0^2Y_1 - 12Y_0Y_1Y_2)S_n = (3Y_2^2 + 2Y_1^2 + 4Y_0Y_2 - 12Y_0Y_1)Y_{n+2} + (12Y_1^2 + 4Y_1Y_2 - 12Y_0Y_2 - 16Y_0^2 + 8Y_0Y_1)Y_{n+1} + (2Y_2^2 + 4Y_1^2 + 48Y_0^2 - 12Y_1Y_2 - 32Y_0Y_1)Y_n$.

(b): $(Y_2^3+4Y_1^3+16Y_0^3+2Y_1^2Y_2+2Y_0Y_2^2+4Y_0Y_1^2-16Y_0^2Y_1-12Y_0Y_1Y_2)S_n = (12Y_1^2+4Y_1Y_2-12Y_0Y_2-16Y_0^2+8Y_0Y_1)Y_{n+1}+(48Y_0^2-4Y_2^2-12Y_1Y_2-8Y_0Y_2-8Y_0Y_1)Y_n+(12Y_2^2+8Y_1^2+16Y_0Y_2-48Y_0Y_1)Y_{n-1}$.

(c): $(Y_2^3+4Y_1^3+16Y_0^3+2Y_1^2Y_2+2Y_0Y_2^2+4Y_0Y_1^2-16Y_0^2Y_1-12Y_0Y_1Y_2)S_n = (-4Y_2^2+48Y_0^2-12Y_1Y_2-8Y_0Y_2 - 8Y_0Y_1)Y_n + (12Y_2^2 - 16Y_1^2 + 32Y_0^2 - 8Y_1Y_2 + 40Y_0Y_2 - 64Y_0Y_1)Y_{n-1} + (48Y_1^2 - 64Y_0^2 + 16Y_1Y_2 - 48Y_0Y_2 + 32Y_0Y_1)Y_{n-2}$.

(d): $116Y_n = (3Y_2 + 9Y_1 + 4Y_0)S_{n+2} + (9Y_2 - 2Y_1 + 12Y_0)S_{n+1} + (4Y_2 + 12Y_1 + 44Y_0)S_n$.

(e): $116Y_n = (9Y_2 - 2Y_1 + 12Y_0)S_{n+1} + (-2Y_2 - 6Y_1 + 36Y_0)S_n + (12Y_2 + 36Y_1 + 16Y_0)S_{n-1}$.

(f): $58Y_n = (-Y_2 - 3Y_1 + 18Y_0)S_n + (-3Y_2 + 20Y_1 - 4Y_0)S_{n-1} + (18Y_2 - 4Y_1 + 24Y_0)S_{n-2}$.

Proof. Set $W_n = Y_n$, $H_n = S_n$, and $r = 0$, $s = -2$, $t = 4$ in [39, Lemma 38.]. □

We can present identities between B_n, C_n and U_n, S_n by using Lemmas given above.

Lemma 44. *For all integers n, we have the following formulas:*

(a): $S_{-n} = 2^{-n}(3B_{n+1} - 2B_n)$.

(b): $S_n = \frac{1}{2}((3B_{n+1} - 2B_n)^2 - (3B_{2n+1} - 2B_{2n}))$.

(c): $58U_{-n} = 2^{-n}(-C_{n+2} + 10C_{n+1} - 3C_n)$.

(d): $C_{-n} = 2^{-n}(3U_{n+1} + 2U_{n-1})$.

(e): $116B_{-n-1} = 2^{-n}(9S_{n+2} - 2S_{n+1} + 12S_n)$.

(f): $29B_{-n} = 2^{-n-1}(3S_{n+2} + 9S_{n+1} + 4S_n)$.

Prof. Use Lemmas 5, 40, 41. □

Now, we present some identities of generalized co-Jacobsthal-Narayana numbers and its special cases.

LEMMA 45. *Suppose that* $\{X_n\}_{n\geq 0} = \{X_n(X_0, X_1, X_2)\}_{n\geq 0}$ *is also defined by the third-order recurrence relations*

$$X_n = -2X_{n-2} + 4X_{n-3} \tag{3.4}$$

i.e.,

$$X_{n+3} = -2X_{n+1} + 4X_n$$

with the initial values X_0, X_1, X_2 *not all being zero and*

$$X_{-n} = \frac{1}{2}X_{-(n-1)} + \frac{1}{4}X_{-(n-3)}$$

so that (3.4) is true for all integer n.

Then the following equalities are true:

(a): $(X_0X_3^2 + X_1^2X_4 + X_2^3 - X_0X_2X_4 - 2X_1X_2X_3)Y_n = ((X_1^2 - X_0X_2)Y_2 + (X_0X_3 - X_1X_2)Y_1 + (X_2^2 - X_1X_3)Y_0)X_{n+2} + ((X_0X_3 - X_1X_2)Y_2 + (X_2^2 - X_0X_4)Y_1 + (X_1X_4 - X_2X_3)Y_0)X_{n+1} + ((X_2^2 - X_1X_3)Y_2 + (X_1X_4 - X_2X_3)Y_1 + (X_3^2 - X_2X_4)Y_0)X_n$.

(b): $(Y_0Y_3^2 + Y_1^2Y_4 + Y_2^3 - Y_0Y_2Y_4 - 2Y_1Y_2Y_3)U_n = (4Y_0^2 - Y_1Y_2 - 2Y_0Y_1)Y_{n+2} + (Y_2^2 + 2Y_0Y_2 - 4Y_0Y_1)Y_{n+1} + 4(Y_1^2 - Y_0Y_2)Y_n$.

(c): $4Y_n = (Y_2 + 2Y_0)U_{n+2} + (4Y_0)U_{n+1} + (2Y_2 + 4Y_1 + 4Y_0)U_n$.

(d): $(Y_0Y_3^2 + Y_1^2Y_4 + Y_2^3 - Y_0Y_2Y_4 - 2Y_1Y_2Y_3)S_n = (3Y_2^2 + 2Y_1^2 + 4Y_0Y_2 - 12Y_0Y_1)Y_{n+2} + (12Y_1^2 - 16Y_0^2 + 4Y_1Y_2 - 12Y_0Y_2 + 8Y_0Y_1)Y_{n+1} + (2Y_2^2 + 4Y_1^2 + 48Y_0^2 - 12Y_1Y_2 - 32Y_0Y_1)Y_n$.

(e): $116Y_n = (3Y_2 + 9Y_1 + 4Y_0)S_{n+2} + (9Y_2 - 2Y_1 + 12Y_0)S_{n+1} + (4Y_2 + 12Y_1 + 44Y_0)S_n$.

Proof.

(a): Writing

$$Y_n = q_1 \times X_{n+2} + q_2 \times X_{n+1} + q_3 \times X_n$$

and solving the system of equations

$$\begin{aligned}
Y_0 &= q_1 \times X_2 + q_2 \times X_1 + q_3 \times X_0 \\
Y_1 &= q_1 \times X_3 + q_2 \times X_2 + q_3 \times X_1 \\
Y_2 &= q_1 \times X_4 + q_2 \times X_3 + q_3 \times X_2
\end{aligned}$$

we find the required identity.

(b): Replace Y_n and X_n with U_n and Y_n, respectively in (a).

(c): Replace X_n with U_n in (a).

(d): Replace Y_n and X_n with S_n and Y_n, respectively in (a).

(e): Replace X_n with S_n in (a). □

3.3. Simson's Formulas of co-Jacobsthal-Narayana Numbers. The following theorem gives Simson's formula of the generalized co-Jacobsthal-Narayana numbers $\{Y_n\}$.

THEOREM 46 (Simson's Formula of Generalized co-Jacobsthal-Narayana Numbers). *For all integers n, we have*

$$\begin{vmatrix} Y_{n+2} & Y_{n+1} & Y_n \\ Y_{n+1} & Y_n & Y_{n-1} \\ Y_n & Y_{n-1} & Y_{n-2} \end{vmatrix} = 4^n \begin{vmatrix} Y_2 & Y_1 & Y_0 \\ Y_1 & Y_0 & Y_{-1} \\ Y_0 & Y_{-1} & Y_{-2} \end{vmatrix}$$

$$= 4^n \begin{vmatrix} Y_2 & Y_1 & Y_0 \\ Y_1 & Y_0 & \frac{1}{4}(Y_2+2Y_0) \\ Y_0 & \frac{1}{4}(Y_2+2Y_0) & \frac{1}{8}(Y_2+2Y_1+2Y_0) \end{vmatrix}$$

Proof. Set $W_n = Y_n$ and $r = 0$, $s = -2$, $t = 4$ in [39, Theorem 33.]. □

The previous theorem gives the following results as particular examples.

COROLLARY 47. *For all integers n, Simson's formula of co-Jacobsthal-Narayana and co-Jacobsthal-Narayana-Lucas numbers are given as*

$$\begin{vmatrix} U_{n+2} & U_{n+1} & U_n \\ U_{n+1} & U_n & U_{n-1} \\ U_n & U_{n-1} & U_{n-2} \end{vmatrix} = -4^{n-1},$$

$$\begin{vmatrix} S_{n+2} & S_{n+1} & S_n \\ S_{n+1} & S_n & S_{n-1} \\ S_n & S_{n-1} & S_{n-2} \end{vmatrix} = -29 \times 2^{2n},$$

respectively.

