NEW CONTEMPORARY CONJECTURES FOR THE RIEMANN HYPOTHESIS

ABSTRACT. We will present two new results for the "Dirichlet eta" function $S(s) = \sum_{n\geq 1} \frac{(-1)^n}{n^s}$ which would lead us to announce some new conjectures equivalent to that of the Riemann hypothesis.

1. Introduction

The Riemann Hypothesis is a conjecture formulated in 1859 by the mathematician Bernhard Riemann, according to which the nontrivial zeros of the Riemann zeta function are infinite and all have a real part equal to 1/2.

His proof would improve knowledge of the distribution of prime numbers and open up new areas of mathematics. Riemann's article (see [4]) on the distribution of prime numbers is his only text dealing with number theory. He develops the properties of the zeta function $C(s) = \sum_{n=1}^{+\infty} \frac{1}{n^s}$ and proves the prime number theorem by admitting several results, including what is now called the Riemann Hypothesis. Hardy then demonstrated that there are infinitely many zeros on the critical line. (see [1], [2]), which gives us hope that the RH might be true...

This paper is a continuation of our last "A Contemporary Conjecture for the Riemann Hypothesis" work already published (see [5]).

Let
$S(s) = \sum_{n=1}^{+\infty} \frac{(-1)^n}{n^s} = -\sum_{n=1}^{+\infty} \frac{(-1)^{n-1}}{n^s}$
so
$S(s) = \rho(s) e^{i\theta(s)} S(1-s)$

Remark 1. *(Functional equation of Hardy)*
We have $\forall s \in \mathbb{C}$ *such* $\operatorname{Re}(s) \in]0,1[$

$$S(s) = \varphi(s) S(1-s)$$

with $\varphi(s) = 2\frac{1-2^{s-1}}{1-2^s}\pi^{s-1}\sin\left(\frac{s}{2}\pi\right)\Gamma(1-s) = \rho(s) e^{i\theta(s)}$.
see [1] *&* [2]

1991 *Mathematics Subject Classification.* 11E45 (primary); 11M41 (secondary).
Key words and phrases. Riemann zeta-function, Hardy's functional equation, Adherent Point, Complex number.

2. Preliminary

Proposition 1. *Let $s = r + ic$, so*

$$\begin{aligned} S &= \sum_{n=1}^{+\infty} (-1)^n \frac{e^{-i\ln(n)c}}{n^r} = C_1 - C_2 \\ S &= \sum_{n=1}^{+\infty} (-1)^n \frac{e^{i\alpha_n}}{n^r} = R' + iI' \end{aligned}$$

and

$$\begin{aligned} C &= \sum_{n=1}^{+\infty} \frac{1}{n^s} = \sum_{n=1}^{+\infty} \frac{e^{-i\ln(n)c}}{(2n)^r} = \sum_{n=1}^{+\infty} \frac{e^{i\alpha_n}}{(2n)^r} \\ C &= C_1 + C_2 = R + iI \end{aligned}$$

with $\alpha_n = -\ln(n)\, c$,

$$C_1 = \sum_{n=1}^{+\infty} \frac{1}{(2n)^s} = \sum_{n=1}^{+\infty} \frac{e^{-i\ln(2n)c}}{(2n)^r} = \sum_{n=1}^{+\infty} \frac{e^{i\alpha_{2n}}}{(2n)^r} = R_1 + iI_1$$

$$C_2 = \sum_{n=1}^{+\infty} \frac{1}{(2n-1)^s} = \sum_{n=1}^{+\infty} \frac{e^{-i\ln(2n-1)c}}{(2n-1)^r} = \sum_{n=1}^{+\infty} \frac{e^{i\alpha_{2n-1}}}{(2n-1)^r} = R_2 + iI_2$$

and

$$\begin{aligned} R_1 &= \sum_{n=1}^{+\infty} \frac{\cos(\alpha_{2n})}{(2n)^r},\ I_1 = \sum_{n=1}^{+\infty} \frac{\sin(\alpha_{2n})}{(2n)^r},\ R_2 = \sum_{n=1}^{+\infty} \frac{\cos(\alpha_{2n-1})}{(2n-1)^r},\ I_2 = \sum_{n=1}^{+\infty} \frac{\sin(\alpha_{2n-1})}{(2n-1)^r} \\ R &= \sum_{n=1}^{+\infty} \frac{\cos(\alpha_n)}{(2n)^r} = R_1 + R_2,\ I = \sum_{n=1}^{+\infty} \frac{\sin(\alpha_n)}{(2n)^r} = I_1 + I_2 \\ R' &= \sum_{n=1}^{+\infty} (-1)^n \frac{\cos(\alpha_n)}{n^r},\ I' = \sum_{n=1}^{+\infty} (-1)^n \frac{\sin(\alpha_n)}{n^r},\ R' = R_1 - R_2,\ I' = I_1 - I_2 \end{aligned}$$

