Anticenter-Symmetric Bialgebras

Abstract

This paper develops a bialgebra theory for anticenter-symmetric algebras by introducing the concept of an *anticenter-symmetric bialgebra*, equivalent to a Manin triple of anticenter-symmetric algebras. A study of this framework leads to the *anticenter-symmetric Yang-Baxter equation* in anticenter-symmetric algebras, analogous to the classical Yang-Baxter equation in Mock Lie algebras and the associative Yang-Baxter equation.

An unexpected finding is that the anticenter-symmetric and associative Yang-Baxter equations share the same form. Additionally, skew-symmetric solutions to the anticenter-symmetric Yang-Baxter equation define anticenter-symmetric bialgebras. To advance the theory, the paper introduces $\mathcal{O}$*-operators* and *pre-anticenter-symmetric algebras*, which facilitate the construction of these solutions and provide a foundation for further exploration.

Keywords. Anticenter-Symmetric Algebras, Pre-Anticenter-Symmetric Algebras, Matched Pairs, Manin Triples, Bialgebras, Yang-Baxter Equation and $\mathcal{O}$-operators

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1 Introduction

Mock-Lie algebras are commutative algebras characterized by their adherence to the Jacobi identity, with significant contributions to their study made by P. Zusmanovich in [14]. These algebras have appeared under various names, reflecting diverse mathematical perspectives.

Their earliest mention was in [12], where an infinite-dimensional solvable but non-nilpotent example was introduced, later reproduced in [13]. They are also referred to as "Jordan algebras of nil index 3" in Jordan-algebraic literature, "Lie–Jordan algebras" in [11], and "Jacobi–Jordan algebras" in recent studies [6] and [1]. The term "mock-Lie" originates from [9], where the operad appears in a classification of quadratic cyclic operads. They possess two particularly noteworthy features:

(a) Algebras associated with the Koszul dual of the Mock-Lie operad can be equivalently characterized in three distinct ways, as detailed in [14] and [7].

(b) As observed in [11], Mock-Lie algebras can also be constructed from antiassociative algebras, paralleling their derivation from associative algebras. This underscores a profound relationship between Mock-Lie and antiassociative algebras.

Significant progress has been made in understanding the cohomology and deformation theories of Mock-Lie algebras. A notable development is the introduction of a cohomology framework based on two operators, referred to as *zigzag cohomology*, which was explored in [4] alongside a detailed examination of low-degree cohomology spaces. Furthermore, [5] investigated Mock-Lie bialgebras, the Yang-Baxter equation, and Manin triples, broadening the algebraic and structural insights into these algebras. The study of Lie-admissible algebras has been of great significance, particularly the bialgebraic exploration of left-symmetric algebras as detailed in [2]. More recently, anti-flexible algebras, also known as center-symmetric algebras, have emerged as another class of

Lie-admissible algebras, with their bialgebraic properties investigated by [8]. In addition, we have recently introduced the concept of anticenter-symmetric Jacobi-Jordan algebras, which we refer to more succinctly as anticenter-symmetric algebras [10]. These algebras belong to the category of Mock-Lie admissible algebras.

The primary aim of our paper is to undertake an algebraic study of these structures; we establishe a bialgebra theory for anti-center-symmetric algebras by defining the concept of an *anticenter-symmetric bialgebra*, linked to a Manin triple of such algebras. This framework introduces the *anticenter-symmetric Yang-Baxter equation*, paralleling the classical Yang-Baxter equation in Mock Lie algebras and the associative Yang-Baxter equation. Remarkably, the anticenter-symmetric and associative Yang-Baxter equations share the same form. Skew-symmetric solutions to the former directly define anticenter-symmetric bialgebras. To support this theory, we introduce $\mathcal{O}$*-operators* and *pre-anticenter-symmetric algebras*, providing tools for constructing solutions.

The paper begins in Section 2 with a review of the bimodules and matched pairs of anti-center-symmetric algebras. Section 3 then focuses on the Manin triple of anti-center-symmetric algebras, providing a deeper understanding of their bialgebraic structural aspects. Section 4 explores a special class of anticenter-symmetric bialgebras, this leads to anticenter-symmetric Yang-Baxter equation.

Section 5 develops the theory of $\mathcal{O}$-operators of anticenter-symmetric algebras and pre-anticenter-symmetric algebras. Finally, Section 6 concludes the paper with reflective remarks that summarize the findings.

2 Bimodules and matched pairs of anticenter-symmetric algebras

Definition 2.1 *[10] $(\mathcal{A}, \cdot)$, is said to be an anticenter-symmetric algebra if $\forall\, x, y, z \in \mathcal{A}$, the antiassociator of the bilinear product $\cdot$ defined by $(\ x,\ y,\ z\ )_{-1} := (\ x \cdot y\ ) \cdot z + x \cdot\ (\ y \cdot z\ )$, is symmetric in x and z, i.e.,*

$$(\ x,\ y,\ z\ )_{-1} = -(\ z,\ y,\ x\ )_{-1}. \tag{2.1}$$

As matter of notation simplification, we will denote $x \cdot y$ by xy if not any confusion.

Definition 2.2 *[10] Let $\mathcal{A}$ be an anticenter-symmetric algebra, V be a vector space. Suppose $l, r : \mathcal{A} \to \mathfrak{gl}(V)$ be two linear maps satisfying: for all $x, y \in \mathcal{A}$,*

$$[l_x, r_y] = -\,[l_y, r_x] \tag{2.2}$$

$$l_{xy} + l_x l_y = -r_{yx} - r_x r_y. \tag{2.3}$$

Then, (l, r, V) (or simply (l, r)) is called bimodule of the anticenter-symmetric algebra $\mathcal{A}$.

Let $(\mathcal{A}, \cdot)$ be an anticenter-symmetric algebra. For any $x, y \in \mathcal{A}$, let L_x and R_x denote the left and right multiplication operators respectively, that is, $L_x(y) = xy$ and $R_x(y) = yx$. Let $L, R : \mathcal{A} \to \mathrm{End}(\mathcal{A})$ be two linear maps with $x \to L_x$ and $x \to R_x$ for any $x \in \mathcal{A}$ respectively.

Example 2.3 *Let $(\mathcal{A}, \cdot)$ be an antisymmetric algebra. Then $(L, R, \mathcal{A})$ is a bimodule of $(\mathcal{A}, \cdot)$, which is called the* **regular bimodule of** *$(\mathcal{A}, \cdot)$.*

Proposition 2.4 *Let $(\mathcal{A}, \cdot)$ be an anticenter-symmetric algebra and V be a vector space over $\mathbb{K}$. Consider two linear maps, $l, r : \mathcal{A} \to \mathfrak{gl}(V)$. Then, (l, r, V) is a bimodule of $\mathcal{A}$ if and only if, the semi-direct sum $\mathcal{A} \oplus V$ of vector spaces is turned into an anticenter-symmetric algebra by defining the multiplication in $\mathcal{A} \oplus V$ by $\forall x_1, x_2 \in \mathcal{A},\ v_1, v_2 \in V$,*

$$(x_1 + v_1) * (x_2 + v_2) = x_1 \cdot x_2 + (l_{x_1} v_2 + r_{x_2} v_1),$$

We denote it by $\mathcal{A} \ltimes^{-1}_{l,r} V$ or simply $\mathcal{A} \ltimes^{-1} V$.

It is known that an anticenter-symmetric algebra is a Mock Lie-admissible algebra ([10]).

Proposition 2.5 *Let $(\mathcal{A}, \cdot)$ be an anticenter-symmetric algebra. Define the anticommutator by*

$$[x, y] = x \cdot y + y \cdot x, \quad \forall x, y \in A. \tag{2.4}$$

Then it is a Mock Lie algebra and we denote it by $(\mathcal{G}(\mathcal{A}), [\ ,\ ])$ or simply $\mathcal{G}(\mathcal{A})$, which is called **the the sub-adjacent Mock Lie algebra of** $(\mathcal{A}, \cdot)$.

Corollary 2.6 *Let $(\mathcal{A}, \cdot)$ be an anticenter-symmetric algebra and V be a vector space over $\mathbb{K}$. Consider two linear maps, $l, r : \mathcal{A} \to \mathfrak{gl}(V)$, such that (l, r, V) is a bimodule of $\mathcal{A}$. Then, the map: $l + r : \mathcal{A} \longrightarrow \mathfrak{gl}(V)\ x \longmapsto l_x + r_x$, is a linear representation of the sub-adjacent Mock Lie algebra of $\mathcal{A}$.*

Proof: Let (l, r, V) be a bimodule of the anticenter-symmetric algebra $\mathcal{A}$. Then, $\forall x, y \in \mathcal{A}$ $[l_x, r_y] = -[l_y, r_x]; l_{xy} + l_x l_y = -r_x r_y - r_{yx}$. Besides, it is a matter of straightforward computation to show that $l + r$ is a linear map on $\mathcal{A}$. Then, we have:

$$\begin{aligned}
[(l + r)(x), (l + r)(y)] &= [l_x + r_x, l_y + r_y] \\
&= [l_x, l_y] + [l_x, r_y] + [r_x, l_y] + [r_x, r_y] \\
&= [l_x, l_y] + [r_x, r_y] \\
&= l_x l_y + l_y l_x + r_x r_y + r_y r_x \\
&= \{l_x l_y + r_x r_y\} + \{l_y l_x + r_y r_x\} \\
&= \{l_{xy} + r_{yx}\} + \{l_{yx} + r_{xy}\} \\
&= (l + r)_{xy} + (l + r)_{yx} = (l + r)_{[x,r]}.
\end{aligned}$$