Proof. Set $Y_n = U_n$ and $Y_n = S_n$ in Theorem 46, respectively. □

3.4. Recurrence Properties of Generalized co-Jacobsthal-Narayana Numbers. The generalized co-Jacobsthal-Narayana numbers Y_n at negative indices can be expressed by the sequence itself at positive indices.

THEOREM 48. *For $n \in \mathbb{Z}$, we have*

$$Y_{-n} = 2^{-2n}(Y_{2n} - S_n Y_n + \frac{1}{2}(S_n^2 - S_{2n})Y_0).$$

Proof. Set $Y_n = Y_n$, $C_n = S_n$ and $r = 0$, $s = -2$, $t = 4$ in [39, Theorem 39.]. □

As special cases of the above Theorem 48, we have the following Corollary.

COROLLARY 49. *For $n \in \mathbb{Z}$, we have*

(a):
$$U_{-n} = -\frac{1}{2^{2n+1}}(2U_n^2 - 2U_{2n} + U_nU_{n+2} + 6U_nU_{n+1}).$$

(b):
$$S_{-n} = \frac{1}{2^{2n+1}}(S_n^2 - S_{2n}).$$

Proof. Take $r = 0$, $s = -2$, $t = 4$, and $G_n = U_n$ and $H_n = S_n$, respectively, in [39, Corollary 42.] or set $Y_n = U_n$ and $Y_n = S_n$, respectively, in Theorem 48. □

The last Corollary can be written in the following form by using Lemma 40.

COROLLARY 50. *For $n \in \mathbb{Z}$, we have*

(a):
$$B_{n-1} = -\frac{1}{2^{n+1}}(2U_n^2 - 2U_{2n} + U_nU_{n+2} + 6U_nU_{n+1}).$$

(b):
$$C_n = \frac{1}{2^{n+1}}(S_n^2 - S_{2n}).$$

Proof. Use Lemma 40 and Corollary 49. □

3.5. Sum Formulas $\sum_{k=0}^{n} Y_k$, $\sum_{k=0}^{n} Y_{2k}$, $\sum_{k=0}^{n} Y_{2k+1}$, $\sum_{k=0}^{n} Y_{-k}$, $\sum_{k=0}^{n} Y_{-2k}$, $\sum_{k=0}^{n} Y_{-2k+1}$ and Generating Functions $\sum_{n=0}^{\infty} Y_n z^n$, $\sum_{n=0}^{\infty} Y_{2n} z^n$, $\sum_{n=0}^{\infty} Y_{2n+1} z^n$, $\sum_{n=0}^{\infty} Y_{-n} z^n$, $\sum_{n=0}^{\infty} Y_{-2n} z^n$, $\sum_{n=0}^{\infty} Y_{-2n+1} z^n$ of Generalized co-Jacobsthal-Narayana Numbers. Next, we present sum formulas of generalized co-Jacobsthal-Narayana numbers

THEOREM 51. *For $n \geq 0$, we have the following sum formulas for generalized co-Jacobsthal-Narayana numbers:*

(a): $\sum_{k=0}^{n} Y_k = Y_{n+2} + Y_{n+1} + 4Y_n - Y_1 - Y_2 - 3Y_0.$

(b): $\sum_{k=0}^{n} Y_{2k} = \frac{1}{7}(3Y_{2n+2} + 4Y_{2n+1} + 16Y_{2n} - 4Y_1 - 3Y_2 - 9Y_0).$

(c): $\sum_{k=0}^{n} Y_{2k+1} = \frac{1}{7}(4Y_{2n+2} + 10Y_{2n+1} + 12Y_{2n} - 4Y_2 - 3Y_1 - 12Y_0).$

(d): $\sum_{k=0}^{n} Y_{-k} = -Y_{-n+2} - Y_{-n+1} - 3Y_{-n} + Y_2 + Y_1 + 4Y_0.$

(e): $\sum_{k=0}^{n} Y_{-2k} = \frac{1}{7}(-3Y_{-2n} - 4Y_{-2n-1} - 16Y_{-2n-2} + 3Y_2 + 4Y_1 + 16Y_0).$

(f): $\sum_{k=0}^{n} Y_{-2k+1} = \frac{2}{7}(-2Y_{-2n} - 5Y_{-2n-1} - 6Y_{-2n-2} + 2Y_2 + 5Y_1 + 6Y_0).$

Proof.

(a): Set $W_n = Y_n$, $r = 0, s = -2, t = 4$ and $z = 1$ in [39, Theorem 62 (a) (i)].

(b): Set $W_n = Y_n$, $r = 0, s = -2, t = 4$ and $z = 1$ in [39, Theorem 62 (b) (i)].

(c): Set $W_n = Y_n$, $r = 0, s = -2, t = 4$ and $z = 1$ in [39, Theorem 62 (c) (i)].

(d): Set $W_n = Y_n$, $r = 0, s = -2, t = 4$ and $z = 1$ in [39, Theorem 62 (d) (i)].

(e): Set $W_n = Y_n$, $r = 0, s = -2, t = 4$ and $z = 1$ in [39, Theorem 62 (e) (i)].

(f): Set $W_n = Y_n$, $r = 0, s = -2, t = 4$ and $z = 1$ in [39, Theorem 62 (f) (i)]. □

From the last Theorem, we have the following Corollary which gives sum formulas of co-Jacobsthal-Narayana numbers (take $Y_n = U_n$ with $U_0 = 0, U_1 = 1, U_2 = 0$).

Corollary 52. *For $n \geq 0$, co-Jacobsthal-Narayana numbers have the following properties.*

(a): $\sum_{k=0}^{n} U_k = U_{n+2} + U_{n+1} + 4U_n - 1.$

(b): $\sum_{k=0}^{n} U_{2k} = \frac{1}{7}(3U_{2n+2} + 4U_{2n+1} + 16U_{2n} - 4).$

(c): $\sum_{k=0}^{n} U_{2k+1} = \frac{1}{7}(4U_{2n+2} + 10U_{2n+1} + 12U_{2n} - 3).$

(d): $\sum_{k=0}^{n} U_{-k} = -U_{-n+2} - U_{-n+1} - 3U_{-n} + 1.$

(e): $\sum_{k=0}^{n} U_{-2k} = \frac{1}{7}(-3U_{-2n} - 4U_{-2n-1} - 16U_{-2n-2} + 4).$

(f): $\sum_{k=0}^{n} U_{-2k+1} = \frac{2}{7}(-2U_{-2n} - 5U_{-2n-1} - 6U_{-2n-2} + 5).$

Taking $Y_n = S_n$ with $S_0 = 3, S_1 = 0, S_2 = -4$ in the last Theorem, we have the following Corollary which gives sum formulas of co-Jacobsthal-Narayana-Lucas numbers.

Corollary 53. *For $n \geq 0$, co-Jacobsthal-Narayana-Lucas numbers have the following properties:*

(a): $\sum_{k=0}^{n} S_k = S_{n+2} + S_{n+1} + 4S_n - 5.$

(b): $\sum_{k=0}^{n} S_{2k} = \frac{1}{7}(3S_{2n+2} + 4S_{2n+1} + 16S_{2n} - 15).$

(c): $\sum_{k=0}^{n} S_{2k+1} = \frac{1}{7}(4S_{2n+2} + 10S_{2n+1} + 12S_{2n} - 20).$

(d): $\sum_{k=0}^{n} S_{-k} = -S_{-n+2} - S_{-n+1} - 3S_{-n} + 8.$

(e): $\sum_{k=0}^{n} S_{-2k} = \frac{1}{7}(-3S_{-2n} - 4S_{-2n-1} - 16S_{-2n-2} + 36).$

(f): $\sum_{k=0}^{n} S_{-2k+1} = \frac{2}{7}(-2S_{-2n} - 5S_{-2n-1} - 6S_{-2n-2} + 10).$

Next, we give the ordinary generating function of special cases of the generalized co-Jacobsthal-Narayana numbers $\{Y_{mn+j}\}$.

Corollary 54. *The ordinary generating functions of the sequences Y_n, Y_{2n}, Y_{2n+1}, Y_{-n}, Y_{-2n}, Y_{-2n+1} are given as follows:*

(a): $(|z| < \min\{|\theta_1|^{-1}, |\theta_2|^{-1}, |\theta_3|^{-1}\} = |\theta_2|^{-1} = |\theta_3|^{-1} \simeq 0.543026\,).$

$$\sum_{n=0}^{\infty} Y_n z^n = \frac{(2Y_0 + Y_2)z^2 + Y_1 z + Y_0}{-4z^3 + 2z^2 + 1}.$$

(b): $(|z| < \min\{|\theta_1|^{-2}, |\theta_2|^{-2}, |\theta_3|^{-2}\} = |\theta_2|^{-2} = |\theta_3|^{-2} \simeq 0.294877)$.

$$\sum_{n=0}^{\infty} Y_{2n}z^n = \frac{(4Y_0+4Y_1+2Y_2)z^2+(4Y_0+Y_2)z+Y_0}{-16z^3+4z^2+4z+1}.$$

(c): $(|z| < \min\{|\theta_1|^{-2}, |\theta_2|^{-2}, |\theta_3|^{-2}\} = |\theta_2|^{-2} = |\theta_3|^{-2} \simeq 0.294877)$.

$$\sum_{n=0}^{\infty} Y_{2n+1}z^n = \frac{Y_1+(4Y_0+2Y_1)z+4(2Y_0+Y_2)z^2}{-16z^3+4z^2+4z+1}.$$

(d): $(|z| < \min\{|\theta_1|, |\theta_2|, |\theta_3|\} = |\theta_1| \simeq 1.179509)$.

$$\sum_{n=0}^{\infty} Y_{-n}z^n = \frac{-Y_1z^2-Y_2z-4Y_0}{z^3+2z-4}.$$

(e): $(|z| < \min\{|\theta_1|^{2}, |\theta_2|^{2}, |\theta_3|^{2}\} = |\theta_1|^{2} \simeq 1.391241)$.

$$\sum_{n=0}^{\infty} Y_{-2n}z^n = \frac{-Y_2z^2-(4Y_1+2Y_2)z-16Y_0}{z^3+4z^2+4z-16}.$$

(f): $(|z| < \min\{|\theta_1|^{2}, |\theta_2|^{2}, |\theta_3|^{2}\} = |\theta_1|^{2} \simeq 1.391241)$.

$$\sum_{n=0}^{\infty} Y_{-2n+1}z^n = \frac{2(Y_1-2Y_0)z^2+4(Y_1-2Y_0-Y_2)z-16Y_1}{z^3+4z^2+4z-16}.$$

Proof. Set $W_n = Y_n$ and $r = 0, s = -2, t = 4$ in [39, Corollary 67.]. □

Now, we consider special cases of the last corollary.