Proposition 2. *Let $s = r + ic = r + i\frac{\alpha}{\ln(2)}$ (since $\alpha = \ln(2)\, c$)*

$$\begin{aligned} C_1 &= \frac{e^{-i\alpha}}{2^r} C \\ C_2 &= \left(1 - \frac{e^{-i\alpha}}{2^r}\right) C \\ S &= \left(2^{1-r} e^{-i\alpha} - 1\right) C \end{aligned}$$

Proof. $\alpha = \ln(2)\, c \Rightarrow e^{-i\ln(2)c} = e^{-i\alpha}$

Therefore,

$$C_1 = \sum_{n=1}^{+\infty} \frac{e^{-i\ln(2n)c}}{(2n)^r} = \frac{e^{-i\ln(2)c}}{2^r} \sum_{n=1}^{+\infty} \frac{e^{-i\ln(n)c}}{n^r}$$

$$C_1 = \frac{e^{-i\ln(2)c}}{2^r} C = \frac{e^{-i\alpha}}{2^r} C$$

$C_2 = C - C_1 = C - \frac{e^{-i\alpha}}{2^r} C \Rightarrow$

$$C_2 = \left(1 - \frac{e^{-i\alpha}}{2^r}\right) C$$

and

$S = C_1 - C_2 = \frac{e^{-i\alpha}}{2^r} C - \left(1 - \frac{e^{-i\alpha}}{2^r}\right) C \Longrightarrow$

$$S = \left(2^{1-r} e^{-i\alpha} - 1\right) C$$

□

3. Our previous contributions

Theorem 1. *(Adherent Point and Closure)*

Let X be a topological space and $A \subseteq X$ be a subset. A point $x \in X$ is said to be an adherent point (Closure point) of A if every open neighborhood of x intersects A. The closure of A, denoted by $\overline{A}$, consists of all adherent points of A.

Theorem 2. *(Adherent Point and Existence of Convergent Sequences)*

Let X be a topological space and $A \subseteq X$ be a subset. A point $x \in X$ is an adherent point (Closure point) of A if and only if there exists a sequence x_n in A such that

$$\lim_{n \to +\infty} x_n = x$$

3.1. The first announcement.

Lemma 1. *Assuming that there exists an s_1 with $r_1 = \mathrm{Re}[s_1] \in \left]0, \frac{1}{2}\right[$ and $\alpha = \ln(2) c > 0$ such that $S^2(s_1) \in IR$, so*

(i) $\exists V(s_1) \subset \mathbb{C}$ such $\forall s \in V(s_1) - \{s_1\}$, $S^2(s) \notin IR$

(ii) $\exists u_n \in V(s_1) - \{s_1\}$ such $\lim u_n = s_1$ (since $s_1 \in \overline{V(s_1) - \{s_1\}}$ with $\overline{A}$ is the adherant of A).

Proof. Obvious.

(i) Reasoning by the absurd.

(ii) Using (i) and the last theorem. □

Lemma 2. *Let $D_1 = \left\{z \in \mathbb{C} / \mathrm{Re}(z) \in]0,1[, \mathrm{Re}(z) \neq \frac{1}{2} \text{ and } \mathrm{Im}(z) \neq 0\right\}$, so $\forall s \in D_1$*

$$S^2(s) \in IR \Leftrightarrow S^2(1-s) \in IR$$

Proof. Since the first lemma:

Assuming that there exists an s_1 with $r_1 = \mathrm{Re}[s_1] \in \left]0, \frac{1}{2}\right[$ and $\alpha = \ln(2) c > 0$ such that $S^2(s_1) \in IR$, so

(i) $\exists V(s_1) \subset \mathbb{C}$ such $\forall s \in V(s_1) - \{s_1\}$, $S^2(s) \notin IR$

(ii) $\exists u_n \in V(s_1) - \{s_1\}$ such $\lim u_n = s_1$

(since $s_1 \in \overline{V(s_1) - \{s_1\}}$ with $\overline{A}$ is the adherant of A).