Therefore, (l, r, V) is a bimodule of $\mathcal{A}$ implies that $l + r$ is a representation of the linear representation of the sub-adjacent Mock Lie algebra of $\mathcal{A}$. □

Theorem 2.7 *[10] Let $(\mathcal{A}, \cdot)$ and $(\mathcal{B}, \circ)$ be two anticenter-symmetric algebras. Suppose that $(l_{\mathcal{A}}, r_{\mathcal{A}}, \mathcal{B})$ and $(l_{\mathcal{B}}, r_{\mathcal{B}}, \mathcal{A})$ are bimodules of $\mathcal{A}$ and $\mathcal{B}$, respectively, obeying the relations:*

$$\begin{aligned}
&r_{\mathcal{A}}(x)(a \circ b) + r_{\mathcal{A}}(l_{\mathcal{B}}(b)x)a + a \circ (r_{\mathcal{A}}(x)b) \\
&+ l_{\mathcal{A}}(r_{\mathcal{B}}(b)x)a + (l_{\mathcal{A}}(x)b) \circ a + l_{\mathcal{A}}(x)(b \circ a) = 0,
\end{aligned} \tag{2.5}$$

$$\begin{aligned}
&r_{\mathcal{B}}(a)(x \cdot y) + r_{\mathcal{B}}(l_{\mathcal{A}}(y)a)x + x \cdot (r_{\mathcal{B}}(a)y) \\
&+ l_{\mathcal{B}}(r_{\mathcal{A}}(y)a)x + (l_{\mathcal{B}}(a)y) \cdot x + l_{\mathcal{B}}(a)(y \cdot x) = 0,
\end{aligned} \tag{2.6}$$

$$\begin{aligned}
&a \circ (l_{\mathcal{A}}(x)b) + (r_{\mathcal{A}}(x)b) \circ a + (r_{\mathcal{A}}(x)a) \circ b + l_{\mathcal{A}}(l_{\mathcal{B}}(a)x)b \\
&+ r_{\mathcal{A}}(r_{\mathcal{B}}(b)x)a + l_{\mathcal{A}}(l_{\mathcal{B}}(b)x)a + b \circ (l_{\mathcal{A}}(x)a) + r_{\mathcal{A}}(r_{\mathcal{B}}(a)x)b = 0,
\end{aligned} \tag{2.7}$$

$$\begin{aligned}
&x \cdot (l_{\mathcal{B}}(a)y) + (r_{\mathcal{B}}(a)y) \cdot x + (r_{\mathcal{B}}(a)x) \cdot y + l_{\mathcal{B}}(l_{\mathcal{A}}(x)a)y \\
&+ r_{\mathcal{B}}(r_{\mathcal{A}}(y)a)x + l_{\mathcal{B}}(l_{\mathcal{A}}(y)a)x + y \cdot (l_{\mathcal{B}}(a)x) + r_{\mathcal{B}}(r_{\mathcal{A}}(x)a)y = 0,
\end{aligned} \tag{2.8}$$

for all $x, y \in A$ and $a, b \in \mathcal{B}$. Then, there is an anticenter-symmetric algebra structure on $\mathcal{A} \oplus \mathcal{B}$ given by:

$$\begin{aligned}
(x + a) * (y + b) &= (x \cdot y + l_{\mathcal{B}}(a)y + r_{\mathcal{B}}(b)x) \\
&+ (a \circ b + l_{\mathcal{A}}(x)b + r_{\mathcal{A}}(y)a).
\end{aligned} \tag{2.9}$$

We denote this anticenter-symmetric algebra by $\mathcal{A} \bowtie^{-1, l_{\mathcal{A}}, r_{\mathcal{A}}}_{l_{\mathcal{B}}, r_{\mathcal{B}}} \mathcal{B}$, or simply by $\mathcal{A} \bowtie^{-1} \mathcal{B}$. Then $(\mathcal{A}, \mathcal{B}, l_{\mathcal{A}}, r_{\mathcal{A}}, l_{\mathcal{B}}, r_{\mathcal{B}})$ satisfying the above conditions is called matched pair of the anticenter-symmetric algebras $\mathcal{A}$ and $\mathcal{B}$.

Definition 2.8 *Let (l, r, V) be a bimodule of an anticenter-symmetric algebra $\mathcal{A}$, where V is a finite dimensional vector space. The dual maps l^*, r^* of the linear maps l, r, are defined, respectively, as: $l^*, r^* : \mathcal{A} \to \mathfrak{gl}(V^*)$ such that: for all $x \in \mathcal{A}, u^* \in V^*, v \in V$,*

$$\begin{array}{rcl} l^* : \mathcal{A} & \longrightarrow & \mathfrak{gl}(V^*) \\ x & \longmapsto & l^*_x : \begin{array}{rcl} V^* & \longrightarrow & V^* \\ u^* & \longmapsto & l^*_x u^* : \begin{array}{rcl} V & \longrightarrow & \mathbb{K} \\ v & \longmapsto & \langle l^*_x u^*, v\rangle := \langle u^*, l_x v\rangle , \end{array} \end{array} \end{array} \tag{2.10}$$

$$\begin{array}{rcl} r^* : \mathcal{A} & \longrightarrow & \mathfrak{gl}(V^*) \\ x & \longmapsto & r^*_x : \begin{array}{rcl} V^* & \longrightarrow & V^* \\ u^* & \longmapsto & r^*_x u^* : \begin{array}{rcl} V & \longrightarrow & \mathbb{K} \\ v & \longmapsto & \langle r^*_x u^*, v\rangle := \langle u^*, r_x v\rangle . \end{array} \end{array} \end{array} \tag{2.11}$$

Proposition 2.9 *Let $(\mathcal{A}, \cdot)$ be an anticenter-symmetric algebra and $l, r : \mathcal{A} \to \mathfrak{gl}(V)$ be two linear maps, where V is a finite dimensional vector space. The following conditions are equivalent:*

1. *(l, r, V) is a bimodule of $\mathcal{A}$.*
2. *(r^*, l^*, V^*) is a bimodule of $\mathcal{A}$.*

Proof:

(1)⇒(2) Suppose that (l, r, V) is a bimodule of $(\mathcal{A}, \cdot)$ and show that (r^*, l^*, V^*) is also a bimodule of $(\mathcal{A}, \cdot)$. We have:

- $$\begin{array}{rl} & \left\langle (r^*_{xy} + r^*_x r^*_y)u^*, v\right\rangle \\ = & \left\langle r^*_{xy}u^*, v\right\rangle + \left\langle (r^*_x r^*_y)u^*, v\right\rangle = \langle r_{xy}(v), u^*\rangle + \langle r_y(r_x(v)), u^*\rangle \\ = & \langle (r_{xy} + r_y r_x)(v), u^*\rangle \; = \langle -(l_{yx} + l_y l_x)(v), u^*\rangle \\ = & -\langle l_{yx}(v), u^*\rangle - \langle (l_y l_x)(v), u^*\rangle \\ = & -\left\langle l^*_{yx}u^*, v\right\rangle - \left\langle (l^*_x l^*_y)u^*, v\right\rangle \\ = & \left\langle -(l^*_{yx} + l^*_x l^*_y)u^*, v\right\rangle . \end{array}$$

  Therefore,

  $$l^*_{yx} + l^*_x l^*_y = -r^*_{xy} - r^*_x r^*_y, \; \forall \, x, y \; \mathcal{A} \tag{2.12}$$

- $$\begin{array}{rl} & \left\langle [l^*_x, r^*_y]u^*, v\right\rangle \\ = & \left\langle l^*_x(r^*_y)u^*, v\right\rangle + \left\langle r^*_y(l^*_x)u^*, v\right\rangle = \left\langle l_x(v), r^*_y u^*\right\rangle + \langle r_y v, l^*_x u^*\rangle \\ = & \langle r_y(l_x(v)), u^*\rangle + \langle l_x(r_y(v)), u^*\rangle = \langle [r_y, l_x]v, u^*\rangle \\ = & \langle -[r_x, l_y]v, u^*\rangle = \langle -(r_x(l_y) + l_y(r_x))v, u^*\rangle \\ = & \left\langle -(l^*_y r^*_x + r^*_x l^*_y)u^*, v\right\rangle = \left\langle -[l^*_y, r^*_x]u^*, v\right\rangle \end{array}$$

  Therefore

  $$[l^*_x, r^*_y] = -[l^*_y, r^*_x], \; \forall \, x, y \; \in \mathcal{A}. \tag{2.13}$$

By considering the relations (2.12) and (2.13), we conclude that (r^*, l^*, V) is a bimodule of $(\mathcal{A}, \cdot)$.

(2)⇒(1) The converse, (i.e., by supposing that (r^*, l^*, V) is a bimodule of $(\mathcal{A}, \cdot)$ then (l, r, V) is also a bimodule of $(\mathcal{A}, \cdot)$), can be proved by direct calculations by using similar relations as for the first part of the proof.

□

3 Manin triple of anticenter-symmetric algebras

In this section, we first give the definition of Manin triple of an anticenter-symmetric algebra and investigate its main properties.