Corollary 55. *The ordinary generating functions of special cases of the generalized co-Jacobsthal-Narayana numbers are given as follows:*

(a): $(|z| < \min\{|\theta_1|^{-1}, |\theta_2|^{-1}, |\theta_3|^{-1}\} = |\theta_2|^{-1} = |\theta_3|^{-1} \simeq 0.543026)$.

$$\sum_{n=0}^{\infty} U_nz^n = \frac{z}{-4z^3+2z^2+1},$$
$$\sum_{n=0}^{\infty} S_nz^n = \frac{2z^2+3}{-4z^3+2z^2+1}.$$

(b): $(|z| < \min\{|\theta_1|^{-2}, |\theta_2|^{-2}, |\theta_3|^{-2}\} = |\theta_2|^{-2} = |\theta_3|^{-2} \simeq 0.294877)$.

$$\sum_{n=0}^{\infty} U_{2n}z^n = \frac{4z^2}{-16z^3+4z^2+4z+1},$$
$$\sum_{n=0}^{\infty} S_{2n}z^n = \frac{4z^2+8z+3}{-16z^3+4z^2+4z+1}.$$

(c): $(|z| < \min\{|\theta_1|^{-2}, |\theta_2|^{-2}, |\theta_3|^{-2}\} = |\theta_2|^{-2} = |\theta_3|^{-2} \simeq 0.294877)$.

$$\sum_{n=0}^{\infty} U_{2n+1}z^n = \frac{2z+1}{-16z^3+4z^2+4z+1},$$
$$\sum_{n=0}^{\infty} S_{2n+1}z^n = \frac{8z^2+12z}{-16z^3+4z^2+4z+1}.$$

(d): $(|z| < \min\{|\theta_1|, |\theta_2|, |\theta_3|\} = |\theta_1| \simeq 1.179509)$.

$$\begin{aligned}\sum_{n=0}^{\infty} U_{-n} z^n &= \frac{-z^2}{z^3+2z-4}, \\ \sum_{n=0}^{\infty} S_{-n} z^n &= \frac{4z-12}{z^3+2z-4}.\end{aligned}$$

(e): $(|z| < \min\{|\theta_1|^2, |\theta_2|^2, |\theta_3|^2\} = |\theta_1|^2 \simeq 1.391241)$.

$$\begin{aligned}\sum_{n=0}^{\infty} U_{-2n} z^n &= \frac{-4z}{z^3+4z^2+4z-16}, \\ \sum_{n=0}^{\infty} S_{-2n} z^n &= \frac{4z^2+8z-48}{z^3+4z^2+4z-16}.\end{aligned}$$

(f): $(|z| < \min\{|\theta_1|^2, |\theta_2|^2, |\theta_3|^2\} = |\theta_1|^2 \simeq 1.391241)$.

$$\begin{aligned}\sum_{n=0}^{\infty} U_{-2n+1} z^n &= \frac{2z^2+4z-16}{z^3+4z^2+4z-16}, \\ \sum_{n=0}^{\infty} S_{-2n+1} z^n &= \frac{-12z^2-8z}{z^3+4z^2+4z-16}.\end{aligned}$$

From the last corollary, we obtain the following results for special cases of z.

Corollary 56. *We have the following infinite sums .*

(a): $z = \frac{1}{2}$.

$$\begin{aligned}\sum_{n=0}^{\infty} \frac{U_n}{2^n} &= \frac{1}{2}, \\ \sum_{n=0}^{\infty} \frac{S_n}{2^n} &= \frac{7}{2}.\end{aligned}$$

(b): $z = \frac{1}{4}$.

$$\begin{aligned}\sum_{n=0}^{\infty} \frac{U_{2n}}{4^n} &= \frac{1}{8}, \\ \sum_{n=0}^{\infty} \frac{S_{2n}}{4^n} &= \frac{21}{8}.\end{aligned}$$

(c): $z = \frac{1}{4}$.

$$\begin{aligned}\sum_{n=0}^{\infty} \frac{U_{2n+1}}{4^n} &= \frac{3}{4}, \\ \sum_{n=0}^{\infty} \frac{S_{2n+1}}{4^n} &= \frac{7}{4}.\end{aligned}$$

(d): $z = 1$.

$$\sum_{n=0}^{\infty} U_{-n} = 1,$$
$$\sum_{n=0}^{\infty} S_{-n} = 8.$$

(e): $z = 1$.

$$\sum_{n=0}^{\infty} U_{-2n} = \frac{4}{7},$$
$$\sum_{n=0}^{\infty} S_{-2n} = \frac{36}{7}.$$

(f): $z = 1$.

$$\sum_{n=0}^{\infty} U_{-2n+1} = \frac{10}{7},$$
$$\sum_{n=0}^{\infty} S_{-2n+1} = \frac{20}{7}.$$

3.6. Sum Formulas $\sum_{k=0}^{n} z^k Y_k^2$, $\sum_{k=0}^{n} z^k Y_{k+1}Y_k$, $\sum_{k=0}^{n} z^k Y_{k+2}Y_k$ and Generating Functions $\sum_{n=0}^{\infty} Y_n^2 z^n$, $\sum_{n=0}^{\infty} Y_{n+1}Y_n z^n$, $\sum_{n=0}^{\infty} Y_{n+2}Y_n z^n$ of Generalized co-Jacobsthal-Narayana Numbers. Next, we present sum formulas of generalized co-Jacobsthal-Narayana numbers.

THEOREM 57. *For $n \geq 0$, we have the following sum formulas for generalized co-Jacobsthal-Narayana numbers:*

(a): $\sum_{k=0}^{n} Y_k^2 = \frac{1}{119}(13Y_{n+3}^2 + 13Y_{n+2}^2 + 89Y_{n+1}^2 + 16Y_{n+2}Y_{n+3} + 64Y_{n+1}Y_{n+3} + 48Y_{n+1}Y_{n+2} - 13Y_2^2 - 13Y_1^2 - 89Y_0^2 - 16Y_1Y_2 - 64Y_0Y_2 - 48Y_0Y_1)$.

(b): $\sum_{k=0}^{n} Y_{k+1}Y_k = \frac{1}{119}(8Y_{n+3}^2 + 8Y_{n+2}^2 + 128Y_{n+1}^2 + 19Y_{n+2}Y_{n+3} + 76Y_{n+1}Y_{n+3} + 57Y_{n+1}Y_{n+2} - 8Y_2^2 - 8Y_1^2 - 128Y_0^2 - 57Y_0Y_1 - 19Y_1Y_2 - 76Y_0Y_2)$.

(c): $\sum_{k=0}^{n} Y_{k+2}Y_k = \frac{1}{119}(6Y_{n+3}^2 + 6Y_{n+2}^2 + 96Y_{n+1}^2 + 44Y_{n+2}Y_{n+3} + 57Y_{n+1}Y_{n+3} + 132Y_{n+1}Y_{n+2} - 6Y_2^2 - 6Y_1^2 - 96Y_0^2 - 44Y_1Y_2 - 57Y_0Y_2 - 132Y_0Y_1)$.

Proof.

(a): Set $W_n = Y_n$, $r = 0$, $s = -2$, $t = 4$ and $z = 1$ in [43, Theorem 2.1 (a) (i)].

(b): Set $W_n = Y_n$, $r = 0$, $s = -2$, $t = 4$ and $z = 1$ in [43, Theorem 2.1 (b) (i)].

(c): Set $W_n = Y_n$, $r = 0$, $s = -2$, $t = 4$ and $z = 1$ in [43, Theorem 2.1 (c) (i)]. □

From the last Theorem, we have the following Corollary which gives sum formulas of co-Jacobsthal-Narayana numbers (take $Y_n = U_n$ with $U_0 = 0, U_1 = 1, U_2 = 0$).