$u_n \in V(s_1) - \{s_1\} \Rightarrow S^2(u_n) \notin IR$

$\Rightarrow \left(S(u_n), \overline{S(u_n)}\right)$ is a basis of $\mathbb{C}$

$\Rightarrow \exists! (a_n, b_n) \in IR^2$ such $S(1-u_n) = a_n S(u_n) + b_n \overline{S(u_n)}$

$S(s) = \varphi(s) S(1-s)$

$\Rightarrow S(1-u_n) = a_n \varphi(u_n) S(1-u_n) + b_n \overline{\varphi(u_n) S(1-u_n)}$

$\Rightarrow [1 - a_n \varphi(u_n)] S(1-u_n) = \left[b_n \overline{\varphi(u_n)}\right] \overline{S(1-u_n)}$

$\Rightarrow \left[1-a_n\overline{\varphi(u_n)}\right]\overline{S(1-u_n)}=\left[b_n\varphi(u_n)\right]S(1-u_n)$
$\Rightarrow b_n\varphi(u_n)S^2(1-u_n)=\left[1-a_n\overline{\varphi(u_n)}\right]|S(1-u_n)|^2$
$\varphi(s)\varphi(1-s)=1\ \forall s$
$\Rightarrow b_nS^2(1-u_n)=\varphi(1-u_n)\left[1-a_n\overline{\varphi(u_n)}\right]|S(1-u_n)|^2$
$\Rightarrow b_nS^2(1-u_n)=\left[\varphi(1-u_n)-a_n\overline{\varphi(u_n)}\varphi(1-u_n)\right]|S(1-u_n)|^2$
$\varphi(s)\varphi(1-s)=1\Rightarrow|\varphi(s)|^2\varphi(1-s)=\overline{\varphi(s)}$
$\Rightarrow\overline{\varphi(s)}=\rho^2\varphi(1-s)$
$\Rightarrow$

$$\begin{aligned}b_nS^2(1-u_n)&=|S(1-u_n)|^2\left[\varphi(1-u_n)-a_n\rho_n^2\varphi^2(1-u_n)\right]\\&=|S(1-u_n)|^2\varphi(1-u_n)\left[1-a_n\rho_n^2\varphi(1-u_n)\right]\end{aligned}$$

$\Rightarrow|b_n|=|\varphi(1-u_n)|\left|1-a_n\rho_n^2\varphi(1-u_n)\right|$ or $|S(1-u_n)|=0$
we have $S^2(u_n)\notin IR\Rightarrow S(u_n)\neq0\Rightarrow|S(1-u_n)|\neq0$
$\Rightarrow|b_n|=|\varphi(1-u_n)|\left|1-a_n\rho_n^2\varphi(1-u_n)\right|$
also $\rho_n\neq0$ since $|S(u_n)|=\rho_n|S(1-u_n)|$
$\Rightarrow|b_n|=\frac{1}{\rho_n}\left|1-a_n\rho_n^2\varphi(1-u_n)\right|$
$\Rightarrow b_n^2\rho_n^2=\left|1-a_n\rho_n^2\varphi(1-u_n)\right|^2$
$\Rightarrow$

