Norm Attainability of Compact Operators: Spectral, Geometric, and Perturbation Insights

Abstract

This paper presents a comprehensive investigation into the norm attainability of compact operators on infinite-dimensional Hilbert spaces, offering novel spectral and geometric insights. We establish necessary and sufficient conditions for norm attainment in terms of the spectral structure of the operator, demonstrating that a compact operator T attains its norm if and only if $\|T\|$ is an eigenvalue of $|T| = \sqrt{T^*T}$ with a corresponding eigenvector of unit norm. Our results extend to Schatten class operators, highlighting the interplay between norm attainment, singular values, and maximizing sequences. Furthermore, we explore norm attainment under perturbations, revealing stability conditions and spectral dominance properties that ensure preservation of norm-attaining behavior. These findings contribute to a deeper understanding of operator theory with applications in quantum mechanics, signal processing, and functional analysis.

keywords{Norm Attainment, Compact Operators, Hilbert Spaces, Spectral Characterization, Singular Values, Eigenvalues and Eigenvectors, Schatten Class Operators, Perturbation Analysis, Operator Theory, Functional Analysis}

Introduction and Preliminaries

Norm attainability plays a crucial role in operator theory, offering significant insights into the spectral and geometric structure of bounded linear operators on Hilbert spaces [1, 3]. The study of norm-attaining operators has evolved considerably from its classical foundations in functional analysis [5, 8], and continues to be an active area of research due to its connections with spectral theory, optimization, and applications across mathematical and physical disciplines [4, 11]. A bounded linear operator T on a Hilbert space $\mathcal{H}$ is said to attain its norm if there exists a unit vector $x \in \mathcal{H}$ such that $\|T\| = \|Tx\|$. This fundamental property has been extensively studied for compact operators, where spectral considerations provide powerful characterization tools [7, 9]. Recent work has particularly focused on the interplay between norm attainment and spectral properties of T, revealing deep connections with singular values, eigenvalues, and maximizing sequences [6, 13]. Building on these foundations, our paper presents novel characterizations of norm attainment for compact operators in infinite-dimensional Hilbert spaces. We establish that a compact operator T attains its norm precisely when its largest singular value appears as an eigenvalue of the absolute operator $|T| = \sqrt{T^*T}$ [2, 14]. This spectral characterization provides new insights into norm-attaining vectors and their relationship with maximizing sequences [10, 15]. A key contribution of our work involves the stability of norm attainment under perturbations, which has important implications for numerical analysis and applications in physics and engineering [8, 12]. We extend these results to Schatten class operators, uncovering new perspectives on their norm-attaining behavior and consequences for functional analysis [1, 4]. Furthermore, we investigate the fundamental relationship between norm attainability and compactness, which naturally leads to extensions for non-compact operators and examinations of norm attainment properties in Banach spaces [3, 11]. These advances contribute significantly to the comprehensive understanding of norm-attaining operators and their wide-ranging applications [5, 9].

Preliminaries

In this section, we establish the fundamental concepts and notations that will be used throughout the paper. We review essential results from functional analysis, operator theory, and spectral theory relevant to norm attainment.

Hilbert Spaces and Operators

Let $\mathcal{H}$ be an infinite-dimensional Hilbert space over $\mathbb{C}$, equipped with the inner product $\langle \cdot, \cdot \rangle$, which induces the norm $\|x\| = \sqrt{\langle x, x \rangle}$ for all $x \in \mathcal{H}$. A bounded linear operator $T : \mathcal{H} \to \mathcal{H}$ satisfies $\|T\| = \sup_{\|x\|=1} \|Tx\|$. A compact operator T is one for which the image of the unit ball in $\mathcal{H}$ under T has compact closure. The class of compact operators includes Hilbert-Schmidt and trace-class operators, which are significant in many applications.

Singular Values and Spectral Properties

For any compact operator T, its singular values are given by the eigenvalues of the positive semi-definite operator $|T| = \sqrt{T^*T}$. The sequence of singular values $s_n(T)$ is non-increasing and converges to zero if T is compact. The largest singular value, denoted $s_1(T) = \|T\|$, plays a crucial role in norm attainability.A key property of compact operators is that they possess at most countably many nonzero eigenvalues, and any accumulation point of the spectrum (if it exists) must be zero. For normal compact operators, the spectrum consists entirely of eigenvalues.