COROLLARY 58. *For $n \geq 0$, co-Jacobsthal-Narayana numbers have the following properties.*

(a): $\sum_{k=0}^{n} U_k^2 = \frac{1}{119}(13U_{n+3}^2 + 13U_{n+2}^2 + 89U_{n+1}^2 + 16U_{n+2}U_{n+3} + 64U_{n+1}U_{n+3} + 48U_{n+1}U_{n+2} - 13)$

(b): $\sum_{k=0}^{n} U_{k+1}U_k = \frac{1}{119}(8U_{n+3}^2 + 8U_{n+2}^2 + 128U_{n+1}^2 + 19U_{n+2}U_{n+3} + 76U_{n+1}U_{n+3} + 57U_{n+1}U_{n+2} - 8)$

(c): $\sum_{k=0}^{n} U_{k+2}U_k = \frac{1}{119}(6U_{n+3}^2 + 6U_{n+2}^2 + 96U_{n+1}^2 + 44U_{n+2}U_{n+3} + 57U_{n+1}U_{n+3} + 132U_{n+1}U_{n+2} - 6)$

Taking $Y_n = S_n$ with $S_0 = 3, S_1 = 0, S_2 = -4$ in the last Theorem, we have the following Corollary which gives sum formulas of co-Jacobsthal-Narayana-Lucas numbers.

Corollary 59. *For $n \geq 0$, co-Jacobsthal-Narayana-Lucas numbers have the following properties:*

(a): $\sum_{k=0}^{n} S_k^2 = \frac{1}{119}(13S_{n+3}^2 + 13S_{n+2}^2 + 89S_{n+1}^2 + 16S_{n+2}S_{n+3} + 64S_{n+1}S_{n+3} + 48S_{n+1}S_{n+2} - 241)$

(b): $\sum_{k=0}^{n} S_{k+1}S_k = \frac{1}{119}(8S_{n+3}^2 + 8S_{n+2}^2 + 128S_{n+1}^2 + 19S_{n+2}S_{n+3} + 76S_{n+1}S_{n+3} + 57S_{n+1}S_{n+2} - 368)$

(c): $\sum_{k=0}^{n} S_{k+2}S_k = \frac{1}{119}(6S_{n+3}^2 + 6S_{n+2}^2 + 96S_{n+1}^2 + 44S_{n+2}S_{n+3} + 57S_{n+1}S_{n+3} + 132S_{n+1}S_{n+2} - 276)$

Next, we give the ordinary generating functions $\sum_{n=0}^{\infty} Y_n^2 z^n$, $\sum_{n=0}^{\infty} Y_{n+1}Y_n z^n$, $\sum_{n=0}^{\infty} Y_{n+2}Y_n z^n$ of the sequences $\{Y_n^2\}$, $\{Y_{n+1}Y_n\}$, $\{Y_{n+2}Y_n\}$.

Theorem 60. *Assume that $|z| < \min\{|\theta_1|^{-2}, |\theta_2|^{-2}, |\theta_3|^{-2}, |\theta_1\theta_2|^{-1}, |\theta_1\theta_3|^{-1}, |\theta_2\theta_3|^{-1}\} = |\theta_2|^{-2} = |\theta_3|^{-2} = |\theta_2\theta_3|^{-1} \simeq 0.294877$. Then the ordinary generating functions of the sequences $\{Y_n^2\}$, $\{Y_{n+1}Y_n\}$, $\{Y_{n+2}Y_n\}$ are given as follows:*

(a): $\sum_{n=0}^{\infty} Y_n^2 z^n = \frac{1}{-256z^6 + 64z^5 + 32z^4 + 40z^3 + 4z^2 - 2z - 1}(16(2Y_0 + Y_2)^2 z^5 + 4(4Y_1^2 + 8Y_0Y_1 + 4Y_1Y_2)z^4 + (24Y_0^2 + 16Y_1Y_0 - 2Y_2^2)z^3 - z^2(-4Y_0^2 + 2Y_1^2 + Y_2^2) - (2Y_0^2 + Y_1^2)z - Y_0^2)$.

(b): $\sum_{n=0}^{\infty} Y_{n+1}Y_n z^n = \frac{1}{-256z^6 + 64z^5 + 32z^4 + 40z^3 + 4z^2 - 2z - 1}(64Y_0(2Y_0+Y_2)z^5 + 4(2Y_0+Y_2)(4Y_1 + 2Y_2)z^4 + (8Y_1^2 + 24Y_0Y_1 + 4Y_1Y_2)z^3 + (4Y_0Y_1 - 4Y_0Y_2)z^2 - Y_1(2Y_0 + Y_2)z - Y_0Y_1)$.

(c): $\sum_{n=0}^{\infty} Y_{n+2}Y_n z^n = \frac{1}{-256z^6 + 64z^5 + 32z^4 + 40z^3 + 4z^2 - 2z - 1}(64Y_1(2Y_0+Y_2)z^5 + 4(16Y_0^2 + 8Y_0Y_1 + 8Y_0Y_2 - 4Y_1Y_2)z^4 + (32Y_0^2 + 24Y_0Y_2 - 16Y_1Y_0 + 4Y_2^2)z^3 + (4Y_1^2 - 4Y_1Y_2 - 8Y_0Y_1 + 2Y_2^2 + 4Y_0Y_2)z^2 - Y_0Y_2 - (-2Y_1^2 + 4Y_0Y_1 + 2Y_0Y_2)z)$.

Proof. Set $Y_n = Y_n$ and $r = 0$, $s = -2$, $t = 4$ in [42, Theorem 3.1] or in [43, Theorem 3.1]. □

Now, we consider special cases of the last Theorem.

Corollary 61. *Assume that $|z| < \min\{|\theta_1|^{-2}, |\theta_2|^{-2}, |\theta_3|^{-2}, |\theta_1\theta_2|^{-1}, |\theta_1\theta_3|^{-1}, |\theta_2\theta_3|^{-1}\} = |\theta_2|^{-2} = |\theta_3|^{-2} = |\theta_2\theta_3|^{-1} \simeq 0.294877$. The ordinary generating functions of the sequences $\{U_n^2\}$, $\{U_{n+1}U_n\}$, $\{U_{n+2}U_n\}$ and $\{S_n^2\}$, $\{S_{n+1}S_n\}$, $\{S_{n+2}S_n\}$ are given as follows:*

(a):

$$\sum_{n=0}^{\infty} U_n^2 z^n = \frac{16z^4 - 2z^2 - z}{-256z^6 + 64z^5 + 32z^4 + 40z^3 + 4z^2 - 2z - 1},$$

$$\sum_{n=0}^{\infty} S_n^2 z^n = \frac{64z^5 + 184z^3 + 20z^2 - 18z - 9}{-256z^6 + 64z^5 + 32z^4 + 40z^3 + 4z^2 - 2z - 1}.$$

(b):

$$\sum_{n=0}^{\infty} U_{n+1}U_n z^n = \frac{8z^3}{-256z^6+64z^5+32z^4+40z^3+4z^2-2z-1},$$
$$\sum_{n=0}^{\infty} S_{n+1}S_n z^n = \frac{384z^5-64z^4+48z^2}{-256z^6+64z^5+32z^4+40z^3+4z^2-2z-1}.$$

(c):

$$\sum_{n=0}^{\infty} U_{n+2}U_n z^n = \frac{4z^2+2z}{-256z^6+64z^5+32z^4+40z^3+4z^2-2z-1},$$
$$\sum_{n=0}^{\infty} S_{n+2}S_n z^n = \frac{192z^4+64z^3-16z^2+24z+12}{-256z^6+64z^5+32z^4+40z^3+4z^2-2z-1}.$$

From the last corollary, we obtain the following results for special cases of z.

Corollary 62. *Some infinite sums of* $\{U_n^2\}$, $\{U_{n+1}U_n\}$, $\{U_{n+2}U_n\}$ *and* $\{S_n^2\}$, $\{S_{n+1}S_n\}$, $\{S_{n+2}S_n\}$ *are given as follows:*

(a): $z=\frac{1}{4}$.

$$\sum_{n=0}^{\infty} \frac{U_n^2}{4^n} = \frac{5}{8},$$
$$\sum_{n=0}^{\infty} \frac{S_n^2}{4^n} = \frac{149}{8}.$$

(b): $z=\frac{1}{4}$.

$$\sum_{n=0}^{\infty} \frac{U_{n+1}U_n}{4^n} = -\frac{1}{4},$$
$$\sum_{n=0}^{\infty} \frac{S_{n+1}S_n}{4^n} = -\frac{25}{4}.$$

(c): $z=\frac{1}{4}$.

$$\sum_{n=0}^{\infty} \frac{U_{n+2}U_n}{4^n} = -\frac{3}{2},$$
$$\sum_{n=0}^{\infty} \frac{S_{n+2}S_n}{4^n} = -\frac{75}{2}.$$

3.7. Generalized co-Jacobsthal-Narayana Numbers by Matrix Methods. In this section, we present matrix representations of the sequences Y_n, U_n and S_n. We also introduce Simson matrix and investigate its properties.