$$b_n^2\rho_n^2=1+a_n^2\rho_n^2-2a_n\rho_n\cos(\theta_n)\qquad(1)$$

$S(1-u_n)=a_nS(u_n)+b_n\overline{S(u_n)}$
$\Rightarrow|S(1-u_n)|^2=\left(a_n^2+b_n^2\right)|S(u_n)|^2+a_nb_n\left(S^2(u_n)+\overline{S^2(u_n)}\right)$
$\Rightarrow\rho_n^2|S(1-u_n)|^2=\left(a_n^2\rho_n^2+b_n^2\rho_n^2\right)|S(u_n)|^2+a_nb_n\rho_n^2\left(S^2(u_n)+\overline{S^2(u_n)}\right)$
$\Rightarrow$
$|S(u_n)|^2=\left(a_n^2\rho_n^2+1+a_n^2\rho_n^2-2a_n\rho_n\cos(\theta_n)\right)|S(u_n)|^2+a_nb_n\rho_n^2\left(S^2(u_n)+\overline{S^2(u_n)}\right)$
$\Rightarrow0=\left(2a_n^2\rho_n^2-2a_n\rho_n\cos(\theta_n)\right)|S(u_n)|^2+a_nb_n\rho_n^2\left(S^2(u_n)+\overline{S^2(u_n)}\right)$
$\Rightarrow a_n\rho_n\left[2(a_n\rho_n-\cos(\theta_n))|S(u_n)|^2+b_n\rho_n\left(S^2(u_n)+\overline{S^2(u_n)}\right)\right]=0$
$\Rightarrow a_n\rho_n=0$ or $2\left[a_n\rho_n-\cos(\theta_n)\right]|S(u_n)|^2+b_n\rho_n\left[S^2(u_n)+\overline{S^2(u_n)}\right]=0$
$2\left[a_n\rho_n-\cos(\theta_n)\right]|S(u_n)|^2+b_n\rho_n\left[S^2(u_n)+\overline{S^2(u_n)}\right]=0\Longrightarrow$
$b_n\rho_n\left[S^2(u_n)+\overline{S^2(u_n)}\right]=-2\left[a_n\rho_n-\cos(\theta_n)\right]|S(u_n)|^2$
$\Rightarrow b_n^2\rho_n^2\left[S^2(u_n)+\overline{S^2(u_n)}\right]^2=4\left[a_n\rho_n-\cos(\theta_n)\right]^2|S(u_n)|^4$
$\Rightarrow\left(1+a_n^2\rho_n^2-2a_n\rho_n\cos(\theta_n)\right)\left[S^2(u_n)+\overline{S^2(u_n)}\right]^2=4\left[a_n\rho_n-\cos(\theta_n)\right]^2|S(u_n)|^4$
$\Rightarrow\left[(a_n\rho_n-\cos(\theta_n))^2+\sin^2(\theta_n)\right]\left[S^2(u_n)+\overline{S^2(u_n)}\right]^2=4\left[a_n\rho_n-\cos(\theta_n)\right]^2|S(u_n)|^4$
$\Rightarrow$
$\sin^2(\theta_n)\left[S^2(u_n)+\overline{S^2(u_n)}\right]^2=\left[a_n\rho_n-\cos(\theta_n)\right]^2\left[4|S(u_n)|^4-\left(S^2(u_n)+\overline{S^2(u_n)}\right)^2\right]$
$\Rightarrow\sin^2(\theta_n)\left[S^2(u_n)+\overline{S^2(u_n)}\right]^2=\left[a_n\rho_n-\cos(\theta_n)\right]^2\left[2|S(u_n)|^4-S^4(u_n)-\overline{S^4(u_n)}\right]$
$\Rightarrow\sin^2(\theta_n)\left[S^2(u_n)+\overline{S^2(u_n)}\right]^2=-\left[a_n\rho_n-\cos(\theta_n)\right]^2\left[S^2(u_n)-\overline{S^2(u_n)}\right]^2$

$\Rightarrow [i\sin(\theta_n)]^2\left[S^2(u_n)+\overline{S^2(u_n)}\right]^2=[a_n\rho_n-\cos(\theta_n)]^2\left[S^2(u_n)-\overline{S^2(u_n)}\right]^2$
$\Rightarrow$

$$[a_n\rho_n-\cos(\theta_n)]\left[S^2(u_n)-\overline{S^2(u_n)}\right]=\pm i\sin(\theta_n)\left[S^2(u_n)+\overline{S^2(u_n)}\right]$$

$\Rightarrow$
$[a_n\rho_n-\cos(\theta_n)\mp i\sin(\theta_n)]\,S^2(u_n)=[a_n\rho_n-\cos(\theta_n)\pm i\sin(\theta_n)]\,\overline{S^2(u_n)}=Z$
as $Z=\overline{Z}$ so $Z=[a_n\rho_n-\cos(\theta_n)\mp i\sin(\theta_n)]\,S^2(u_n)\in\mathbb{R}$
$\Rightarrow$

$$\left(a_n\rho_n-e^{\pm i\theta_n}\right)S^2(u_n)\in\mathbb{R}$$

$\Rightarrow$

$$S^2(u_n)=K_n\left(a_n\rho_n-e^{\mp i\theta_n}\right)$$

with $K_n\in\mathbb{R}$
$\Rightarrow$

$$\mathrm{Im}\left[S^2(u_n)\right]=\pm K_n\sin(\theta_n)$$

since $\lim u_n=s_1$ & $S^2(s_1)\in IR^*$ so
$\lim[\sin(\theta_n)]=\sin(\theta(s_1))=0$
$\Rightarrow$