Definition 3.1 *A Manin triple of anticenter-symmetric algebras is a triple* $(\mathcal{A}, \mathcal{A}^+, \mathcal{A}^-)$ *equipped with a nondegenerate symmetric bilinear form* $\mathfrak{B}(\ ,\ )$ *on* $\mathcal{A}$ *which is invariant, i.e.,* $\forall x, y, z \in \mathcal{A}$, $\mathfrak{B}(x * y, z) = \mathfrak{B}(x, y * z)$, *satisfying:*

1. $\mathcal{A} = \mathcal{A}^+ \oplus \mathcal{A}^-$ *as* $\mathbb{K}$*-vector space;*
2. $\mathcal{A}^+$ *and* $\mathcal{A}^-$ *are anticenter-symmetric subalgebras of* $\mathcal{A}$*;*
3. $\mathcal{A}^+$ *and* $\mathcal{A}^-$ *are isotropic with respect to* $\mathfrak{B}(,)$*, that is* $\mathfrak{B}(\mathcal{A}^+; \mathcal{A}^+) = \mathfrak{B}(\mathcal{A}^-; \mathcal{A}^-) = 0$.

Definition 3.2 *Two Manin triples* $(\mathcal{A}_1, \mathcal{A}_1^+, \mathcal{A}_1^-, \mathfrak{B}_1)$ *and* $(\mathcal{A}_2, \mathcal{A}_2^+, \mathcal{A}_2^-, \mathfrak{B}_2)$ *of anticenter-symmetric algebras* $\mathcal{A}_1$ *and* $\mathcal{A}_2$ *are homomorphic (isomorphic) if there is a homomorphism (isomorphism)* $\varphi : \mathcal{A}_1 \to \mathcal{A}_2$ *such that:* $\varphi(\mathcal{A}_1^+) \subset \mathcal{A}_2^+$, $\varphi(\mathcal{A}_1^-) \subset \mathcal{A}_2^-$, $\mathfrak{B}_1(x, y) = \mathfrak{B}_2(\varphi(x), \varphi(y))$.

In particular, if $(\mathcal{A}, \cdot)$ is an anticenter-symmetric algebra, and if there exists an anticenter-symmetric algebra structure on its dual space $\mathcal{A}^*$ denoted $(\mathcal{A}^*, \circ)$, then there is a anticenter-symmetric algebra structure on the direct sum of the underlying vector spaces of $\mathcal{A}$ and $\mathcal{A}^*$ (see Theorem 2.7) such that $(\mathcal{A} \oplus \mathcal{A}^*, \mathcal{A}, \mathcal{A}^*)$ is the associated Manin triple with the invariant bilinear symmetric form given by

$$\mathfrak{B}_d(x + a^*, y + b^*) =< x, b^* > + < y, a^* >,\ \forall x, y \in \mathcal{A}; a^*, b^* \in \mathcal{A}^*, \tag{3.1}$$

called the standard Manin triple of the anticenter-symmetric algebra $\mathcal{A}$.

Theorem 3.3 *Let* $(\mathcal{A}, \cdot)$ *and* $(\mathcal{A}^*, \circ)$ *be two anticenter-symmetric algebras. Then, the sixtuple* $(\mathcal{A}, \mathcal{A}^*, R^*_\cdot, L^*_\cdot; R^*_\circ, L^*_\circ)$ *is a matched pair of anticenter-symmetric algebras* $\mathcal{A}$ *and* $\mathcal{A}^*$ *if and only if* $(\mathcal{A} \oplus \mathcal{A}^*, \mathcal{A}, \mathcal{A}^*)$ *is their standard Manin triple.*

Proof:

By considering that $(\mathcal{A}, \mathcal{A}^*, R^*_\cdot, L^*_\cdot; R^*_\circ, L^*_\circ)$ is a matched pair of anticenter-symmetric algebras, it follows that the bilinear product $*$ defined in the Theorem 2.7 is anticenter-symmetric on the direct sum of underlying vectors spaces, $\mathcal{A} \oplus \mathcal{A}^*$.

We have $\forall x, y, z \in \mathcal{A}; a, b, c \in \mathcal{A}^*$.

$$\begin{array}{rcl}
\mathfrak{B}_d((x + a) * (y + b), z + c) &=& \langle xy + R^*_\circ(a)y + L^*_\circ(b)x, c\rangle + \langle z, a \circ b + R^*_\cdot(x)b + L^*_\cdot(y)a\rangle \\
&=& \langle xy, c\rangle + \langle R^*_\circ(a)y, c\rangle + \langle L^*_\circ(b)x, c\rangle + \langle z, a \circ b\rangle + \langle z, R^*_\cdot(x)b\rangle \\
&+& \langle z, L^*_\cdot(y)a\rangle = \langle xy, c\rangle + \langle y, R_a(c)\rangle + \langle x, L_b(c)\rangle + \langle z, a \circ b\rangle \\
&+& \langle R_x(z), b\rangle + \langle L_y(z), a\rangle = \langle xy, c\rangle + \langle y, c \circ a\rangle \\
&+& \langle x, b \circ c\rangle + \langle z, a \circ b\rangle + \langle zx, b\rangle + \langle yz, a\rangle .
\end{array}$$

$$\begin{array}{rcl}
\mathfrak{B}_d\left((x + a), (y + b) * (z + c)\right) &=& \langle x, b \circ c + R^*_\cdot(y)c + L^*_\cdot(z)b\rangle + < yz + R^*_\circ(b)z \\
&+& L^*_\circ(c)y, a > + \langle x, b \circ c\rangle + \langle x, R^*_\cdot(y)c\rangle + \langle x, L^*_\cdot(z)b\rangle \\
&+& \langle yz, a\rangle + \langle R^*_\circ(b)z, a\rangle + \langle L^*_\circ(c)y, a\rangle \\
&=& \langle x, b \circ c\rangle + \langle R_y(x), c\rangle + \langle L_z(x), b\rangle \\
&+& \langle yz, a\rangle + \langle z, R_b(a)\rangle + \langle y, L_c(a)\rangle \\
&=& \langle x, b \circ c\rangle + \langle xy, c\rangle + \langle zx, b\rangle + \langle yz, a\rangle \\
&+& \langle z, a \circ b\rangle + \langle y, c \circ a\rangle .
\end{array}$$

Therefore, the following relation

$$\mathfrak{B}_d((x + a) * (y + b), (z + c)) = \mathfrak{B}_d\left((x + a), (y + b) * (z + c)\right) \tag{3.2}$$

holds, which expresses the invariance of the standard bilinear form on $\mathcal{A} \oplus \mathcal{A}^*$. Therefore, $(\mathcal{A} \oplus \mathcal{A}^*, \mathcal{A}, \mathcal{A}^*)$ is the standard Manin triple of the anticenter-symmetric algebras $\mathcal{A}$ and $\mathcal{A}^*$. □

Proposition 3.4 *Let* $(\mathcal{A},\cdot)$ *be an anticenter-symmetric algebra. Suppose that there exists an anticenter-symmetric algebra structure "*$\circ$*" on the dual space* $\mathcal{A}^*$.
Then, $(\mathcal{A},\mathcal{A}^*,R_\cdot^*,L_\cdot^*,R_\circ^*,L_\circ^*)$ *is a matched pair of anticenter-symmetric algebras if and only if for any* $x,y\in\mathcal{A},a\in\mathcal{A}^*$,

$$R_\circ^*(a)(x\cdot y)+L_\circ^*(a)(y\cdot x)+L_\circ^*(R_\cdot^*(x)a)y+y\cdot(L_\circ^*(a)x)+R_\circ^*(L_\cdot^*(x)a)y+(R_\circ^*(a)x)\cdot y=0, \tag{3.3}$$

$$\begin{aligned} &y\cdot(R_\circ^*(a)x)+x\cdot(R_\circ^*(a)y)+(L_\circ^*(a)x)\cdot y+(L_\circ^*(a)y)\cdot x\\ &+L_\circ^*(L_\cdot^*(x)a)y+R_\circ^*(R_\cdot^*(y)a)x+R_\circ^*(R_\cdot^*(x)a)y+L_\circ^*(L_\cdot^*(y)a)x=0. \end{aligned} \tag{3.4}$$

Proof: Obviously, Eq. (3.3) is exactly Eq. (2.6) and Eq. (3.4) is exactly Eq. (2.8) in the case $l_A=R_\cdot^*,r_A=L_\cdot^*$, $l_B=l_{A^*}=R_\circ^*,r_B=r_{A^*}=L_\circ^*$. For any $x,y\in A,a,b\in A^*$, we have:

$$\begin{aligned} &\langle R_\circ^*(a)(x\cdot y),b\rangle=\langle x\cdot y,R_\circ(a)b\rangle=\langle x\cdot y,b\circ a\rangle=\langle L_\cdot(x)y,b\circ a\rangle=\langle y,L_\cdot^*(x)(b\circ a)\rangle\,;\\ &\langle L_\circ^*(a)(y\cdot x),b\rangle=\langle y\cdot x,L_\circ(a)b\rangle=\langle y\cdot x,a\circ b\rangle=\langle R_\cdot(x)y,a\circ b\rangle=\langle y,R_\cdot^*(x)(a\circ b)\rangle\,;\\ &\langle L_\circ^*(R_\cdot^*(x)a)y,b\rangle=\langle y,L_\circ(R_\cdot^*(x)a)b\rangle=\langle y,(R_\cdot^*(x)a)\circ b\rangle\,;\\ &\langle y\cdot(L_\circ^*(a)x),b\rangle=\langle R_\cdot(L_\circ^*(a)x)y,b\rangle=\langle y,R_\cdot^*(L_\circ^*(a)x)b\rangle\,;\\ &\langle R_\circ^*(L_\cdot^*(x)a)y,b\rangle=\langle y,R_\circ(L_\cdot^*(x)a)b\rangle=\langle y,b\circ(L_\cdot^*(x)a)\rangle\,;\\ &\langle (R_\circ^*(a)x)\cdot y,b\rangle=\langle L_\cdot(R_\circ^*(a)x)y,b\rangle=\langle y,L_\cdot^*(R_\circ^*(a)x)b\rangle\,. \end{aligned}$$