3.7.1. *Matrix Representations of the Sequences Y_n, U_n and S_n.* We define the square matrix K of order 3 as:

$$K = \begin{pmatrix} 0 & -2 & 4 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

such that $\det K = 4$. Some properties of matrix K^n can be given as

$$\begin{aligned} K^n &= -2K^{n-2} + 4K^{n-3}, \\ K^{n+m} &= K^n K^m = K^m K^n, \end{aligned}$$

for all integers m and n. Note that we have the following formulas:

$$\begin{pmatrix} Y_{n+2} \\ Y_{n+1} \\ Y_n \end{pmatrix} = \begin{pmatrix} 0 & -2 & 4 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} Y_{n+1} \\ Y_n \\ Y_{n-1} \end{pmatrix},$$

and

$$\begin{pmatrix} Y_{n+2} \\ Y_{n+1} \\ Y_n \end{pmatrix} = \begin{pmatrix} 0 & -2 & 4 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^n \begin{pmatrix} Y_2 \\ Y_1 \\ Y_0 \end{pmatrix},$$

and

$$\begin{pmatrix} U_{n+2} \\ U_{n+1} \\ U_n \end{pmatrix} = \begin{pmatrix} 0 & -2 & 4 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} U_{n+1} \\ U_n \\ U_{n-1} \end{pmatrix}.$$

We also define

$$N_n = \begin{pmatrix} U_{n+1} & -2U_n + 4U_{n-1} & 4U_n \\ U_n & -2U_{n-1} + 4U_{n-2} & 4U_{n-1} \\ U_{n-1} & -2U_{n-2} + 4U_{n-3} & 4U_{n-2} \end{pmatrix}$$

and

$$M_n = \begin{pmatrix} Y_{n+1} & -2Y_n + 4Y_{n-1} & 4Y_n \\ Y_n & -2Y_{n-1} + 4Y_{n-2} & 4Y_{n-1} \\ Y_{n-1} & -2sY_{n-2} + 4Y_{n-3} & 4Y_{n-2} \end{pmatrix}.$$

THEOREM 63. *For all integers m, n, we have the following properties:*

(a): $N_n = K^n$, *i.e.,*

$$\begin{pmatrix} 0 & -2 & 4 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^n = \begin{pmatrix} U_{n+1} & -2U_n + 4U_{n-1} & 4U_n \\ U_n & -2U_{n-1} + 4U_{n-2} & 4U_{n-1} \\ U_{n-1} & -2U_{n-2} + 4U_{n-3} & 4U_{n-2} \end{pmatrix}.$$

(b): $M_1 K^n = K^n M_1$.

(c): $M_{n+m} = M_n N_m = N_m M_n$, *i.e.,*

$$\begin{pmatrix} Y_{n+m+1} & -2Y_{n+m}+4Y_{n+m-1} & 4Y_{n+m} \\ Y_{n+m} & -2Y_{n+m-1}+4Y_{n+m-2} & 4Y_{n+m-1} \\ Y_{n+m-1} & -2Y_{n+m-2}+4Y_{n+m-3} & 4Y_{n+m-2} \end{pmatrix}$$

$$= \begin{pmatrix} Y_{n+1} & -2Y_n+4Y_{n-1} & 4Y_n \\ Y_n & -2Y_{n-1}+4Y_{n-2} & 4Y_{n-1} \\ Y_{n-1} & -2Y_{n-2}+4Y_{n-3} & 4Y_{n-2} \end{pmatrix} \begin{pmatrix} U_{m+1} & -2U_m+4U_{m-1} & 4U_m \\ U_m & -2U_{m-1}+4U_{m-2} & 4U_{m-1} \\ U_{m-1} & -2U_{m-2}+4U_{m-3} & 4U_{m-2} \end{pmatrix}$$

$$= \begin{pmatrix} U_{m+1} & -2U_m+4U_{m-1} & 4U_m \\ U_m & -2U_{m-1}+4U_{m-2} & 4U_{m-1} \\ U_{m-1} & -2U_{m-2}+4U_{m-3} & 4U_{m-2} \end{pmatrix} \begin{pmatrix} Y_{n+1} & -2Y_n+4Y_{n-1} & 4Y_n \\ Y_n & -2Y_{n-1}+4Y_{n-2} & 4Y_{n-1} \\ Y_{n-1} & -2Y_{n-2}+4Y_{n-3} & 4Y_{n-2} \end{pmatrix}.$$

(d):

$$K^n = U_{n-1}K^2 + (-2U_{n-2}+4U_{n-3})K + 4U_{n-2}I$$

i.e.,

$$K^n = \frac{1}{4}((U_{n+2}+2U_n)K^2 + 4U_nK + (2U_{n+2}+4U_{n+1}+4U_n)I)$$

that is,

$$K^n = \frac{1}{4}(U_{n+2}(K^2+2I) + 4U_{n+1}I + U_n(2K^2+4K+4I))$$

where

$$I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Proof. Set $W_n = Y_n$, $r = 0$, $s = -2$, $t = 4$ and $G_n = U_n$ in [39, Theorem 51.]. □

Next, we present matrix formulas for the generalized co-Jacobsthal-Narayana and co-Jacobsthal-Narayana-Lucas numbers.

Corollary 64. *For all integers n, we have the following formulas for generalized co-Jacobsthal-Narayana numbers and co-Jacobsthal-Narayana-Lucas numbers.*

(a): *Generalized co-Jacobsthal-Narayana numbers.*

$$\begin{pmatrix} 0 & -2 & 4 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^n = \frac{1}{\Lambda_Y(0)} \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

where

$$a_{11} = (4Y_0^2 - Y_1Y_2 - 2Y_0Y_1)Y_{n+3} + (Y_2^2 + 2Y_0Y_2 - 4Y_0Y_1)Y_{n+2} + 4(Y_1^2 - Y_0Y_2)Y_{n+1}$$
$$a_{21} = (4Y_0^2 - Y_1Y_2 - 2Y_0Y_1)Y_{n+2} + (Y_2^2 + 2Y_0Y_2 - 4Y_0Y_1)Y_{n+1} + 4(Y_1^2 - Y_0Y_2)Y_n$$
$$a_{31} = (4Y_0^2 - Y_1Y_2 - 2Y_0Y_1)Y_{n+1} + (Y_2^2 + 2Y_0Y_2 - 4Y_0Y_1)Y_n + 4(Y_1^2 - Y_0Y_2)Y_{n-1}$$

$a_{12} = -2((4Y_0^2 - Y_1Y_2 - 2Y_0Y_1)Y_{n+2} + (Y_2^2 + 2Y_0Y_2 - 4Y_0Y_1)Y_{n+1} + 4(Y_1^2 - Y_0Y_2)Y_n) + 4((4Y_0^2 - Y_1Y_2 - 2Y_0Y_1)Y_{n+1} + (Y_2^2 + 2Y_0Y_2 - 4Y_0Y_1)Y_n + 4(Y_1^2 - Y_0Y_2)Y_{n-1})$

$a_{22} = -2((4Y_0^2 - Y_1Y_2 - 2Y_0Y_1)Y_{n+1} + (Y_2^2 + 2Y_0Y_2 - 4Y_0Y_1)Y_n + 4(Y_1^2 - Y_0Y_2)Y_{n-1}) + 4((4Y_0^2 - Y_1Y_2 - 2Y_0Y_1)Y_n + (Y_2^2 + 2Y_0Y_2 - 4Y_0Y_1)Y_{n-1} + 4(Y_1^2 - Y_0Y_2)Y_{n-2})$

$a_{32} = -2((4Y_0^2 - Y_1Y_2 - 2Y_0Y_1)Y_n + (Y_2^2 + 2Y_0Y_2 - 4Y_0Y_1)Y_{n-1} + 4(Y_1^2 - Y_0Y_2)Y_{n-2}) + 4((4Y_0^2 - Y_1Y_2 - 2Y_0Y_1)Y_{n-1} + (Y_2^2 + 2Y_0Y_2 - 4Y_0Y_1)Y_{n-2} + 4(Y_1^2 - Y_0Y_2)Y_{n-3})$

$a_{13} = 4((4Y_0^2 - Y_1Y_2 - 2Y_0Y_1)Y_{n+2} + (Y_2^2 + 2Y_0Y_2 - 4Y_0Y_1)Y_{n+1} + 4(Y_1^2 - Y_0Y_2)Y_n)$

$a_{23} = 4((4Y_0^2 - Y_1Y_2 - 2Y_0Y_1)Y_{n+1} + (Y_2^2 + 2Y_0Y_2 - 4Y_0Y_1)Y_n + 4(Y_1^2 - Y_0Y_2)Y_{n-1})$

$a_{33} = 4((4Y_0^2 - Y_1Y_2 - 2Y_0Y_1)Y_n + (Y_2^2 + 2Y_0Y_2 - 4Y_0Y_1)Y_{n-1} + 4(Y_1^2 - Y_0Y_2)Y_{n-2})$

$\Lambda_Y(0) = Y_2^3 + 4Y_1^3 + 16Y_0^3 + 2Y_0Y_2^2 + 2Y_2Y_1^2 + 4Y_0Y_1^2 - 16Y_0^2Y_1 - 12Y_2Y_1Y_0$

(b): *co-Jacobsthal-Narayana-Lucas numbers.*

$$\begin{pmatrix} 0 & -2 & 4 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^n = \frac{1}{464}\begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix}$$

where

$b_{11} = 36S_{n+3} - 8S_{n+2} + 48S_{n+1}$

$b_{21} = 36S_{n+2} - 8S_{n+1} + 48S_n$

$b_{31} = 36S_{n+1} - 8S_n + 48S_{n-1}$

$b_{12} = -72S_{n+2} + 160S_{n+1} - 128S_n + 192S_{n-1}$

$b_{22} = -72S_{n+1} + 160S_n - 128S_{n-1} + 192S_{n-2}$

$b_{32} = -72S_n + 160S_{n-1} - 128S_{n-2} + 192S_{n-3}$

$b_{13} = 144S_{n+2} - 32S_{n+1} + 192S_n$

$b_{23} = 144S_{n+1} - 32S_n + 192S_{n-1}$

$b_{33} = 144S_n - 32S_{n-1} + 192S_{n-2}$

Proof. Set $W_n = Y_n, r = 0,\ s = -2,\ t = 4$ and then take $Y_n = S$, respectively, in [39, Corollary 52.]. □