$$\theta(s_1)\equiv 0\ \ [\pi]$$

as $S(s)=\varphi(s)\,S(1-s)$ & $\varphi(s)=\rho(s)\,e^{i\theta(s)}$
$\Rightarrow S(s_1)=\rho(s_1)\,e^{i\theta(s_1)}S(1-s_1)$
$\Rightarrow S^2(s_1)=\rho^2(s_1)\,e^{2i\theta(s_1)}S^2(1-s_1)$
$\theta(s_1)\equiv 0\ \ [\pi]\Rightarrow S^2(s_1)=\rho^2(s_1)\,S^2(1-s_1)$
so
$S^2(s)\in IR\Leftrightarrow S^2(1-s)\in IR$
Another proof :
If $\lim a_n=a$ and $\lim b_n=b$, and as we have

$$b_nS^2(1-u_n)=|S(1-u_n)|^2\left[\varphi(1-u_n)-a_n\rho_n^2\varphi^2(1-u_n)\right]$$

with $\rho_n=|\varphi(u_n)|$
where $n\longrightarrow+\infty$ we would have
$bS^2(1-s_1)=|S(1-s_1)|^2\left[\varphi(1-s_1)-a\overline{\varphi(s_1)}\varphi(1-s_1)\right]$
$\Rightarrow$

$$bS^2(1-s_1)=|S(1-s_1)|^2\left[\varphi(1-s_1)-a\rho^2\varphi^2(1-s_1)\right]$$

with $S(s_1)=\varphi(s_1)\,S(1-s_1)=\rho e^{i\theta(s_1)}S(1-s_1)$
$\Rightarrow S^2(1-s_1)=\rho^{-2}e^{-i2\theta(s_1)}S^2(s_1)$ (or $\rho=0$)
$(S(s_1)\neq 0\Leftrightarrow\rho\neq 0)$

$$\Rightarrow\begin{cases} b\rho^{-2}e^{-i2\theta(s_1)}S^2(s_1)=|S(1-s_1)|^2\left[\varphi(1-s_1)-a\rho^2\varphi^2(1-s_1)\right]\\ \text{or}\\ S^2(s_1)=S^2(1-s_1)=0\end{cases}$$

Moreover
$S^2(1-s_1)=\rho^{-2}e^{-i2\theta(s_1)}S^2(s_1)$ and $S^2(s_1)\in IR\Rightarrow|S(1-s_1)|^2=\pm\rho^{-2}S^2(s_1)$
$\Rightarrow\pm be^{-i2\theta(s_1)}=\varphi(1-s_1)-a\rho^2\varphi^2(1-s_1)$ or $S^2(s_1)=S^2(1-s_1)=0$
$\varphi(s_1)=\rho e^{i\theta(s_1)}$ and $\varphi(s_1)\,\varphi(1-s_1)=1$
$\Rightarrow\pm b\rho^2=\varphi(s_1)-a\rho^2$ or $S^2(s_1)=S^2(1-s_1)=0$
$\Rightarrow(a\pm b)\,\rho^2=\varphi(s_1)$ or $S^2(s_1)=S^2(1-s_1)=0$
$(a\pm b)\,\rho^2=\varphi(s_1)=\rho e^{i\theta(s_1)}\Rightarrow$

$$(|a\pm b|\,\rho=1 \text{ and } \theta(s_1)\equiv 0\,[\pi])\ \text{ or } \rho=0$$

$\Rightarrow S^2(1-s_1) = \rho^{-2}e^{-i2\theta(s_1)}S^2(s_1) = \rho^{-2}S^2(s_1) \in IR$ or $S(s_1) = 0$
Conclusion: $S^2(s_1) \in IR \Rightarrow S^2(1-s_1) \in IR$ or $S^2(s_1) = S^2(1-s_1) = 0$ □

Remark 2. *It's obvious if $s \in IR$, $S(s) \in IR$ and $S(1-s) \in IR$.*

Lemma 3. *Let $D = \{z \in \mathbb{C}/\operatorname{Re}(z) \in]0,1[\}$, so $\forall s \in D$*

$$S^2(s) \in IR \Leftrightarrow S^2(1-s) \in IR$$

Proof. Since $D - D_1 = \left\{z \in \mathbb{C}/\operatorname{Re}(z) = \frac{1}{2} \text{ and } \operatorname{Im}(z) \neq 0\right\}$
$\operatorname{Re}(s) = \frac{1}{2} \Rightarrow 1 - s = \overline{s}$
so $S(1-s) = S(\overline{s}) = \overline{S(s)}$
and $S^2(s) \in IR \Leftrightarrow S^2(1-s) \in IR$ □