Then Eq. (2.5) holds if and only if Eq. (2.6) holds. Similarly, Eq. (2.7) holds if and only if Eq. (2.8) holds. Therefore the conclusion holds. □

Let V be a vector space. Let $\sigma:V\otimes V\to V\otimes V$ be the *flip* defined as

$$\sigma(x\otimes y)=y\otimes x,\quad \forall x,y\in V. \tag{3.5}$$

Theorem 3.5 *Let* $(\mathcal{A},\cdot)$ *be an anticenter-symmetric algebra. Suppose there is an anticenter-symmetric algebra structure "*$\circ$*" on its dual space* $\mathcal{A}^*$ *given by a linear map* $\Delta^*:\mathcal{A}^*\otimes\mathcal{A}^*\to\mathcal{A}^*$. *Then* $(\mathcal{A},\mathcal{A}^*,R_\cdot^*,L_\cdot^*,R_\circ^*,L_\circ^*)$ *is a matched pair of anticenter-symmetric algebras if and only if* $\Delta:A\to\mathcal{A}\otimes\mathcal{A}$ *satisfies the following two conditions:*

$$\Delta(x\cdot y)+\sigma\Delta(y\cdot x)=-(\sigma(\mathrm{id}\otimes L_\cdot(y))+R_\cdot(y)\otimes\mathrm{id})\Delta(x)-(\sigma(R_\cdot(x)\otimes\mathrm{id})+\mathrm{id}\otimes L_\cdot(x))\Delta(y), \tag{3.6}$$

$$\begin{aligned} &(\sigma(\mathrm{id}\otimes R_\cdot(y))+\mathrm{id}\otimes R_\cdot(y)+\sigma(L_\cdot(y)\otimes\mathrm{id})+L_\cdot(y)\otimes\mathrm{id})\Delta(x)=\\ &(-\sigma(\mathrm{id}\otimes R_\cdot(x))-\mathrm{id}\otimes R_\cdot(x)-\sigma(L_\cdot(x)\otimes\mathrm{id})-L_\cdot(x)\otimes\mathrm{id})\Delta(y), \end{aligned} \tag{3.7}$$

for any $x,y\in\mathcal{A}$.

Proof: For any $x,y\in\mathcal{A}$ and any $a,b\in\mathcal{A}^*$, we have

$$\begin{aligned} &\langle\Delta(x\cdot y),a\otimes b\rangle=\langle x\cdot y,a\cdot b\rangle,=\langle L_\circ^*(a)(x\cdot y),b\rangle,\\ &\langle\sigma\Delta(y\cdot x),a\otimes b\rangle=\langle y\cdot x,b\circ a\rangle=\langle R_\circ^*(a)(y\cdot x),b\rangle,\\ &\langle\sigma(\mathrm{id}\otimes L_\cdot(y))\Delta(x),a\otimes b\rangle=\langle x,b\circ(L_\cdot^*(y)a)\rangle=\langle R_\circ^*(L_\cdot^*(y)a)x,b\rangle,\\ &\langle(R_\cdot(y)\otimes\mathrm{id})\Delta(x),a\otimes b\rangle=\langle x,(R_\cdot^*(y)a)\circ b\rangle=\langle L_\circ^*(R_\cdot^*(y)a)x,b\rangle,\\ &\langle\sigma(R_\cdot(x)\otimes\mathrm{id})\Delta(y),a\otimes b\rangle=\langle y,(R_\cdot^*(x)b)\circ a\rangle=\langle(R_\circ^*(a)y)\cdot x,b\rangle,\\ &\langle(\mathrm{id}\otimes L_\cdot(x))\Delta(y),a\otimes b\rangle=\langle y,a\circ(L_\cdot^*(x)b)\rangle=\langle x\cdot(L_\circ^*(a)y),b\rangle. \end{aligned}$$

Then Eq. (3.3) is equivalent to Eq. (3.6). Moreover, we have

$$\begin{aligned} &\langle\sigma(\mathrm{id}\otimes R_\cdot(y))\Delta(x),a\otimes b\rangle=\langle x,b\circ(R_\cdot^*(y)a)\rangle=\langle R_\circ^*(R_\cdot^*(y)a)x,b\rangle,\\ &\langle(\mathrm{id}\otimes R_\cdot(y))\Delta(x),a\otimes b\rangle=\langle x,a\circ(R_\cdot^*(y)b)\rangle=\langle(L_\circ^*(a)x)\cdot y,b\rangle,\\ &\langle\sigma(L_\cdot(y)\otimes\mathrm{id})\Delta(x),a\otimes b\rangle=\langle x,(L_\cdot^*(y)b)\circ a\rangle=\langle y\cdot(R_\circ^*(a)x),b\rangle,\\ &\langle(L_\cdot(y)\otimes\mathrm{id})\Delta(x),a\otimes b\rangle=\langle x,(L_\cdot^*(y)a)\circ b\rangle=\langle L_\circ^*(L_\cdot^*(y)a)x,b\rangle. \end{aligned}$$

Then Eq. (3.4) is equivalent to Eq. (3.7). Hence the conclusion holds. □

Remark 3.6 *From the symmetry of the anticenter-symmetric algebras $(\mathcal{A}, \cdot)$ and $(\mathcal{A}^*, \circ)$ in the standard Manin triple of anticenter-symmetric algebras associated to $\mathfrak{B}_d$, we also can consider a linear map $\gamma : \mathcal{A}^* \to \mathcal{A}^* \otimes \mathcal{A}^*$ such that $\gamma^* : \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$ gives the anticenter-symmetric algebra structure "$\cdot$" on $\mathcal{A}$. It is straightforward to show that Δ satisfies Eqs.* (3.6) *and* (3.7) *if and only if γ satisfies*

$$\gamma(a \circ b) + \sigma\gamma(b \circ a) = (\sigma(\mathrm{id} \otimes L_\circ(b)) + R_\circ(b) \otimes \mathrm{id})\gamma(a) + (\sigma(R_\circ(a) \otimes \mathrm{id}) + \mathrm{id} \otimes L_\circ(a))\gamma(b), \quad (3.8)$$

$$\begin{aligned}&(\sigma(\mathrm{id} \otimes R_\circ(b)) + \mathrm{id} \otimes R_\circ(b) + \sigma(L_\circ(b) \otimes \mathrm{id}) + (L_\circ(b) \otimes \mathrm{id}))\gamma(a) + \\ &((L_\circ(a) \otimes \mathrm{id}) + \sigma(L_\circ(a) \otimes \mathrm{id}) + \sigma(\mathrm{id} \otimes R_\circ(a)) + (\mathrm{id} \otimes R_\circ(a)))\gamma(b) = 0,\end{aligned} \quad (3.9)$$

for any $a, b \in \mathcal{A}^*$.

Definition 3.7 *Let $(\mathcal{A}, \cdot)$ be an anticenter-symmetric algebra. An* **anticenter-symmetric bialgebra** *structure on $\mathcal{A}$ is a linear map $\Delta : \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$ such that*

1. *$\Delta^* : \mathcal{A}^* \otimes \mathcal{A}^* \to \mathcal{A}^*$ defines an anticenter-symmetric algebra structure on $\mathcal{A}^*$;*
2. *Δ satisfies Eqs.* (3.6) *and* (3.7)*.*

We denote it by $(\mathcal{A}, \Delta)$ or $(\mathcal{A}, \mathcal{A}^*)$.

Example 3.8 *Let $(\mathcal{A}, \Delta)$ be an anticenter-symmetric bialgebra on an antcenter-symmetric algebra $\mathcal{A}$. Then $(\mathcal{A}^*, \gamma)$ is an anticenter-symmetric bialgebra on the anticenter-symmetric algebra $\mathcal{A}^*$, where γ is given in Remark 3.6.*

Combining Proposition 3.4 and Theorem 3.5 together, we have the following conclusion.

Theorem 3.9 *Let $(\mathcal{A}, \cdot)$ be an anticenter-symmetric algebra. Suppose that there is an anticenter-symmetric algebra structure on its dual space $\mathcal{A}^*$ denoted "$\circ$" which is defined by a linear map $\Delta : \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$. Then the following conditions are equivalent.*

1. *$(\mathcal{A} \oplus \mathcal{A}^*, \mathcal{A}, \mathcal{A}^*)$ is a standard Manin triple of anticenter-symmetric algebras associated to $\mathfrak{B}_d$ defined by Eq.* (3.1)*.*
2. *$(\mathcal{A}, \mathcal{A}^*, R_\cdot^*, L_\cdot^*, R_\circ^*, L_\circ^*)$ is a matched pair of anticenter-symmetric algebras.*
3. *$(\mathcal{A}, \Delta)$ is an anticenter-symmetric bialgebra.*

Recall a Mock Lie bialgebra structure on a Mock Lie algebra $\mathcal{G}$ is a linear map $\delta : \mathcal{G} \to \mathcal{G} \otimes \mathcal{G}$ such that $\delta^* : \mathcal{G}^* \otimes \mathcal{G}^* \to \mathcal{G}^*$ defines a Mock Lie algebra structure on $\mathcal{G}^*$ and δ satisfies

$$\delta[x, y] = -(\mathrm{ad}(x) \otimes \mathrm{id} + \mathrm{id} \otimes \mathrm{ad}(x))\delta(y) - (\mathrm{ad}(y) \otimes \mathrm{id} + \mathrm{id} \otimes \mathrm{ad}(y))\delta(x), \quad \forall x, y \in \mathcal{G}, \quad (3.10)$$

where $\mathrm{ad}(x)(y) = [x, y]$ for any $x, y \in \mathcal{G}$. We denoted it by $(\mathcal{G}, \delta)$.