Note that, a_{12}, a_{22}, a_{32} and b_{12}, b_{22}, b_{32} can be written in the following form:

$a_{12} = (-2Y_2^2 + 16Y_0^2 - 4Y_1Y_2 - 4Y_0Y_2)Y_{n+1} + (4Y_2^2 - 8Y_1^2 + 16Y_0^2 - 4Y_1Y_2 + 16Y_0Y_2 - 24Y_0Y_1)Y_n + 4(4Y_1^2 - 8Y_0^2 + 2Y_1Y_2 - 4Y_0Y_2 + 4Y_0Y_1)Y_{n-1}$

$a_{22} = (-2Y_2^2 + 16Y_0^2 - 4Y_1Y_2 - 4Y_0Y_2)Y_n + (4Y_2^2 - 8Y_1^2 + 16Y_0^2 - 4Y_1Y_2 + 16Y_0Y_2 - 24Y_0Y_1)Y_{n-1} + 4(4Y_1^2 - 8Y_0^2 + 2Y_1Y_2 - 4Y_0Y_2 + 4Y_0Y_1)Y_{n-2}$

$a_{32} = (-2Y_2^2 + 16Y_0^2 - 4Y_1Y_2 - 4Y_0Y_2)Y_{n-1} + (4Y_2^2 - 8Y_1^2 + 16Y_0^2 - 4Y_1Y_2 + 16Y_0Y_2 - 24Y_0Y_1)Y_{n-2} + 4(4Y_1^2 - 8Y_0^2 + 2Y_1Y_2 - 4Y_0Y_2 + 4Y_0Y_1)Y_{n-3}$

and

$b_{12} = 160S_{n+1} + 16S_n - 96S_{n-1}$

$b_{22} = 160S_n + 16S_{n-1} - 96S_{n-2}$

$b_{32} = 160S_{n-1} + 16S_{n-2} - 96S_{n-3}$.

Now, we present an identity for Y_{n+m}.

THEOREM 65. *(Honsberger's Identity) For all integers m and n, we have*

$$\begin{aligned} Y_{n+m} &= Y_n U_{m+1} + Y_{n-1}(-2U_m + 4U_{m-1}) + 4Y_{n-2}U_m \\ &= Y_n U_{m+1} + (-2Y_{n-1} + 4Y_{n-2})\, U_m + 4Y_{n-1}U_{m-1} \end{aligned}$$

Proof. Set $W_n = Y_n$, $r = 0,\ s = -2,\ t = 4$ and then $G_n = U_n$ in [39, Theorem 53.]. □

As special cases of the last Theorem, we have the following corollary.

COROLLARY 66. *For all integers m, n, we have the following properties:*

$$\begin{aligned} U_{n+m} &= U_n U_{m+1} + U_{n-1}(-2U_m + 4U_{m-1}) + 4U_{n-2}U_m \\ S_{n+m} &= S_n U_{m+1} + S_{n-1}(-2U_m + 4U_{m-1}) + 4S_{n-2}U_m \end{aligned}$$

Next, we present identies for Y_{mn+j} and its special cases.

COROLLARY 67. *For all integers m, n, j, we have the following properties:*

$$\begin{aligned} Y_{mn+j} &= U_{mn-1}Y_{j+2} + (-2U_{mn-2} + 4U_{mn-3})\, Y_{j+1} + 4U_{mn-2}Y_j \\ U_{mn+j} &= U_{mn-1}U_{j+2} + (-2U_{mn-2} + 4U_{mn-3})\, U_{j+1} + 4U_{mn-2}U_j \\ S_{mn+j} &= U_{mn-1}S_{j+2} + (-2U_{mn-2} + 4U_{mn-3})\, S_{j+1} + 4U_{mn-2}S_j \end{aligned}$$

Proof. Set $r = 0,\ s = -2,\ t = 4$ and $W_n = Y_n$, then take $Y_n = U_n$ and $Y_n = S_n$, respectively, in [39, Corollary 55.]. □

3.7.2. *Simson Matrix and its Properties.* For $n \in \mathbb{Z}$, we define

$$f_Y(n) = \begin{pmatrix} Y_{n+2} & Y_{n+1} & Y_n \\ Y_{n+1} & Y_n & Y_{n-1} \\ Y_n & Y_{n-1} & Y_{n-2} \end{pmatrix}.$$

We call this matrix as Simson matrix of the sequence Y_n. Similarly, as special cases of Y_n, Simson matrices of the sequences U_n and S_n are

$$f_U(n) = \begin{pmatrix} U_{n+2} & U_{n+1} & U_n \\ U_{n+1} & U_n & U_{n-1} \\ U_n & U_{n-1} & U_{n-2} \end{pmatrix} \text{ and } f_S(n) = \begin{pmatrix} S_{n+2} & S_{n+1} & S_n \\ S_{n+1} & S_n & S_{n-1} \\ S_n & S_{n-1} & S_{n-2} \end{pmatrix}$$

respectively.

LEMMA 68. *For all integers n, m and j, the followings hold.*

(a): $f_Y(n) = -2f_Y(n-2) + 4f_Y(n-3)$.

(b): $f_Y(n) = Kf_Y(n-1)$ *and* $f_Y(n) = K^n f_Y(0)$, *i.e.,*

$$\begin{pmatrix} Y_{n+2} & Y_{n+1} & Y_n \\ Y_{n+1} & Y_n & Y_{n-1} \\ Y_n & Y_{n-1} & Y_{n-2} \end{pmatrix} = \begin{pmatrix} 0 & -2 & 4 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} Y_{n+1} & Y_n & Y_{n-1} \\ Y_n & Y_{n-1} & Y_{n-2} \\ Y_{n-1} & Y_{n-2} & Y_{n-3} \end{pmatrix}$$

and

$$\begin{pmatrix} Y_{n+2} & Y_{n+1} & Y_n \\ Y_{n+1} & Y_n & Y_{n-1} \\ Y_n & Y_{n-1} & Y_{n-2} \end{pmatrix} = \begin{pmatrix} 0 & -2 & 4 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^n \begin{pmatrix} Y_2 & Y_1 & Y_0 \\ Y_1 & Y_0 & Y_{-1} \\ Y_0 & Y_{-1} & Y_{-2} \end{pmatrix}.$$

(c): $f_Y(n+m) = K^n f_Y(m)$ *and* $f_Y(n+m) = K^m f_Y(n)$ *i.e.,*

$$\begin{pmatrix} Y_{n+m+2} & Y_{n+m+1} & Y_{n+m} \\ Y_{n+m+1} & Y_{n+m} & Y_{n+m-1} \\ Y_{n+m} & Y_{n+m-1} & Y_{n+m-2} \end{pmatrix} = \begin{pmatrix} 0 & -2 & 4 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^n \begin{pmatrix} Y_{m+2} & Y_{m+1} & Y_m \\ Y_{m+1} & Y_m & Y_{m-1} \\ Y_m & Y_{m-1} & Y_{m-2} \end{pmatrix},$$

and

$$\begin{pmatrix} Y_{m+n+2} & Y_{m+n+1} & Y_{m+n} \\ Y_{m+n+1} & Y_{m+n} & Y_{m+n-1} \\ Y_{m+n} & Y_{m+n-1} & Y_{m+n-2} \end{pmatrix} = \begin{pmatrix} 0 & -2 & 4 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^m \begin{pmatrix} Y_{n+2} & Y_{n+1} & Y_n \\ Y_{n+1} & Y_n & Y_{n-1} \\ Y_n & Y_{n-1} & Y_{n-2} \end{pmatrix},$$

and $f_Y(n) = K^m f_Y(n-m)$, *i.e.,*

$$\begin{pmatrix} Y_{n+2} & Y_{n+1} & Y_n \\ Y_{n+1} & Y_n & Y_{n-1} \\ Y_n & Y_{n-1} & Y_{n-2} \end{pmatrix} = \begin{pmatrix} 0 & -2 & 4 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^m \begin{pmatrix} Y_{n-m+2} & Y_{n-m+1} & Y_{n-m} \\ Y_{n-m+1} & Y_{n-m} & Y_{n-m-1} \\ Y_{n-m} & Y_{n-m-1} & Y_{n-m-2} \end{pmatrix}.$$

Proof. Set $W_n = Y_n$, and $r = 0$, $s = -2$, $t = 4$ in [39, Lemma 56.]. □

Taking the determinant of both sides of the identities given in the last Lemma, we obtain the following Theorem.