Claim 1. *Let $D = \{z \in \mathbb{C}/\operatorname{Re}(z) \in]0,1[\}$, so $\forall s \in D$*

$$\begin{aligned} S(s) &\in IR \Leftrightarrow S(1-s) \in IR \\ S(s) &\in iIR \Leftrightarrow S(1-s) \in iIR \end{aligned}$$

Proof. $S(s) \in IR$ or $S(s) \in iIR \Rightarrow S^2(s) \in IR \Rightarrow \theta(s) \equiv 0\,[\pi]$
$S(s) = \varphi(s)\,S(1-s) = \rho e^{i\theta(s)}S(1-s_1) = \pm\rho S(1-s_1)$ □

3.2. **Other results.** Let be $S = S(s)$ and $S' = S(1-s)$such $S^2(s) \in IR$
so $C_1 = \frac{e^{-i\alpha}}{2^r}C$, $C_2 = \left(1 - \frac{e^{-i\alpha}}{2^r}\right)C$ and $S = \left(2^{1-r}e^{-i\alpha} - 1\right)C$
with $C_2 = C - C_1$, $S = C_1 - C_2$
&
$C_1' = \frac{e^{i\alpha}}{2^{1-r}}C'$, $C_2' = \left(1 - \frac{e^{i\alpha}}{2^{1-r}}\right)C'$ and $S' = \left(2^r e^{i\alpha} - 1\right)C'$
with $C_2' = C' - C_1'$, $S' = C_1' - C_2'$
as $1 - s = r' + ic' = 1 - r - ic$, $\alpha = \ln(2)\,c \Rightarrow r' = 1 - r$, $c' = -c \Rightarrow r' = 1 - r$, $\alpha' = -\alpha$

Remark 3. *Let be $S = S(s)$ such $S^2(s) \in IR$, so*
1) $2C_1C_1' = CC'$
2) $C_1\overline{C}' = 2^{1-2r}C\overline{C}_1'$
3) $2C_1\overline{C}_1' = e^{-i2\alpha}C\overline{C}'$
4) $2C_1C_2' = SC'$
5) $2C_2C_2' = SS' \in IR$
6) $SC_1' = CC_2'$ *(&* $S'C_1 = C'C_2$*)*

Proof. 1) $C_1C_1' = \frac{e^{-i\alpha}}{2^r}C\frac{e^{i\alpha}}{2^{1-r}}C' = \frac{CC'}{2}$
2) $C_1' = \frac{e^{i\alpha}}{2^{1-r}}C' \Rightarrow C' = 2^{1-r}e^{-i\alpha}C_1'$
$C_1\overline{C}' = \left(\frac{e^{-i\alpha}}{2^r}C\right)\left(\overline{2^{1-r}e^{-i\alpha}C_1'}\right)$
$C_1\overline{C}' = \left(\frac{e^{-i\alpha}}{2^r}C\right)\left(2^{1-r}e^{i\alpha}\overline{C_1'}\right) = 2^{1-2r}C\overline{C}_1'$
3) $2C_1\overline{C}_1' = 2\left(\frac{e^{-i\alpha}}{2^r}C\right)\left(\overline{\frac{e^{i\alpha}}{2^{1-r}}C'}\right) = 2\frac{e^{-i\alpha}}{2^r}C\frac{e^{-i\alpha}}{2^{1-r}}\overline{C'}$
$2C_1\overline{C}_1' = e^{-i2\alpha}C\overline{C}'$
4) If $S \in IR$ we have $S = 2C_1 - C = 2\overline{C}_1 - \overline{C} = \overline{S} \in IR$
$\Longrightarrow 2C_1 + \overline{C} = 2\overline{C}_1 + C$
$\Longrightarrow 2C_1C' + \overline{C}C' = 2\overline{C}_1C' + CC'$
$\Longrightarrow 2C_1C' + \overline{C}C' = 2\overline{C}_1C' + 2C_1C_1'$