Norm-Attaining Operators

An operator T is said to attain its norm if there exists a unit vector $x \in \mathcal{H}$ such that $\|Tx\| = \|T\|$. The study of norm-attaining operators is deeply connected to the geometry of Hilbert spaces, the structure of eigenvectors, and the distribution of singular values. If T is compact, norm attainment is equivalent to $\|T\|$ being an eigenvalue of $|T|$.

Schatten Classes and Perturbation Theory

The Schatten p-classes, denoted $\mathcal{S}_p$, consist of compact operators whose singular values satisfy $\sum_n s_n(T)^p < \infty$ for $1 \leq p < \infty$. These classes generalize the Hilbert-Schmidt and trace-class operators and provide a rich framework for studying norm attainment. Perturbation theory plays an essential role in analyzing norm-attaining operators under small modifications. If T is a norm-attaining operator, we explore conditions under which small perturbations $T+S$ retain norm attainability. This is particularly important in applications where exact operator structures are subject to variations.

Main Results and Discussions

Theorem 1. *Let T be a compact operator on an infinite-dimensional Hilbert space $\mathcal{H}$. Then T attains its norm if and only if $\|T\|$ is an eigenvalue of $|T| = \sqrt{T^*T}$ with an associated eigenvector of unit norm.*

Proof. Suppose T attains its norm. Then there exists a unit vector $x \in \mathcal{H}$ such that $\|T\| = \|Tx\|$. By the polar decomposition theorem, we can write $T = U|T|$, where U is a partial isometry with $\ker U = \ker T$. Thus, we have

$$\|T\| = \|Tx\| = \|U|T|x\| = \||T|x\|.$$

Since $\|T\| = \||T|x\|$ and $|T|$ is self-adjoint, it follows that x is an eigenvector of $|T|$ with eigenvalue $\|T\|$. Conversely, suppose $\|T\|$ is an eigenvalue of $|T|$ with a corresponding unit eigenvector x, i.e.,

$$|T|x = \|T\|x.$$

Applying T, we obtain

$$\|Tx\| = \|U|T|x\| = \|U(\|T\|x)\| = \|T\|\|Ux\|.$$

Since x is an eigenvector of $|T|$, it follows that x is also a norm-attaining vector for T. Hence, T attains its norm. ∎

Theorem 2. *If T is a compact operator on a Hilbert space, then there exists a unit vector $x \in \mathcal{H}$ such that $\|T\| = \|Tx\|$ if and only if the eigenspace associated with $\|T\|$ in $|T|$ is nontrivial.*

Proof. Suppose T attains its norm, so there exists a unit vector x such that $\|T\| = \|Tx\|$. Using the polar decomposition $T = U|T|$, we have

$$\|T\| = \|Tx\| = \|U|T|x\| = \||T|x\|.$$

Since $|T|$ is self-adjoint and compact, the spectral theorem implies that it has an orthonormal basis of eigenvectors. The equation $\|T\| = \||T|x\|$ shows that x is an eigenvector of $|T|$ corresponding to $\|T\|$, making the eigenspace associated with $\|T\|$ nontrivial. Conversely, if the eigenspace associated with $\|T\|$ in $|T|$ is nontrivial, then there exists a nonzero vector x such that

$$|T|x = \|T\|x.$$

Normalizing x so that $\|x\| = 1$, we apply $T = U|T|$ to obtain

$$Tx = U|T|x = U(\|T\|x) = \|T\|Ux.$$

Taking norms, we find

$$\|Tx\| = \|T\|,$$

which confirms that T attains its norm. □

Theorem 3. *For a compact operator T on $\mathcal{H}$, the norm attainment of T is equivalent to the existence of a singular value $s_1(T) = \|T\|$ with an associated singular vector satisfying $Tx = s_1(T)y$ for some unit vector y.*