Theorem 69. *For all integers n and m, the following identities hold.*

(a): *Catalan's Identity:*

$$\det(f_Y(n+m)) = 4^n \det(f_Y(m)) \quad \text{and} \quad \det(f_Y(n)) = 4^m \det(f_Y(n-m)),$$

i.e.,

$$\begin{vmatrix} Y_{n+m+2} & Y_{n+m+1} & Y_{n+m} \\ Y_{n+m+1} & Y_{n+m} & Y_{n+m-1} \\ Y_{n+m} & Y_{n+m-1} & Y_{n+m-2} \end{vmatrix} = 4^n \begin{vmatrix} Y_{m+2} & Y_{m+1} & Y_m \\ Y_{m+1} & Y_m & Y_{m-1} \\ Y_m & Y_{m-1} & Y_{m-2} \end{vmatrix},$$

and

$$\begin{vmatrix} Y_{n+2} & Y_{n+1} & Y_n \\ Y_{n+1} & Y_n & Y_{n-1} \\ Y_n & Y_{n-1} & Y_{n-2} \end{vmatrix} = 4^m \begin{vmatrix} Y_{n-m+2} & Y_{n-m+1} & Y_{n-m} \\ Y_{n-m+1} & Y_{n-m} & Y_{n-m-1} \\ Y_{n-m} & Y_{n-m-1} & Y_{n-m-2} \end{vmatrix}.$$

(b): *(see Theorem 46) Simson's (or Cassini's) Identity:*

$$\det(f_Y(n)) = 4^n \det(f_Y(0)),$$

i.e.,

$$\begin{vmatrix} Y_{n+2} & Y_{n+1} & Y_n \\ Y_{n+1} & Y_n & Y_{n-1} \\ Y_n & Y_{n-1} & Y_{n-2} \end{vmatrix} = 4^n \begin{vmatrix} Y_2 & Y_1 & Y_0 \\ Y_1 & Y_0 & Y_{-1} \\ Y_0 & Y_{-1} & Y_{-2} \end{vmatrix}.$$

Proof. Set $W_n = Y_n$, and $r = 0$, $s = -2$, $t = 4$ in [39, Theorem 57.]. □

From the last Theorem, we have the following Corollary which gives determinantal formulas of co-Jacobsthal-Narayana numbers (take $Y_n = U_n$ with $U_0 =, U_1 =, U_2 =$).

Corollary 70. *For all integers n and m, the following identities hold.*

(a): *Catalan's Identity:*

$$\det(f_U(n+m)) = 4^n \det(f_U(m)) \quad and \quad \det(f_U(n)) = 4^m \det(f_U(n-m)),$$

i.e.,

$$\begin{vmatrix} U_{n+m+2} & U_{n+m+1} & U_{n+m} \\ U_{n+m+1} & U_{n+m} & U_{n+m-1} \\ U_{n+m} & U_{n+m-1} & U_{n+m-2} \end{vmatrix} = 4^n \begin{vmatrix} U_{m+2} & U_{m+1} & U_m \\ U_{m+1} & U_m & U_{m-1} \\ U_m & U_{m-1} & U_{m-2} \end{vmatrix},$$

and

$$\begin{vmatrix} U_{n+2} & U_{n+1} & U_n \\ U_{n+1} & U_n & U_{n-1} \\ U_n & U_{n-1} & U_{n-2} \end{vmatrix} = 4^m \begin{vmatrix} U_{n-m+2} & U_{n-m+1} & U_{n-m} \\ U_{n-m+1} & U_{n-m} & U_{n-m-1} \\ U_{n-m} & U_{n-m-1} & U_{n-m-2} \end{vmatrix}.$$

(b): *Simson's (or Cassini's) Identity:*

$$\det(f_U(n)) = 4^n \det(f_U(0)),$$

i.e.,

$$\begin{vmatrix} U_{n+2} & U_{n+1} & U_n \\ U_{n+1} & U_n & U_{n-1} \\ U_n & U_{n-1} & U_{n-2} \end{vmatrix} = -4^{n-1}.$$

Taking $Y_n = S_n$ with $S_0 =, S_1 =, S_2 =$ in the last Theorem, we have the following Corollary which gives determinantal formulas of co-Jacobsthal-Narayana-Lucas numbers.

Corollary 71. *For all integers n and m, the following identities hold.*

(a): *Catalan's Identity:*

$$\det(f_S(n+m)) = 4^n \det(f_S(m)) \quad and \quad \det(f_S(n)) = 4^m \det(f_S(n-m))$$

i.e.,

$$\begin{vmatrix} S_{n+m+2} & S_{n+m+1} & S_{n+m} \\ S_{n+m+1} & S_{n+m} & S_{n+m-1} \\ S_{n+m} & S_{n+m-1} & S_{n+m-2} \end{vmatrix} = 4^n \begin{vmatrix} S_{m+2} & S_{m+1} & S_{m} \\ S_{m+1} & S_{m} & S_{m-1} \\ S_{m} & S_{m-1} & S_{m-2} \end{vmatrix},$$

and

$$\begin{vmatrix} S_{n+2} & S_{n+1} & S_{n} \\ S_{n+1} & S_{n} & S_{n-1} \\ S_{n} & S_{n-1} & S_{n-2} \end{vmatrix} = 4^m \begin{vmatrix} S_{n-m+2} & S_{n-m+1} & S_{n-m} \\ S_{n-m+1} & S_{n-m} & S_{n-m-1} \\ S_{n-m} & S_{n-m-1} & S_{n-m-2} \end{vmatrix}.$$

(b): *Simson's (or Cassini's) Identity:*

$$\det(f_S(n)) = 4^n \det(f_S(0)),$$

i.e.,

$$\begin{vmatrix} S_{n+2} & S_{n+1} & S_{n} \\ S_{n+1} & S_{n} & S_{n-1} \\ S_{n} & S_{n-1} & S_{n-2} \end{vmatrix} = -29 \times 2^{2n}.$$

4. Conclusions

In the literature, there have been so many studies of the sequences of numbers and the sequences of numbers were widely used in many research areas, such as physics, engineering, architecture, nature and art. Sequences of integer number such as Fibonacci, Lucas, Pell, Jacobsthal are the most well-known second order recurrence sequences. The Fibonacci numbers are perhaps most famous for appearing in the rabbit breeding problem, introduced by Leonardo de Pisa in 1202 in his book called Liber Abaci. The Fibonacci and Lucas sequences are sources of many nice and interesting identities. For rich applications of these second order sequences in science and nature, one can see the citations in [9,10,11].

As third order sequences, we introduce the generalized Jacobsthal-Narayana sequence and co-Jacobsthal-Narayana sequence (and their's two special cases) and we present Binet's formulas, generating functions, Simson formulas, the sum formulas, some identities, recurrence properties and matrices represantations for these sequences.

We now present some third order recurrence sequences (of numbers and polynomials) as follows.

- For generalized third-order Pell numbers, see [23].
- For generalized Narayana numbers, see [24].
- For generalized Jacobsthal-Padovan numbers, see [25].
- For generalized Pell-Padovan numbers, see [26].
- For generalized Tribonacci numbers, see [27].
- For generalized Padovan numbers, see [28].
- For generalized third-order Jacobsthal numbers, see [14] and [5].
- For generalized Graham numbers, see [29].

- For generalized reverse 3-primes numbers, see [30].
- For generalized Guglielmo numbers, see [31].
- For generalized Woodall numbers, see [32].
- For generalized Leonardo numbers, see [33].
- For generalized Ernst numbers, see [34].
- For generalized Edouard numbers, see [35].
- For generalized John numbers, see [36].
- For generalized Pisano numbers, see [37].
- For generalized Bigollo numbers, see [38].
- For generalized Tribonacci polynomials, see [39].
- For generalized Horadam-Leonardo polynomials, see [40] and [41].

Third order sequences have many applications. We now present some of them.

- For the applications of third order Jacobsthal numbers and Tribonacci numbers to quaternions, see [4] and [3], respectively.
- For the application of Tribonacci numbers to special matrices, see [46].
- For the applications of Padovan numbers and Tribonacci numbers to coding theory, see [16] and [1], respectively.
- For the application of Pell-Padovan numbers to groups, see [6].
- For the application of adjusted Jacobsthal-Padovan numbers to the exact solutions of some difference equations, see [2].
- For the application of Gaussian Tribonacci numbers to various graphs, see [44].
- For the application of third-order Jacobsthal numbers to hyperbolic numbers, see [7].
- For the application of Narayan numbers to finite groups see [12].
- For the application of generalized third-order Jacobsthal sequence to binomial transform, see [17].
- For the application of generalized Generalized Padovan numbers to Binomial Transform, see [18].
- For the application of generalized Tribonacci numbers to Gaussian numbers, see [19].
- For the application of generalized Tribonacci numbers to Sedenions, see [20].
- For the application of Tribonacci and Tribonacci-Lucas numbers to matrices, see [21].
- For the application of generalized Tribonacci numbers to circulant matrix, see [22].
- For the application of Tribonacci and Tribonacci-Lucas numbers to hybrinomials, see [45].
- For the application of hyperbolic Leonardo and hyperbolic Francois numbers to quaternions, see [8].

References

[1] Basu, M., Das, M., Tribonacci Matrices and a New Coding Theory, Discrete Mathematics, Algorithms and Applications, 6 (1), 1450008, (17 pages), 2014.

[2] Göcen, M., The Exact Solutions of Some Difference Equations Associated with Adjusted Jacobsthal-Padovan Numbers, Kırklareli University Journal of Engineering and Science 8(1), 1-14, 2022. DOI: 10.34186/klujes.1078836

[3] Cerda-Morales, G., On a Generalization of Tribonacci Quaternions, Mediterranean Journal of Mathematics 14:239, 1-12, 2017.

[4] Cerda-Morales, G., Identities for Third Order Jacobsthal Quaternions, Advances in Applied Clifford Algebras 27(2), 1043–1053, 2017.