$\Longrightarrow 2C_1C' - 2C_1C_1' = 2\overline{C}_1C' - \overline{C}C'$
$\Longrightarrow 2C_1\left(C' - C_1'\right) = \left(2\overline{C}_1 - \overline{C}\right)C'$
$\Longrightarrow 2C_1C_2' = \overline{S}C' = SC'$.
If $S \in iIR$ we have $S = 2C_1 - C = \overline{C} - 2\overline{C}_1 = -\overline{S} \in iIR$
$\Longrightarrow 2C_1 - \overline{C} = -2\overline{C}_1 + C$
$\Longrightarrow 2C_1C' - \overline{C}C' = -2\overline{C}_1C' + CC'$
$\Longrightarrow 2C_1C' - \overline{C}C' = -2\overline{C}_1C' + 2C_1C_1'$
$\Longrightarrow 2C_1C' - 2C_1C_1' = -2\overline{C}_1C' + \overline{C}C'$
$\Longrightarrow 2C_1\left(C' - C_1'\right) = \left(-2\overline{C}_1 + \overline{C}\right)C'$
$\Longrightarrow 2C_1C_2' = -\overline{S}C' = SC'$.
5) $2C_1C_2' = SC' \Rightarrow 2\left(S + C_2\right)C_2' = SC'$
$\Rightarrow 2SC_2' + 2C_2C_2' = SC' \Rightarrow S\left(C' - S'\right) + 2C_2C_2' = SC'$
$\Rightarrow SC' - SS' + 2C_2C_2' = SC'$
$\Rightarrow SS' = 2C_2C_2'$.
6) is 4)+5) □

Lemma 4.

$$S^2\left(s\right) \in IR \Rightarrow C_2\left(s\right)C_2\left(1-s\right) \in IR$$

Proof. Since the last Claim $\forall s \in D = \{z \in \mathbb{C}/\operatorname{Re}(z) \in]0,1[\}$

$$\begin{aligned} S\left(s\right) &\in IR \Leftrightarrow S\left(1-s\right) \in IR \\ S\left(s\right) &\in iIR \Leftrightarrow S\left(1-s\right) \in iIR \end{aligned}$$

$\Longrightarrow SS' = S\left(s\right)S\left(1-s\right) \in IR$
and from the last Remark 5)
$2C_2C_2' = SS' \in IR$. □

3.3. **The second announcement.**

Claim 2.

$$\exists s_0/S\left(s_0\right) \in iIR \Longleftrightarrow \exists s_1 \in \left(r_0, s_0\right]/S\left(s_1\right) = 0$$

such $r_0 = \operatorname{Re}\left(s_0\right) \in \left]0, \frac{1}{2}\right[\cup \left]\frac{1}{2}, 1\right[$ *and* $\left(r_0, s_0\right] = \{r_0 + ic \in \mathbb{C}/0 \prec c \leq c_0\}$

Proof. Assuming that
$\exists s' = r' + ic'$ such $S\left(s'\right) \in iIR^*$
with $r' \in \left]0, \frac{1}{2}\right[$
Let

$$c_0 = \min\left\{c \in IR^+/\exists n \in IN^*, S^n\left(r' + ic\right) \in iIR^*\right\} \qquad (*)$$

so $\exists m \in IN^*, S^m\left(s_0\right) \in iIR^*$ with $s_0 = r' + ic_0$ $\left(c_0 \leq c'\right)$
$\Rightarrow S^{2m}\left(s_0\right) \in IR_*^-$
without forgetting $S^{2m}\left(r'\right) \in IR_*^+$,
since

$$S\left(r\right) = \sum_{n \geq 1} \frac{\left(-1\right)^n}{n^r} \in IR, \ \forall r \in IR_*^+$$

Let now $S^{2m}\left(s\right) = R\left(s\right) + iI\left(s\right)$
we have $R\left(s_0\right) \prec 0$ and $R\left(r'\right) \succ 0$, so
$\exists s_1 = r' + ic_1 \in \left(r', s_0\right)$ such $R\left(s_1\right) = 0$
$\Rightarrow S^{2m}\left(s_1\right) \in iIR$ with $0 \prec c_1 \prec c_0$

$0 \prec c_1 \prec c_0 \Rightarrow S^{2m}(s_1) \notin iIR_*$ (since $(*)$)

$$\begin{aligned} S^{2m}(s_1) &\in iIR \;\&\; S^{2m}(s_1) \notin iIR_* \Rightarrow S^{2m}(s_1) = 0 \\ &\Rightarrow S(s_1) = 0 \end{aligned}$$