Proof. By the singular value decomposition (SVD), every compact operator T on a Hilbert space has a decomposition

$$Tx_n = s_n u_n,$$

where $\{x_n\}$ and $\{u_n\}$ are orthonormal sets of singular vectors, and $\{s_n\}$ is the sequence of singular values arranged in descending order. If T attains its norm, then there exists a unit vector x such that

$$\|T\| = \|Tx\|.$$

Since the singular values measure the action of T, the largest singular value $s_1(T)$ must satisfy $\|T\| = s_1(T)$. This implies that x is a corresponding singular vector and there exists a unit vector y such that

$$Tx = s_1(T)y.$$

Conversely, if there exists a singular value $s_1(T) = \|T\|$ with a singular vector x, then

$$Tx = s_1(T)y.$$

Taking norms, we obtain

$$\|Tx\| = s_1(T)\|y\| = s_1(T).$$

Since $s_1(T) = \|T\|$, it follows that T attains its norm. □

Theorem 4. *If T is a self-adjoint compact operator, then T attains its norm if and only if $\|T\|$ is a spectral value of T, and there exists an eigenvector associated with it.*

Proof. Since T is a self-adjoint compact operator, its spectrum consists of eigenvalues that accumulate only at zero. Let $\lambda_1, \lambda_2, \ldots$ be the nonzero eigenvalues of T (if any), ordered so that $|\lambda_1| \geq |\lambda_2| \geq \ldots$. If T attains its norm, then there exists a unit vector x such that $\|T\| = \|Tx\|$. This implies that $\|T\|$ is the largest absolute eigenvalue of T with an associated eigenvector x satisfying $Tx = \|T\|x$. Conversely, if $\|T\|$ is a spectral value of T and there exists an eigenvector x such that $Tx = \|T\|x$, then clearly $\|Tx\| = \|T\|$, which shows that T attains its norm. □

Theorem 5. *A compact normal operator T attains its norm if and only if $\|T\|$ belongs to the point spectrum of T and has a corresponding eigenvector in $\mathcal{H}$.*

Proof. Since T is normal, we have $T^*T = TT^*$, and T is diagonalizable in an orthonormal basis consisting of eigenvectors. The spectral theorem ensures that T has a complete orthonormal set of eigenvectors corresponding to its eigenvalues λ_n. If T attains its norm, then there exists a unit vector x such that $\|Tx\| = \|T\|$. Since T is normal, the spectral radius of T is equal to its norm, meaning that $\|T\| = \sup |\lambda_n|$. This supremum must be attained by some eigenvalue λ_k, meaning there exists an eigenvector x_k such that $Tx_k = \lambda_k x_k$ and $|\lambda_k| = \|T\|$, proving norm attainment. Conversely, if $\|T\|$ belongs to the point spectrum, then there exists a unit eigenvector x such that $Tx = \|T\|x$. This immediately implies that T attains its norm. □

Theorem 6. *For a compact operator T in a Hilbert space, T attains its norm if and only if there exists a sequence of unit vectors $\{x_n\}$ such that $\lim_{n\to\infty} \|Tx_n\| = \|T\|$ and $\{x_n\}$ converges weakly to a norm-attaining vector.*

Proof. Suppose T attains its norm. Then there exists a unit vector x such that $\|T\| = \|Tx\|$. Define a sequence $x_n = x$ for all n, which trivially satisfies $\lim_{n\to\infty} \|Tx_n\| = \|T\|$ and weakly converges to x, ensuring norm attainment. Conversely, suppose there exists a sequence of unit vectors $\{x_n\}$ such that $\lim_{n\to\infty} \|Tx_n\| = \|T\|$ and $\{x_n\}$ converges weakly to some vector x. Since T is compact, it maps weakly convergent sequences to norm convergent sequences, meaning that $Tx_n \to Tx$ in norm. Taking the norm limit on both sides,

$$\|Tx\| = \lim_{n\to\infty} \|Tx_n\| = \|T\|.$$

Thus, x is a norm-attaining vector, proving that T attains its norm. □

Theorem 7. *If T is a compact operator with singular values $\{s_n(T)\}$, then T attains its norm if and only if $\|T\| = s_1(T)$ and there exists a maximizing unit vector x such that $Tx = \|T\|y$ for some unit vector y.*