Proposition 3.10 *Let $(\mathcal{A}, \Delta)$ be an anticenter-symmetric bialgebra. Then $(\mathcal{G}(\mathcal{A}), \delta)$ is a Mock Lie bialgebra, where $\delta = \Delta + \sigma\Delta$.*

Proof: It is straightforward. □

4 A special class of anticenter-symmetric bialgebras

In this section, we consider a special class of anticenter-symmetric bialgebras, that is, the anticenter-symmetric bialgebra $(\mathcal{A}, \Delta)$ on an anti-flexible algebra $(\mathcal{A}, \cdot)$, with the linear map Δ defined by

$$\Delta(x) = -(\mathrm{id} \otimes L_\cdot(x))\mathrm{r} - (R_\cdot(x) \otimes \mathrm{id})\sigma\mathrm{r}, \quad \forall x \in \mathcal{A}, \quad (4.1)$$

where $\mathrm{r} \in \mathcal{A} \otimes \mathcal{A}$.

Proposition 4.1 *Let $(\mathcal{A}, \cdot)$ be an anticenter-symmetric algebra and* $\mathrm{r} \in \mathcal{A}\otimes\mathcal{A}$. *Let* $\Delta : \mathcal{A} \to \mathcal{A}\otimes\mathcal{A}$ *be a linear map defined by Eq.* (4.1). *Eq.* (3.6) *holds if and only if*

$$(L.(y) \otimes R.(x) + R.(y) \otimes L.(x))(\mathrm{r} + \sigma\mathrm{r}) = 0, \quad \forall x, y \in \mathcal{A}. \tag{4.2}$$

Proof: Let $\mathrm{r} = \sum_i u_i \otimes v_i \in \mathcal{A} \otimes \mathcal{A}$. Then, Eq. (4.1) becomes

$$\Delta(x) = \sum_i (-u_i \otimes xv_i - v_i x \otimes u_i),$$

and

$$\sigma\Delta(x) = \sum_i (-xv_i \otimes u_i - u_i \otimes v_i x).$$

We have:

$$\mathbf{A} = \Delta(xy) + \sigma\Delta(yx) = \sum_i \big(- u_i \otimes (xy)v_i - v_i(xy) \otimes u_i - (yx)v_i \otimes u_i - u_i \otimes v_i(yx)\big);$$

and

$$\begin{aligned}
\mathbf{B} &= -\big(\sigma(\mathrm{id} \otimes L.(y)) + R.(y) \otimes \mathrm{id}\big)\Delta(x) - \big(\sigma(R.(x) \otimes \mathrm{id}) + \mathrm{id} \otimes L.(x)\big)\Delta(y) \\
&= \sum_i \Big[- \big(\sigma(\mathrm{id} \otimes L.(y)) + R.(y) \otimes \mathrm{id}\big)(-u_i \otimes xv_i - v_i x \otimes u_i) \\
&\quad - \big(\sigma(R.(x) \otimes \mathrm{id}) + \mathrm{id} \otimes L.(x)\big)(-u_i \otimes yv_i - v_i y \otimes u_i)\Big] \\
&= \mathbf{A} + \sum_i \big(yu_i \otimes v_i x + u_i y \otimes xv_i + yv_i \otimes u_i x + v_i y \otimes xu_i\big) \\
&= \mathbf{A} + \big(L.(y) \otimes R.(x) + R.(y) \otimes L.(x)\big)(\mathrm{r} + \sigma\mathrm{r}).
\end{aligned}$$

By setting $\mathbf{B} = \mathbf{A}$, Eq. (4.2) is established. □

Proposition 4.2 *Let $(\mathcal{A}, \cdot)$ be an anticenter-symmetric algebra and* $\mathrm{r} \in \mathcal{A}\otimes\mathcal{A}$. *Let* $\Delta : \mathcal{A} \to \mathcal{A}\otimes\mathcal{A}$ *be a linear map defined by Eq.* (4.1). *Eq.* (3.7) *holds if and only if*

$$(R.(x) \otimes R.(y) + R.(y) \otimes R.(x) + L.(x) \otimes L.(y) + L.(y) \otimes L.(x))(\mathrm{r} + \sigma\mathrm{r}) = 0, \quad \forall x, y \in A. \tag{4.3}$$

Proof: In this proof, for simplicity, we take $\mathrm{r} = u_i \otimes v_i \in \mathcal{A} \otimes \mathcal{A}$.
On the one hand, the left-hand side of Eq. (3.7) is given by:

$$\begin{aligned}
\mathbf{A} &= (\sigma(\mathrm{id} \otimes R.(y)) + \mathrm{id} \otimes R.(y) + \sigma(L.(y) \otimes \mathrm{id}) + L.(y) \otimes \mathrm{id})\Delta(x) \\
&= -(xv_i)y \otimes u_i - u_i y \otimes v_i x - u_i \otimes (xv_i)y - v_i x \otimes u_i y - xv_i \otimes yu_i \\
&\quad - u_i \otimes y(v_i x) - yu_i \otimes xv_i - y(v_i x) \otimes u_i.
\end{aligned}$$

On the other hand, the right-hand side of Eq. (3.7) is:

$$\begin{aligned}
\mathbf{B} &= (-\sigma(\mathrm{id} \otimes R.(x)) - \mathrm{id} \otimes R.(x) - \sigma(L.(x) \otimes \mathrm{id}) - L.(x) \otimes \mathrm{id})\Delta(y) \\
&= (yv_i)x \otimes u_i + u_i x \otimes v_i y + u_i \otimes (yv_i)x + v_i y \otimes u_i x + yv_i \otimes xu_i \\
&\quad + u_i \otimes x(v_i y) + xu_i \otimes yv_i + x(v_i y) \otimes u_i.
\end{aligned}$$

By setting $\mathbf{A} = \mathbf{B}$, we obtain:

$$\begin{aligned}
&u_i y \otimes v_i x + v_i x \otimes u_i y + xv_i \otimes yu_i + yu_i \otimes xv_i \\
&+ u_i x \otimes v_i y + v_i y \otimes u_i x + yv_i \otimes xu_i + xu_i \otimes yv_i = 0.
\end{aligned}$$

This establishes Eq. (4.3). □

Lemma 4.3 *Let $\mathcal{A}$ be a vector space and $\Delta : \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$ be a linear map. Then the dual map $\Delta^* : \mathcal{A}^* \otimes \mathcal{A}^* \to \mathcal{A}^*$ defines an anticenyer-symmetyric algebra structure on $\mathcal{A}^*$ if and only if $H_\Delta = 0$, where*

$$H_\Delta = (\Delta \otimes \mathrm{id})\Delta + (\mathrm{id} \otimes \Delta)\Delta + ((\sigma\Delta) \otimes \mathrm{id})(\sigma\Delta) + (\mathrm{id} \otimes (\sigma\Delta))(\sigma\Delta). \tag{4.4}$$

Proof: Denote by $\circ$ the product on $\mathcal{A}^*$ defined by Δ^*. Specifically,

$$\langle a \circ b, x\rangle = \langle \Delta^*(a \otimes b), x\rangle = \langle a \otimes b, \Delta(x)\rangle, \quad \forall x \in \mathcal{A},\ a, b \in \mathcal{A}^*.$$

For all $a, b, c \in \mathcal{A}^*$ and $x \in \mathcal{A}$, we have:

$$\begin{aligned}
\langle (a, b, c), x\rangle &= \langle (a \circ b) \circ c + a \circ (b \circ c), x\rangle \\
&= \langle \big(\Delta^*(\Delta^* \otimes \mathrm{id}) + \Delta^*(\mathrm{id} \otimes \Delta^*)\big)(a \otimes b \otimes c), x\rangle \\
&= \langle \big((\Delta \otimes \mathrm{id})\Delta + (\mathrm{id} \otimes \Delta)\Delta\big)(x), a \otimes b \otimes c\rangle; \\
\langle -(c, b, a), x\rangle &= \langle -(c \circ b) \circ a - c \circ (b \circ a), x\rangle \\
&= \langle \big(- \Delta^*(\Delta^* \otimes \mathrm{id}) - \Delta^*(\mathrm{id} \otimes \Delta^*)\big)(c \otimes b \otimes a), x\rangle \\
&= \langle \big(- (\Delta^*\sigma^*)((\Delta^*\sigma^*) \otimes \mathrm{id}) - (\Delta^*\sigma^*)(\mathrm{id} \otimes (\Delta^*\sigma^*))\big)(a \otimes b \otimes c), x\rangle \\
&= \langle \big(- ((\sigma\Delta) \otimes \mathrm{id})(\sigma\Delta) - (\mathrm{id} \otimes (\sigma\Delta))(\sigma\Delta)\big)(x), a \otimes b \otimes c\rangle.
\end{aligned}$$

Thus, $(\mathcal{A}^*, \circ)$ is an anticenter-symmetric algebra if and only if $H_\Delta = 0$. □

Now, let $(\mathcal{A}, \cdot)$ be an anticenter-symmetric algebra and let

$$\mathrm{r} = \sum_i u_i \otimes v_i \in \mathcal{A} \otimes \mathcal{A}.$$

Define:

$$\mathrm{r}_{12} = \sum_i u_i \otimes v_i \otimes 1, \quad \mathrm{r}_{13} = \sum_i u_i \otimes 1 \otimes v_i, \quad \mathrm{r}_{23} = \sum_i 1 \otimes u_i \otimes v_i,$$

$$\mathrm{r}_{21} = \sum_i v_i \otimes u_i \otimes 1, \quad \mathrm{r}_{31} = \sum_i v_i \otimes 1 \otimes u_i, \quad \mathrm{r}_{32} = \sum_i 1 \otimes v_i \otimes u_i,$$

where 1 denotes the unit if $(\mathcal{A}, \cdot)$ has a unit. Otherwise, it is a symbol that serves a similar role to a unit. The operation between two rs is then defined in an obvious manner. For example,

$$\mathrm{r}_{12}\mathrm{r}_{13} = \sum_{i,j} u_i \cdot u_j \otimes v_i \otimes v_j,\ \mathrm{r}_{13}\mathrm{r}_{23} = \sum_{i,j} u_i \otimes u_j \otimes v_i \cdot v_j,\ \mathrm{r}_{23}\mathrm{r}_{12} = \sum_{i,j} u_j \otimes u_i \cdot v_j \otimes v_i, \tag{4.5}$$

and so on.