Conclusions:
- $\exists s_0 / S(s_0) \in iIR \Rightarrow S(s_0) \in iIR^*$ or $S(s_0) = 0 \Rightarrow \exists s_1 \in (r_0, s_0] / S(s_1) = 0$.
- The other implication is obvious:
$\exists s_1 \in (r_0, s_0] / S(s_1) = 0 \Rightarrow \exists s_1 \in (r_0, s_1] / S(s_1) = 0$
with $s_1 = s_0$ $S(s_0) = S(s_1) = 0 \Rightarrow \exists s_0 / S(s_0) \in iIR$. □

4. News contributions

Corollary 1.

$$(\exists m_0 \;\&\; \exists s_0 \text{ such } S^{m_0}(s_0) \in iIR) \Longleftrightarrow \exists s_1 \in (r_0, s_0] / S(s_1) = 0$$

such $r_0 = \mathrm{Re}(s_0) \in \left]0, \frac{1}{2}\right[\cup \left]\frac{1}{2}, 1\right[$ *and* $(r_0, s_0] = \{r_0 + ic \in \mathbb{C} / 0 \prec c \leq c_0\}$

Proof. Same proof as the previous one. □

Conjecture 1. $\forall r \in]0,1[$ *and* $s = r + ic$

$$\begin{aligned} r &\neq \frac{1}{2} \Rightarrow \mathrm{Re}[S(s)] \neq 0 \\ \mathrm{Re}[S(s)] &= 0 \Longrightarrow r = \frac{1}{2} \end{aligned}$$

Proof. $\mathrm{Re}[S(s)] = 0 \Longrightarrow S(s) \in iIR \Longrightarrow \exists s_1 \in (r, s] / S(s_1) = 0$ (according to the last Claim)

and according to the Riemann hypothesis:
$S(s_1) = 0 \Longrightarrow \mathrm{Re}(s_1) = \frac{1}{2}$
$\Longrightarrow r = \frac{1}{2}$
since $s_1 \in (r, s]$ & $s = r + ic$. □

Conjecture 2.

$$\left(r = \frac{1}{2} \;\&\; \theta\left(\frac{1}{2} + ic\right) = (2k+1)\pi \right) \Longrightarrow S\left(\frac{1}{2} + ic\right) = 0$$

Proof. Assuming that
$\exists s = \frac{1}{2} + ic$ such $\theta\left(\frac{1}{2} + ic\right) = (2k+1)\pi$ & $S\left(\frac{1}{2} + ic\right) \neq 0$
$S(s) = \varphi(s) S(1-s)$, $1 - s = \overline{s}$, $\varphi(s) = \rho e^{i\theta(s)} = -1$
$\Longrightarrow S(s) = -\overline{S(s)}$
$\Longrightarrow S(s) \in iIR^*$
$\Longrightarrow S^2(s) \in IR_*^-$
$\forall r \in \left]0, \frac{1}{2}\right[$, $S(r) \in IR^* \Longrightarrow S^2(r) \in IR_*^+$
Let now $S^2 = R + iI$
we have $R(s) \prec 0$ and $R(r) \succ 0$, so
$\exists s_0 = r_0 + ic_0 \in (r, s)$ such $R(s_0) = 0$
$\Longrightarrow \exists s_0 = r_0 + ic_0$ such $r_0 \in \left]0, \frac{1}{2}\right[$ & $S^2(s_0) \in iIR$
since $r_0 + ic_0 \in \left(r, \frac{1}{2} + ic\right) \Longrightarrow r < r_0 < \frac{1}{2}$
we have seen in the last corollary that
$(\exists m_0$ & $\exists s_0$ such $S^{m_0}(s_0) \in iIR) \Longrightarrow \exists s_1 \in (r_0, s_0] / S(s_1) = 0$
with $m_0 = 2$ & $s_0 = r_0 + ic_0$

$s_1 \in (r_0, s_0] \Longrightarrow \mathrm{Re}\,(s_1) = r_0 \in \left]0, \frac{1}{2}\right[$
Absurd according to the Riemann hypothesis.
$\Longrightarrow S\left(\frac{1}{2} + ic\right) = 0.$ □

5. Conclusions

We have two new conjectures based on the Riemann hypothesis, and so this is a new way to see if this hypothesis is correct, and if not, we also have a useful new method for determining a counterexample.

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