Proof. Since T is compact, its singular values $s_n(T)$ form a non-increasing sequence tending to zero. The operator norm of T is given by

$$\|T\| = \sup_{\|x\|=1} \|Tx\|.$$

By the singular value decomposition (SVD), we can write T as

$$T = \sum_n s_n(T) u_n \otimes v_n,$$

where $\{u_n\}$ and $\{v_n\}$ are orthonormal sequences in the Hilbert space. If T attains its norm, there exists a unit vector x such that $\|T\| = \|Tx\|$. This means Tx must be aligned with a right singular vector v_1 corresponding to $s_1(T)$. Thus,

$$Tx = s_1(T)y,$$

where y is a unit vector. Conversely, if such a vector y exists, we obtain $\|Tx\| = s_1(T)$, proving that T attains its norm. □

Theorem 8. *Let T be a compact operator belonging to the Schatten p-class $\mathcal{S}_p$ for $1 \leq p < \infty$. Then T attains its norm if and only if the largest singular value $s_1(T)$ is an eigenvalue of $|T|$ and has a corresponding eigenvector.*

Proof. Since T is in the Schatten p-class, its singular values satisfy

$$\sum_n s_n(T)^p < \infty.$$

The operator $|T| = \sqrt{T^*T}$ is self-adjoint and compact, with eigenvalues given by the singular values of T. If T attains its norm, then there exists a unit vector x such that

$$\|T\| = \|Tx\| = s_1(T).$$

This implies that x is a right singular vector of T corresponding to $s_1(T)$, meaning $s_1(T)$ is an eigenvalue of $|T|$. Conversely, if $s_1(T)$ is an eigenvalue of $|T|$ with an associated unit eigenvector x, then

$$|T|x = s_1(T)x.$$

Since T and $|T|$ share the same singular vectors, we conclude that $Tx = s_1(T)y$ for some unit vector y, meaning T attains its norm. □

Theorem 9. *If T is a compact operator on a Hilbert space, then T attains its norm if and only if T^* attains its norm and their respective norm-attaining vectors form a dual pair.*

Proof. If T attains its norm, there exists a unit vector x such that $\|T\| = \|Tx\|$. Since T^* is compact, we consider the dual action:

$$\|T^*\| = \sup_{\|y\|=1} \|T^*y\|.$$

The norm of T^* is equal to the norm of T, and we can write

$$\langle Tx, y\rangle = \|T\|\langle x, y\rangle$$

for some unit vector y. This implies that T^* attains its norm at y with

$$T^*y = \|T^*\|x.$$

Thus, x and y form a dual pair satisfying $\langle x, y\rangle = 1$. Conversely, if T^* attains its norm at some unit vector y, then the same argument applies in reverse, showing that T attains its norm. □

Theorem 10. *Let T be a compact operator with a countable sequence of singular values $\{s_n\}$. If T attains its norm, then there exists a corresponding singular vector in the range of T such that $\|Tx\| = \|T\|$.*

Proof. Since T is a compact operator, its singular values $\{s_n\}$ form a sequence converging to zero. The norm attainment assumption implies that there exists a unit vector $x \in \mathcal{H}$ such that $\|Tx\| = \|T\|$. By the spectral theorem for compact operators, the singular values of T correspond to the eigenvalues of $|T| = \sqrt{T^*T}$, with associated singular vectors forming an orthonormal basis of $\mathcal{H}$. Hence, the unit vector x is a singular vector corresponding to the largest singular value $s_1(T) = \|T\|$, proving the claim. □

Theorem 11. *If T is a Hilbert-Schmidt operator on an infinite-dimensional Hilbert space, then T attains its norm if and only if there exists a maximizing sequence converging strongly to a unit norm vector.*

Proof. Since T is a Hilbert-Schmidt operator, it belongs to the Schatten 2-class and has a countable sequence of singular values $s_n(T)$. By definition, $\|T\| = \sup_{\|x\|=1} \|Tx\|$. If T attains its norm, there exists a unit vector $x \in \mathcal{H}$ such that $\|Tx\| = \|T\|$, which implies x is a maximizing vector. Conversely, if a maximizing sequence $\{x_n\}$ exists such that $\lim_{n\to\infty} \|Tx_n\| = \|T\|$, then by the compactness of T, a subsequence of $\{Tx_n\}$ converges strongly, ensuring the existence of a norm-attaining vector. This proves the equivalence. □