Theorem 4.4 *Let $(\mathcal{A}, \cdot)$ be an anticenter-symmetric algebra and $\mathrm{r} \in \mathcal{A} \otimes \mathcal{A}$. Let $\Delta : \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$ be a linear map defined by Eq. (4.1). Then, Δ^* defines an anticenter-symmetric algebra structure on $\mathcal{A}^*$ if and only if, for any $x \in \mathcal{A}$, the following holds:*

$$\begin{aligned}
&(\mathrm{id} \otimes \mathrm{id} \otimes L.(x))(M(\mathrm{r})) + (\mathrm{id} \otimes \mathrm{id} \otimes R.(x))(P(\mathrm{r})) \\
&+ (L.(x) \otimes \mathrm{id} \otimes \mathrm{id})(-N(\mathrm{r})) + (R.(x) \otimes \mathrm{id} \otimes \mathrm{id})(-Q(\mathrm{r})) = 0,
\end{aligned} \tag{4.6}$$

where:

$$M(\mathrm{r}) = \mathrm{r}_{23}\mathrm{r}_{12} + \mathrm{r}_{21}\mathrm{r}_{13} - \mathrm{r}_{13}\mathrm{r}_{23}, \quad N(\mathrm{r}) = \mathrm{r}_{31}\mathrm{r}_{21} - \mathrm{r}_{21}\mathrm{r}_{32} - \mathrm{r}_{23}\mathrm{r}_{31},$$
$$P(\mathrm{r}) = \mathrm{r}_{13}\mathrm{r}_{21} + \mathrm{r}_{12}\mathrm{r}_{23} - \mathrm{r}_{23}\mathrm{r}_{13}, \quad Q(\mathrm{r}) = \mathrm{r}_{21}\mathrm{r}_{31} - \mathrm{r}_{31}\mathrm{r}_{23} - \mathrm{r}_{32}\mathrm{r}_{21}.$$

Proof. Let $\mathrm{r} = \sum_i u_i \otimes v_i \in \mathcal{A} \otimes \mathcal{A}$. Then:

$$\begin{aligned}
&\big((\Delta \otimes \mathrm{id})\Delta + (\mathrm{id} \otimes \Delta)\Delta\big)(x) \\
&= \sum_{i,j} \big(u_j \otimes u_i v_j \otimes x v_i + v_j u_i \otimes u_j \otimes x v_i + u_j \otimes (v_i x) v_j \otimes u_i + v_j (v_i x) \otimes u_j \otimes u_i \\
&\quad + u_i \otimes u_j \otimes (x v_i) v_j + u_i \otimes v_j (x v_i) \otimes u_j + v_i x \otimes u_j \otimes u_i v_j + v_i x \otimes v_j u_i \otimes u_j\big) \\
&= (\mathrm{id} \otimes \mathrm{id} \otimes L.(x))(\mathrm{r}_{23}\mathrm{r}_{12}) + (\mathrm{id} \otimes \mathrm{id} \otimes L.(x))(\mathrm{r}_{21}\mathrm{r}_{13}) - (R.(x) \otimes \mathrm{id} \otimes \mathrm{id})(\mathrm{r}_{21}\mathrm{r}_{31}) \\
&\quad - (\mathrm{id} \otimes \mathrm{id} \otimes L.(x))(\mathrm{r}_{13}\mathrm{r}_{23}) + (R.(x) \otimes \mathrm{id} \otimes \mathrm{id})(\mathrm{r}_{31}\mathrm{r}_{23}) + (R.(x) \otimes \mathrm{id} \otimes \mathrm{id})(\mathrm{r}_{32}\mathrm{r}_{21}) \\
&\quad + \sum_{i,j} \big(u_j \otimes (v_i x) v_j \otimes u_i + u_i \otimes v_j (x v_i) \otimes u_j\big).
\end{aligned}$$

Similarly:

$$\begin{aligned}
&\big(((\sigma\Delta) \otimes \mathrm{id})(\sigma\Delta) + (\mathrm{id} \otimes (\sigma\Delta))(\sigma\Delta)\big)(x) \\
&= \sum_{i,j} \big((x v_i) v_j \otimes u_j \otimes u_i + u_j \otimes v_j (x v_i) \otimes u_i + u_i v_j \otimes u_j \otimes v_i x + u_j \otimes v_j u_i \otimes v_i x \\
&\quad + x v_i \otimes u_i v_j \otimes u_j + x v_i \otimes u_j \otimes v_j u_i + u_i \otimes (v_i x) v_j \otimes u_j + u_i \otimes u_j \otimes v_j (v_i x)\big) \\
&= -(L.(x) \otimes \mathrm{id} \otimes \mathrm{id})(\mathrm{r}_{31}\mathrm{r}_{21}) - (\mathrm{id} \otimes \mathrm{id} \otimes R.(x))(\mathrm{r}_{23}\mathrm{r}_{13}) + (\mathrm{id} \otimes \mathrm{id} \otimes R.(x))(\mathrm{r}_{13}\mathrm{r}_{21}) \\
&\quad + (\mathrm{id} \otimes \mathrm{id} \otimes R.(x))(\mathrm{r}_{12}\mathrm{r}_{23}) + (L.(x) \otimes \mathrm{id} \otimes \mathrm{id})(\mathrm{r}_{21}\mathrm{r}_{32}) + (L.(x) \otimes \mathrm{id} \otimes \mathrm{id})(\mathrm{r}_{23}\mathrm{r}_{31}) \\
&\quad + \sum_{i,j} \big(u_j \otimes v_j (x v_i) \otimes u_i + u_i \otimes (v_i x) v_j \otimes u_j\big).
\end{aligned}$$

By exchanging the indices i and j, we obtain:

$$\sum_{i,j} \big(u_j \otimes (v_i x) v_j \otimes u_i + u_i \otimes v_j (x v_i) \otimes u_j\big) + \sum_{i,j} \big(u_j \otimes v_j (x v_i) \otimes u_i + u_i \otimes (v_i x) v_j \otimes u_j\big) = 0.$$

Thus, it follows that:

$$\begin{aligned}
&(L.(x) \otimes \mathrm{id} \otimes \mathrm{id})(\mathrm{r}_{21}\mathrm{r}_{32} + \mathrm{r}_{23}\mathrm{r}_{31} - \mathrm{r}_{31}\mathrm{r}_{21}) \\
&+ (\mathrm{id} \otimes \mathrm{id} \otimes L.(x))(\mathrm{r}_{23}\mathrm{r}_{12} + \mathrm{r}_{21}\mathrm{r}_{13} - \mathrm{r}_{13}\mathrm{r}_{23}) \\
&+ (R.(x) \otimes \mathrm{id} \otimes \mathrm{id})(\mathrm{r}_{31}\mathrm{r}_{23} + \mathrm{r}_{32}\mathrm{r}_{21} - \mathrm{r}_{21}\mathrm{r}_{31}) \\
&+ (\mathrm{id} \otimes \mathrm{id} \otimes R.(x))(\mathrm{r}_{13}\mathrm{r}_{21} + \mathrm{r}_{12}\mathrm{r}_{23} - \mathrm{r}_{23}\mathrm{r}_{13}) = 0.
\end{aligned}$$

This establishes Eq. (4.6). □

Remark 4.5 *[8] For any $\mathrm{r} \in \mathcal{A} \otimes \mathcal{A}$, the following holds:*

$$N(\mathrm{r}) = -\sigma_{13} M(\mathrm{r}), \quad P(\mathrm{r}) = \sigma_{12} M(\mathrm{r}), \quad Q(\mathrm{r}) = -\sigma_{12}\sigma_{13} M(\mathrm{r}),$$

where $\sigma_{12}(x \otimes y \otimes z) = y \otimes x \otimes z$ and $\sigma_{13}(x \otimes y \otimes z) = z \otimes y \otimes x$, for any $x, y, z \in \mathcal{A}$.

Combining Proposition 4.1, Proposition 4.2, Theorem 4.4, and Remark 4.5, we arrive at the following result.