[5] Cerda-Morales G., On the Third-Order Jacobsthal and Third-Order Jacobsthal–Lucas Sequences and Their Matrix Representations, Mediterranean Journal of Mathematics, Mediterr. J. Math. (2019) 16:32 https://doi.org/10.1007/s00009-019-1319-9 1660-5446/19/020001-12, 2019.

[6] Deveci, Ö., Shannon, A.G., Pell–Padovan-Circulant Sequences and Their Applications, Notes on Number Theory and Discrete Mathematics, 23(3), 100–114, 2017.

[7] Dikmen, C.M., Altınsoy, M.,On Third Order Hyperbolic Jacobsthal Numbers, Konuralp Journal of Mathematics, 10(1), 118-126, 2022.

[8] Dişkaya, O., Menken, H., Catarino, P.M.M.C., On the Hyperbolic Leonardo and Hyperbolic Francois Quaternions, Journal of New Theory, 42, 74-85, 2023.

[9] Koshy, T., Fibonacci and Lucas Numbers with Applications, Volume 1 (Pure and Applied Mathematics: A Wiley Series of Texts, Monographs and Tracts), Second Edition, John Wiley&Sons, New York, 2018.

[10] Koshy, T., Fibonacci and Lucas Numbers with Applications, Volume 2 (Pure and Applied Mathematics: A Wiley Series of Texts, Monographs and Tracts), John Wiley&Sons, New York, 2019.

[11] Koshy, T., Pell and Pell-Lucas Numbers with Applications, Springer, New York, 2014.

[12] Kuloğlu, B., Özkan, E., Shannon, A.G., The Narayana Sequence in Finite Groups, Fibonacci Quarterly. 60(5), 212–221, 2022.

[13] Petroudi., S.H.J., Dasdemir, A., Pirouz, M., On Jacobsthal–Narayana and Jacobsthal-Narayana-Lucas Sequences, Konuralp Journal of Mathematics, 12 (1), 55-61, 2024.

[14] Polatlı, E.E., Soykan, Y., On Generalized Third-Order Jacobsthal Numbers, Asian Research Journal of Mathematics, 17(2), 1-19, 2021. DOI: 10.9734/ARJOM/2021/v17i230270

[15] Sloane, N.J.A., The on-line encyclopedia of integer sequences, http://oeis.org/

[16] Shtayat, J., Al-Kateeb, A., An Encoding-Decoding algorithm based on Padovan numbers, arXiv:1907.02007, 2019.

[17] Soykan, Y., Taşdemir, E., Göcen, M., Binomial Transform of the Generalized Third-Order Jacobsthal Sequence, Asian-European Journal of Mathematics, 15(12), 2022. https://doi.org/10.1142/S1793557122502242.

[18] Soykan, Y., Taşdemir, E., Okumuş, İ., A Study on Binomial Transform of the Generalized Padovan Sequence, Journal of Science and Arts, 22(1), 63-90, 2022. https://doi.org/10.46939/J.Sci.Arts-22.1-a06

[19] Soykan, Y., Taşdemir, E., Okumuş, İ., Göcen, M., Gaussian Generalized Tribonacci Numbers, Journal of Progressive Research in Mathematics(JPRM), 14 (2), 2373-2387, 2018.

[20] Soykan, Y., Okumuş, İ., Taşdemir, E., On Generalized Tribonacci Sedenions, Sarajevo Journal of Mathematics, 16(1), 103-122, 2020. ISSN 2233-1964, DOI: 10.5644/SJM.16.01.08

[21] Soykan, Y., Matrix Sequences of Tribonacci and Tribonacci-Lucas Numbers, Communications in Mathematics and Applications, 11(2), 281-295, 2020. DOI: 10.26713/cma.v11i2.1102

[22] Soykan, Y. Explicit Euclidean Norm, Eigenvalues, Spectral Norm and Determinant of Circulant Matrix with the Generalized Tribonacci Numbers, Earthline Journal of Mathematical Sciences, 6(1), 131-151, 2021. https://doi.org/10.34198/ejms.6121.131151

[23] Soykan, Y., On Generalized Third-Order Pell Numbers, Asian Journal of Advanced Research and Reports, 6(1), 1-18, 2019.

[24] Soykan Y., On Generalized Narayana Numbers, Int. J. Adv. Appl. Math. and Mech. 7(3), 43-56, 2020.

[25] Soykan, Y., A Study on Generalized Jacobsthal-Padovan Numbers, Earthline Journal of Mathematical Sciences, 4(2), 227-251, 2020. https://doi.org/10.34198/ejms.4220.227251

[26] Soykan, Y., Generalized Pell-Padovan Numbers, Asian Journal of Advanced Research and Reports, 11(2), 8-28, 2020. DOI: 10.9734/AJARR/2020/v11i230259

[27] Soykan, Y., On Four Special Cases of Generalized Tribonacci Sequence: Tribonacci-Perrin, modified Tribonacci, modified Tribonacci-Lucas and adjusted Tribonacci-Lucas Sequences, Journal of Progressive Research in Mathematics, 16(3), 3056-3084, 2020.

[28] Soykan, Y., On Generalized Padovan Numbers, International Journal of Advances in Applied Mathematics and Mechanics, 10(4), 72-90, 2023.

[29] Soykan, Y., On Generalized Grahaml Numbers, Journal of Advances in Mathematics and Computer Science, 35(2), 42-57, 2020. DOI: 10.9734/JAMCS/2020/v35i230248.

[30] Soykan Y., On Generalized Reverse 3-primes Numbers, Journal of Scientific Research and Reports, 26(6), 1-20, 2020. DOI: 10.9734/JSRR/2020/v26i630267

[31] Soykan, Y., Generalized Guglielmo Numbers: An Investigation of Properties of Triangular, Oblong and Pentagonal Numbers via Their Third Order Linear Recurrence Relations, Earthline Journal of Mathematical Sciences, 9(1), 1-39, 2022. https://doi.org/10.34198/ejms.9122.139

[32] Soykan, Y., İrge, V., Generalized Woodall Numbers: An Investigation of Properties of Woodall and Cullen Numbers via Their Third Order Linear Recurrence Relations, Universal Journal of Mathematics and Applications, 5(2), 69-81, 2022. DOI: https://doi.org/10.32323/ujma.1057287

[33] Soykan, Y., Generalized Leonardo Numbers, Journal of Progressive Research in Mathematics, 18(4), 58-84, 2021.

[34] Soykan, Y., Generalized Ernst Numbers, Asian Journal of Pure and Applied Mathematics, 4(3), 1-15, 2022.

[35] Soykan, Y., Generalized Edouard Numbers, International Journal of Advances in Applied Mathematics and Mechanics, 9(3), 41-52, 2022.

[36] Soykan, Y., Generalized John Numbers, Journal of Progressive Research in Mathematics, 19(1), 17-34, 2022.

[37] Soykan, Y., Okumuş, İ, Taşdemir, E., Generalized Pisano Numbers, Notes on Number Theory and Discrete Mathematics, 28(3), 477-490, 2022. DOI: 10.7546/nntdm.2022.28.3.477-490

[38] Soykan, Y., Okumuş, İ., Bilgin, N.G., On Generalized Bigollo Numbers, Asian Research Journal of Mathematics, 19(8), 72-88, 2023. DOI: 10.9734/ARJOM/2023/v19i8690

[39] Soykan, Y., Generalized Tribonacci Polynomials, Earthline Journal of Mathematical Sciences, 13(1), 1-120, 2023. https://doi.org/10.34198/ejms.13123.1120

[40] Soykan, Y., Generalized Horadam-Leonardo Numbers and Polynomials, Asian Journal of Advanced Research and Reports, 17(8), 128-169, 2023. https://doi.org/10.9734/ajarr/2023/v17i8511

[41] Soykan, Y., Interrelations between Horadam and Generalized Horadam-Leonardo Polynomials via Identities, International Journal of Advances in Applied Mathematics and Mechanics, 11(1), 42-55, 2023. ISSN: 2347-2529

[42] Soykan, Y., Sums and Generating Functions of Squares of Generalized Tribonacci Polynomials: Closed Formulas of $\sum_{k=0}^{n} z^k W_k^2$ and $\sum_{n=0}^{\infty} W_n^2 z^n$, International Journal of Mathematics, Statistics and Operations Research, 3(2), 281-300, 2023. https://doi.org/10.47509/IJMSOR.2023.v03i02.06

[43] Soykan, Y., Sums and Generating Functions of Squares of Special Cases of Generalized Tribonacci Polynomials: Closed Formulas of $\sum_{k=0}^{n} z^k W_k^2$ and $\sum_{n=0}^{\infty} W_n^2 z^n$, International Journal of Advances in Applied Mathematics and Mechanics, 12(3), 1-72, 2025.

[44] Sunitha, K., Sheriba. M., Gaussian Tribonacci R-Graceful Labeling of Some Tree Related Graphs, Ratio Mathematica, 44, 188-196, 2022.

[45] Taşyurdu, Y., Polat, Y.E., Tribonacci and Tribonacci-Lucas Hybrinomials, Journal of Mathematics Research; 13(5), 2021.

[46] Yilmaz, N., Taskara, N., Tribonacci and Tribonacci-Lucas Numbers via the Determinants of Special Matrices, Applied Mathematical Sciences, 8(39), 1947-1955, 2014.