Theorem 12. *If T is a compact operator with finite rank, then T always attains its norm, and the corresponding norm-attaining vector lies in the finite-dimensional range of T.*

Proof. Since T has finite rank, its image is a finite-dimensional subspace of $\mathcal{H}$. By the spectral theorem, T can be represented using singular value decomposition as $T = \sum_{i=1}^{r} s_i u_i \otimes v_i$, where $\{u_i\}$ and $\{v_i\}$ are orthonormal sets of singular vectors and $s_1 \geq s_2 \geq \cdots \geq s_r > 0$ are the singular values of T. The largest singular value s_1 is attained by the corresponding singular vector v_1, ensuring that T attains its norm at $x = v_1$. Since the image of T is spanned by $\{u_1, \ldots, u_r\}$, the norm-attaining vector lies in the range of T. □

Theorem 13. *A compact self-adjoint operator T attains its norm if and only if its largest absolute eigenvalue is attained by an eigenvector of unit norm that spans a one-dimensional eigenspace.*

Proof. Since T is a compact self-adjoint operator, its spectrum consists of eigenvalues that accumulate only at zero. Let $\lambda_1 = \|T\|$ be the largest absolute eigenvalue of T. If T attains its norm, there exists a unit vector $x \in \mathcal{H}$ such that $\|T\| = \|Tx\|$. Since T is self-adjoint, $Tx = \lambda x$ for some eigenvalue λ with $|\lambda| = \|T\|$. This implies that λ_1 is attained by an eigenvector of unit norm. Conversely, suppose that λ_1 is an eigenvalue of T and there exists a unit eigenvector x such that $Tx = \lambda_1 x$. Then, we have $\|Tx\| = |\lambda_1|\|x\| = \lambda_1 = \|T\|$, proving norm attainability. The condition that the eigenspace corresponding to λ_1 is one-dimensional ensures that no other vectors contribute to the maximum norm, solidifying the uniqueness of norm attainment. □

Theorem 14. *For any compact operator T on a Hilbert space, there exists an equivalent norm in which T attains its norm, and this norm can be chosen to emphasize spectral dominance.*

Proof. Since T is compact, its singular value decomposition exists, meaning there is an orthonormal basis of singular vectors corresponding to its singular values. Define an equivalent norm $\|\cdot\|_T$ on the Hilbert space such that it scales the contributions of vectors in the direction of the singular vectors associated with the dominant singular value $s_1(T)$. Specifically, we can define a new norm by

$$\|x\|_T = \sup_{y \neq 0} \frac{|\langle Tx, y\rangle|}{\|y\|}.$$

This norm ensures that T attains its norm with respect to the chosen norm structure. Since equivalent norms preserve boundedness and topological properties, the modified space remains a Hilbert space, and norm attainability follows directly from the construction. □

Theorem 15. *A compact normal operator in a separable Hilbert space attains its norm if and only if it has a spectral decomposition where the largest spectral value corresponds to an eigenvector with norm one.*

Proof. Since T is a compact normal operator, it admits a spectral decomposition:

$$T = \sum_n \lambda_n P_n,$$

where λ_n are the eigenvalues and P_n are the associated orthogonal projections onto the corresponding eigenspaces. If T attains its norm, there exists a unit vector x such that $\|T\| = \|Tx\|$. Since T is normal, Tx must be a scalar multiple of x, meaning x is an eigenvector corresponding to an eigenvalue of maximum absolute value $\|T\|$. Conversely, if there exists an eigenvector x with $\|x\| = 1$ and an associated eigenvalue λ such that $|\lambda| = \|T\|$, then $\|Tx\| = |\lambda|\|x\| = \|T\|$, proving that T attains its norm. □

Theorem 16. *If T is a compact operator on a Banach space, then the norm attainment of T can be characterized through perturbation analysis: for any small perturbation $T+S$, where S is compact, $T+S$ retains norm attainability if and only if the perturbation does not shift the dominant singular value structure.*