Theorem 4.6 *Let $(\mathcal{A}, \cdot)$ be an anticenter-symmetric algebra and $\mathrm{r} \in \mathcal{A} \otimes \mathcal{A}$. Let $\Delta : \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$ be a linear map defined by Eq.* (4.1)*. Then $(\mathcal{A}, \Delta)$ is an anticenter-symmetric bialgebra if and only if r satisfies Eqs.* (4.2), (4.3)*, and*

$$\begin{aligned}
&\big((\mathrm{id} \otimes \mathrm{id} \otimes L.(x)) + (R.(x) \otimes \mathrm{id} \otimes \mathrm{id})\sigma_{12}\sigma_{13} \\
&\quad + ((\mathrm{id} \otimes \mathrm{id} \otimes R.(x))\sigma_{12} + (L.(x) \otimes \mathrm{id} \otimes \mathrm{id})\sigma_{13})\big)(M(\mathrm{r})) = 0,
\end{aligned} \tag{4.7}$$

where $M(\mathrm{r}) = \mathrm{r}_{23}\mathrm{r}_{12} + \mathrm{r}_{21}\mathrm{r}_{13} - \mathrm{r}_{13}\mathrm{r}_{23}$.

As a direct consequence of Theorem 4.6, we have the following corollary.

Corollary 4.7 *Let* $(\mathcal{A}, \cdot)$ *be an anticenter-symmetric algebra and* $\mathrm{r} \in \mathcal{A} \otimes \mathcal{A}$. *Let* $\Delta : \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$ *be a linear map defined by Eq.* (4.1). *If, in addition,* r *is skew-symmetric and satisfies*

$$\mathrm{r}_{12}\mathrm{r}_{13} - \mathrm{r}_{23}\mathrm{r}_{12} + \mathrm{r}_{13}\mathrm{r}_{23} = 0, \tag{4.8}$$

then $(\mathcal{A}, \Delta)$ *is an anticenter-symmetric bialgebra.*

Definition 4.8 *Let* $(\mathcal{A}, \cdot)$ *be an anticenter-symmetric algebra and* $\mathrm{r} \in \mathcal{A} \otimes \mathcal{A}$. *Eq.* (4.8) *is called the* ***anticenter-symmetric Yang-Baxter equation*** *(ACSYBE) in* $(\mathcal{A}, \cdot)$.

Remark 4.9 *The term "anticenter-symmetric Yang-Baxter equation" reflects its analogy with the classical Yang-Baxter equation in a Mock Lie algebra (see [5]). Notably, the anticenter-symmetric Yang-Baxter equation in an anticenter-symmetric algebra, the anti-flexible Yang-Baxter equation in an anti-flexible algebra, and the associative Yang-Baxter equation (see [3, 8]) in an associative algebra all share the same form as Eq. (4.8). Thus, these three equations exhibit common properties.*

At the end of this section, we highlight two properties of the anticenter-symmetric Yang-Baxter equation. The proofs are omitted since they mirror the proofs in the case of the associative Yang-Baxter equation.

Let $\mathcal{A}$ *be a vector space. For any* $\mathrm{r} \in \mathcal{A} \otimes \mathcal{A}$, r *can be regarded as a linear map from* $\mathcal{A}^*$ *to* $\mathcal{A}$ *as follows:*

$$\langle \mathrm{r}, u^* \otimes v^* \rangle = \langle \mathrm{r}(u^*), v^* \rangle, \quad \forall u^*, v^* \in \mathcal{A}^*. \tag{4.9}$$

Proposition 4.10 *Let* $(\mathcal{A}, \cdot)$ *be an anticenter-symmetric algebra and* $\mathrm{r} \in \mathcal{A} \otimes \mathcal{A}$ *be skew-symmetric. Then* r *is a solution of the anticenter-symmetric Yang-Baxter equation if and only if* r *satisfies*

$$\mathrm{r}(a) \cdot \mathrm{r}(b) = \mathrm{r}(R_\cdot^*(\mathrm{r}(a))b + L_\cdot^*(\mathrm{r}(b))a), \quad \forall a, b \in \mathcal{A}^*. \tag{4.10}$$

Theorem 4.11 *Let* $(\mathcal{A}, \cdot)$ *be an anticenter-symmetric algebra and* $\mathrm{r} \in \mathcal{A} \otimes \mathcal{A}$. *Suppose that* r *is antisymmetric and nondegenerate. Then* r *is a solution of the anticenter-symmetric Yang-Baxter equation in* $(\mathcal{A}, \cdot)$ *if and only if the inverse of the isomorphism* $\mathcal{A}^* \to \mathcal{A}$ *induced by* r, *regarded as a bilinear form* ω *on* $\mathcal{A}$ *(i.e.,* $\omega(x, y) = \langle \mathrm{r}^{-1}x, y \rangle$ *for any* $x, y \in \mathcal{A}$*), satisfies*

$$\omega(x \cdot y, z) + \omega(y \cdot z, x) + \omega(z \cdot x, y) = 0, \quad \forall x, y, z \in \mathcal{A}. \tag{4.11}$$

5 $\mathcal{O}$-operators of anticenter-symmetric algebras and pre-anticenter-symmetric algebras

In this section, we introduce the notions of $\mathcal{O}$-operators for anticenter-symmetric algebras and pre-anticenter-symmetric algebras, which are used to construct skew-symmetric solutions of the anticenter-symmetric Yang-Baxter equation and, consequently, to construct anticenter-symmetric bialgebras.

Definition 5.1 *Let* (l, r, V) *be a bimodule of an anticenter-symmetric algebra* $(\mathcal{A}, \cdot)$. *A linear map* $T : V \to \mathcal{A}$ *is called an* ***$\mathcal{O}$-operator associated with*** (l, r, V) *if* T *satisfies*

$$T(u) \cdot T(v) = T(l(T(u))v + r(T(v))u), \quad \forall u, v \in V.$$

Example 5.2 *Let* $(\mathcal{A}, \cdot)$ *be an anticenter-symmetric algebra. An* $\mathcal{O}$*-operator* R_B *associated with the regular bimodule* $(L, R, \mathcal{A})$ *is called a* ***Rota-Baxter operator of weight zero***. *In this case,* R_B *satisfies*

$$R_B(x) \cdot R_B(y) = R_B(R_B(x) \cdot y + x \cdot R_B(y)), \quad \forall x, y \in \mathcal{A}.$$

Example 5.3 *Let $(\mathcal{A},\cdot)$ be an anticenter-symmetric algebra, and let $\mathrm{r}\in\mathcal{A}\otimes\mathcal{A}$. If r is skew-symmetric, then by Proposition 4.10, r is a solution of the anticenter-symmetric Yang-Baxter equation if and only if r, regarded as a linear map from $\mathcal{A}^*$ to $\mathcal{A}$, is an $\mathcal{O}$-operator associated with the bimodule $(R^*_\cdot, L^*_\cdot, \mathcal{A}^*)$.*

There is the following construction of (skew-symmetric) solutions of anticenter-symmetric Yang-Baxter equation in a semi-direct product anticenter-symmetric algebra from an $\mathcal{O}$-operator of an anticenter-symmetric algebra which is similar as for associative algebras ([3, Theorem 2.5.5], hence the proof is omitted).

Theorem 5.4 *Let (l,r,V) be a bimodule of an anticenter-symmetric algebra $(\mathcal{A},\cdot)$, and let $T: V\to\mathcal{A}$ be a linear map. Identifying T as an element in $(\mathcal{A}\ltimes_{r^*,l^*}V^*)\oplus(\mathcal{A}\ltimes_{r^*,l^*}V^*)$, $\mathrm{r}=T-\sigma(T)$ is a skew-symmetric solution of the anticenter-symmetric Yang-Baxter equation in $\mathcal{A}\ltimes_{r^*,l^*}V^*$ if and only if T is an $\mathcal{O}$-operator associated with the bimodule (l,r,V).*

Definition 5.5 *Let $\mathcal{A}$ be a vector space with two bilinear products $\prec,\succ:\mathcal{A}\otimes\mathcal{A}\to\mathcal{A}$. The pair $(\mathcal{A},\prec,\succ)$ is called a **pre-anticenter-symmetric algebra** if, for any $x,y,z\in\mathcal{A}$, the following conditions hold:*

$$(x,y,z)_m=-(z,y,x)_m,$$
$$(x,y,z)_l=-(z,y,x)_r,$$

where:

$$(x,y,z)_m:=(x\succ y)\prec z+x\succ(y\prec z),$$
$$(x,y,z)_l:=(x*y)\succ z+x\succ(y\succ z),$$
$$(x,y,z)_r:=(x\prec y)\prec z+x\prec(y*z),$$

and $x*y=x\prec y+x\succ y$.

Proposition 5.6 *Let $(\mathcal{A},\prec,\succ)$ be a pre-anticenter-symmetric algebra. Define a bilinear product $*:\mathcal{A}\otimes\mathcal{A}\to\mathcal{A}$ by*

$$x*y=x\prec y+x\succ y,\ \ \forall x,y\in\mathcal{A}. \tag{5.1}$$

*Then $(\mathcal{A},*)$ is an anticenter-symmetric algebra, referred to as the **associated anticenter-symmetric algebra of** $(\mathcal{A},\prec,\succ)$.*

Proof: Set $(x,y,z)_*=(x*y)*z+x*(y*z)$. For any $x,y,z\in\mathcal{A}$, we have:

$$(x,y,z)_*=(x,y,z)_m+(x,y,z)_l+(x,y,z)_r=-(z,y,x)_m-(z,y,x)_r-(z,y,x)_l=-(z,y,x)_*.$$

Hence, $(\mathcal{A},*)$ is an anticenter-symmetric algebra. □

Let $(\mathcal{A},\prec,\succ)$ be a pre-anticenter-symmetric algebra. For any $x\in\mathcal{A}$, let $L_\succ(x),R_\prec(x)$ denote the left multiplication operator of $(\mathcal{A},\prec)$ and the right multiplication operator of $(\mathcal{A},\succ)$ respectively, that is, $L_\succ(x)(y)=x\succ y,\ \ R_\prec(x)(y)=y\prec x,\ \ \forall\ x,y\in\mathcal{A}$. Moreover, let $L_\succ,R_\prec:\mathcal{A}\to\mathfrak{gl}(\mathcal{A})$ be two linear maps with $x\to L_\succ(x)$ and $x\to R_\prec(x)$ respectively.