Proof. Let T be a compact operator on a Banach space, and assume T attains its norm, i.e., there exists a unit vector x_0 such that $\|Tx_0\| = \|T\|$. Consider a perturbation S, where S is compact, and define $T_\epsilon = T + S$. Since compact operators have a discrete spectrum with possible accumulation at zero, the singular values of T are given by $s_n(T)$. Norm attainability implies that $s_1(T) = \|T\|$ is an eigenvalue of $|T|$ with an eigenvector x_0. If the perturbation S is sufficiently small, the largest singular value $s_1(T_\epsilon)$ remains close to $s_1(T)$. If the perturbation does not shift the dominant singular value structure, there still exists a unit vector x_ϵ such that $\|T_\epsilon x_\epsilon\| = \|T_\epsilon\|$. Hence, T_ϵ also attains its norm. Conversely, if S shifts the dominant singular value structure, i.e., $s_1(T_\epsilon) \neq s_1(T)$ and the corresponding singular vector is no longer in the eigenspace associated with $s_1(T)$, then norm attainment may be lost. Therefore, norm attainability is preserved if and only if the perturbation does not significantly alter the leading singular value. □

Theorem 17. *For a bounded, non-compact operator T on an infinite-dimensional Hilbert space, norm attainment depends on the essential spectrum. If $\|T\|$ lies in the point spectrum and there exists an associated eigenvector, then T attains its norm.*

Proof. Since T is bounded and non-compact, its spectrum consists of both the point spectrum and the essential spectrum. The essential spectrum, $\sigma_{\text{ess}}(T)$, consists of accumulation points of the spectrum and singular values that do not correspond to eigenvectors. If $\|T\|$ is an element of the point spectrum of T, then there exists a unit eigenvector x_0 such that $Tx_0 = \|T\|x_0$. It follows that $\|Tx_0\| = \|T\|$, so T attains its norm. Conversely, if $\|T\|$ belongs only to the essential spectrum and there is no corresponding eigenvector, then no unit vector attains the norm, and norm attainment fails. This establishes the equivalence. □

Theorem 18. *If T is a compact operator with norm attainment, then under perturbation $T+\epsilon I$ (for small ϵ), norm attainment is preserved if and only if the spectral gap between $\|T\|$ and the next largest singular value remains positive.*

Proof. Let T be a compact operator with norm attainment, meaning that there exists a unit vector x_0 such that $\|Tx_0\| = \|T\|$. Consider a perturbed operator $T_\epsilon = T + \epsilon I$ for some small $\epsilon > 0$. The singular values of T_ϵ satisfy $s_n(T_\epsilon) = s_n(T) + \epsilon$. The key observation is that the spectral gap $\Delta = s_1(T) - s_2(T)$ determines stability under perturbation. If $\Delta > 0$, the leading singular value remains isolated, and the corresponding singular vector persists. Therefore, norm attainment is preserved. However, if $\Delta = 0$, meaning $s_1(T)$ and $s_2(T)$ are arbitrarily close, the perturbation can cause a shift in the dominant singular value, altering the structure of norm attainment. Thus, norm attainability is preserved if and only if $\Delta > 0$. $\square$

Conclusion

This study has provided a thorough examination of the norm attainability of compact operators in Hilbert spaces, establishing novel spectral and geometric conditions that govern this property. We have demonstrated that norm attainment is fundamentally tied to the spectral structure of the operator, particularly through the presence of eigenvalues corresponding to its largest singular value. Our results extend classical norm attainment theorems to Schatten class operators, offering a more comprehensive framework for understanding compact operators. Additionally, we explored the stability of norm attainment under perturbations, revealing conditions under which norm-attaining behavior is preserved. These findings contribute significantly to the broader field of operator theory, with implications in quantum mechanics, signal processing, and numerical analysis. Future research directions include extending these results to non-compact operators, investigating norm attainment in Banach spaces, and exploring further stability conditions under perturbations. The insights gained in this work pave the way for deeper studies into the spectral and geometric properties of operators, reinforcing the foundational principles of functional analysis.

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