Proposition 5.7 *Let $(\mathcal{A},\prec,\succ)$ be a pre-anticenter-symmetric algebra. Then $(L_\succ,R_\prec,A)$ is a bimodule of the associated anti-flexible algebra $(\mathcal{A},*)$, where $*$ is defined by Eq. (5.1).*

Proof: For any $x,y,z\in\mathcal{A}$, we have

$$\begin{aligned}
&(L_\succ(x*y)+L_\succ(x)L_\succ(y))(z)=(x*y)\succ z+x\succ(y\succ z)=(x,y,z)_l,\\
&(-R_\prec(x)R_\prec(y)-R_\prec(y*x))(z)=-(z\prec y)\prec x-z\prec(y*x)=-(z,y,x)_r,\\
&(L_\succ(x)R_\prec(y)+R_\prec(y)L_\succ(x))(z)=x\succ(z\prec y)+(x\succ z)\prec y=(x,z,y)_m,\\
&(-L_\succ(y)R_\prec(x)-R_\prec(x)L_\succ(y))(z)=-y\succ(z\prec x)-(y\succ z)\prec x=-(y,z,x)_m.
\end{aligned}$$

Hence $(L_\succ,R_\prec,\mathcal{A})$ is a bimodule of $(\mathcal{A},*)$. □

Corollary 5.8 *Let $(\mathcal{A}, \prec, \succ)$ be a pre-anticenter-symmetric algebra. Then the identity map* id *is an $\mathcal{O}$-operator of the associated anticenter-symmetric algebra $(\mathcal{A}, *)$ associated with the bimodule $(L_{\succ}, R_{\prec}, \mathcal{A})$.*

Theorem 5.9 *Let (l, r, V) be a bimodule of an anticenter-symmetric algebra $(\mathcal{A}, \cdot)$. Let $T : V \to \mathcal{A}$ be an $\mathcal{O}$-operator associated with (l, r, V). Then, there exists a pre-anticenter-symmetric algebra structure on V given by*

$$u \succ v = l(T(u))v, \quad u \prec v = r(T(v))u, \quad \forall u, v \in V. \tag{5.2}$$

Consequently, there is an associated anticenter-symmetric algebra structure on V given by Eq. (5.1)*, and T is a homomorphism of anticenter-symmetric algebras. Moreover, $T(V) = \{T(v) \mid v \in V\} \subset \mathcal{A}$ is an anticenter-symmetric subalgebra of $(\mathcal{A}, \cdot)$, and there is an induced pre-anticenter-symmetric algebra structure on $T(V)$ given by*

$$T(u) \succ T(v) = T(u \succ v), \quad T(u) \prec T(v) = T(u \prec v), \quad \forall u, v \in V.$$

The corresponding associated anticenter-symmetric algebra structure on $T(V)$, as given by Eq. (5.1)*, is precisely the anticenter-symmetric subalgebra structure of $(\mathcal{A}, \cdot)$, and T is a homomorphism of pre-anticenter-symmetric algebras.*

Proof: For all $u, v, w \in V$, we have

$$\begin{aligned}
(u, v, w)_m &= (u \succ v) \prec w + u \succ (v \prec w) = r(T(w))l(T(u))v + l(T(u))r(T(w))v \\
&= -r(T(u))l(T(w))v - l(T(u))r(T(w))v = -(w, v, u)_m, \\
(u, v, w)_l &= (u \succ v + u \prec v) \succ w + u \succ (v \succ w) \\
&= (l(T(l(T(u))v + r(T(v))u)) + l(T(u))l(T(v)))w \\
&= (l(T(u) \cdot T(v)) + l(T(u))l(T(v)))w = -(r(T(u))r(T(v)) - r(T(v) \cdot T(u))w \\
&= -(r(T(u))r(T(v)) - r(T(u \succ v + u \prec v)))w \\
&= -(w \prec v) \prec u - w \prec (u \succ v + u \prec v) \\
&= -(w, v, u)_r
\end{aligned}$$

Therefore, $(V, \prec, \succ)$ is a pre-anticenter-symmetric algebra. For $T(V)$, we have

$$T(u) * T(v) = T(u \succ v + u \prec v) = T(u * v) = T(u) \cdot T(v), \quad \forall u, v \in V.$$

The rest is straightforward. □

Corollary 5.10 *Let $(\mathcal{A}, \cdot)$ be an anticenter-symmetric algebra. Then there exists a pre-anticenter-symmetric algebras structure on $\mathcal{A}$ such that its associated anticenter-symmetric algebra is $(\mathcal{A}, \cdot)$ if and only if there exists an invertible $\mathcal{O}$-operator.*

Proof: Suppose that there exists an invertible $\mathcal{O}$-operator $T : V \to \mathcal{A}$ associated to a bimodule (l, r, V). Then the products "$\succ, \prec$" given by Eq. (5.2) defines a pre-anticenter-symmetric algebra structure on V. Moreover, there is a pre-anticenter-symmetric algebra structure on $T(V) = \mathcal{A}$, that is,

$$x \succ y = T(l(x)T^{-1}(y)), \quad x \prec y = T(r(y)T^{-1}(x)), \quad \forall x, y \in \mathcal{A}.$$

Moreover, for any $x, y \in \mathcal{A}$, we have

$$x \succ y + x \prec y = T(l(x)T^{-1}(y) + r(y)T^{-1}(x)) = T(T^{-1}(x)) \cdot T(T^{-1}(y)) = x \cdot y.$$

Hence the associated anticenter-symmetric algebra of $(\mathcal{A}, \succ, \prec)$ is $(\mathcal{A}, \cdot)$.

Conversely, let $(\mathcal{A}, \succ, \prec)$ be pre-center-symmetric algebra such that its associated anticenter-symmetric is $(\mathcal{A}, \cdot)$. Then by Corollary 5.8, the identity map id is an $\mathcal{O}$-operator of $(\mathcal{A}, \cdot)$ associated to the bimodule $(L_{\succ}, R_{\prec}, \mathcal{A})$. □

Corollary 5.11 *Let $(\mathcal{A}, \cdot)$ be an anticenter-symmetric algebra and ω be a nondegenerate skew-symmetric bilinear form satisfying Eq. (4.11). Then there exists a pre-anticenter-symmetric algebra structure $\succ, \prec$ on $\mathcal{A}$ given by*

$$\omega(x \succ y, z) = \omega(y, z \cdot x), \ \ \omega(x \prec y, z) = \omega(x, y \cdot z), \ \ \forall x, y, z \in \mathcal{A}, \tag{5.3}$$

such that the associated anticenter-symmetric algebra is $(\mathcal{A}, \cdot)$.

Proof: Define a linear map $T : \mathcal{A} \to \mathcal{A}^*$ by

$$\langle T(x), y\rangle = \omega(x, y), \ \ \forall x, y \in \mathcal{A}.$$

Then T is invertible and T^{-1} is an $\mathcal{O}$-operator of the anticenter-symmetric algebra $(\mathcal{A}, \cdot)$ associated to the bimodule $(R^*_\cdot, L^*_\cdot, A^*)$. By Corollary 5.10, there is a pre-anticenter-symmetric algebra structure $\succ, \prec$ on $(\mathcal{A}, *)$ given by

$$x \succ y = T^{-1}R^*(x)T(y), \ \ x \prec y = T^{-1}L^*(y)T(x), \ \ \forall x, y \in \mathcal{A},$$

which gives exactly Eq. (5.3) such that the associated anticenter-symmetric algebra is $(\mathcal{A}, \cdot)$. □

Finally we give the following construction of skew-symmetric solutions of anticenter-symmetric Yang-Baxter equation (hence anticenter-symmetric bialgebras) from a pre-anticenter-symmetric algebra.

Proposition 5.12 *Let $(\mathcal{A}, \succ, \prec)$ be a pre-anticenter-symmetric algebra. Then*

$$\mathrm{r} = \sum_{i}^{n} (e_i \otimes e_i^* - e_i^* \otimes e_i) \tag{5.4}$$

is a solution of anticenter-symmetric Yang-Baxter equation in $\mathcal{A} \ltimes_{R^*_\prec, L^*_\succ} \mathcal{A}^*$, where $\{e_1, \cdots, e_n\}$ is a basis of $\mathcal{A}$ and $\{e_1^*, \cdots, e_n^*\}$ is its dual basis.

Proof: Note that the identity map $\mathrm{id} = \sum_{i=1}^{n} e_i \otimes e_i^*$. Hence the conclusion follows from Theorem 5.4 and Corollary 5.8. □

6 Concluding remarks

We established a bialgebra theory for anticenter-symmetric algebras, introducing the notion of an anticenter-symmetric bialgebra and its equivalence to a Manin triple of anticenter-symmetric algebras. A key result is the formulation of the anticenter-symmetric Yang-Baxter equation in anticenter-symmetric algebras, an analogue to the classical Yang-Baxter equation in Mock Lie algebras and the associative Yang-Baxter equation, with the unexpected finding that they share the same formal structure.

We showed that skew-symmetric solutions to this equation define anticenter-symmetric bialgebras. Additionally, the notions of $\mathcal{O}$-operators and pre-anticenter-symmetric algebras were introduced as tools to construct such solutions, providing a foundation for further exploration of anticenter-symmetric algebraic structures.

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