# Application of M.C.A method to the solution of fractional-order integro-differential equations

# Abstract

In this paper we solve fractional-order integro-differential equations of Fredholm type and volterra type. For the solution we use a new approach to Adomian's decompositional method, which we call the MCA method. This method is a combination of the «Constant Method» and Adomian's method[5].


# Introduction

Many physical phenomena can be modeled by integro-differential equations. An integro-differential equation is an equation that involves both the dereferences of a function and its integrals. It is used in many fields, notably physics, astrophysics, electricity and economics.

Several numerical methods can be used to solve these equations, including the Adomian method [17] [15],[11] [3],[14] [5] the SBA method [3] [2], [4] [6] [7], [8], [9] [10] and the Mellin-SBA method[11].

In this paper we present a new approach to solving fractional-order integrodifferential equations.

**Key words :** partial differential equation, fractional integral, fractional derivative, fractional integro-differential equations.

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# 0.1 Preliminary notions

In this section, we give the basic definitions of fractional analysis [11], [9] [7] [12], [13].

## 0.1.1 special funtions

### 0.1.1.1 Gamma function

Where $x$ is a strictly positive real number (or a complex number with a positive real part), the Gamma function is the function defined on $]0, +\infty[$ by :

$$\Gamma(x) = \int_0^{+\infty} t^{x-1} e^{-t} dt \tag{1}$$

$$\Gamma(\frac{1}{2}) = \int_0^{+\infty} t^{\frac{-1}{2}} e^{-t} dt$$

$\Gamma(\frac{1}{2}) = \sqrt{\pi}$

$\Gamma(1,5) = 0.5\Gamma(0.5) = 0.5\sqrt{\pi}$

The Gamma function can be seen as a generalization of the factorial function.

we have :

$$\Gamma(x+1) = x\Gamma(x) \tag{2}$$

in particular :

$$\Gamma(n+1) = n!$$

### 0.1.2 beta function

Let $x$ and $y$ be two strictly positive real numbers, and the Beta function is defined by :

$$B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}dt \tag{3}$$

For all strictly positive real numbers $x$ and $y$, we have :

$$B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} \tag{4}$$

$$B(x,y) = B(y,x) \tag{5}$$

### 0.1.3 Mittag-Leffler function

The Mittag-Leffler function, known as $E_{alpha,\beta}(z)$, is a special function that applies in the complex plane and depends on two real parameters $\alpha$ and $\beta$. It is defined by :

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{+\infty} \frac{z^k}{\Gamma(\alpha k + \beta)} \tag{6}$$

we note $E_\alpha(z)$ if $\beta = 1$.

$$E_\alpha(z) = \sum_{k=0}^{+\infty} \frac{z^k}{\Gamma(\alpha k + 1)}. \tag{7}$$

The Mittag-Leffler function is a convergent serie. we have :

$$E_{1,1}(x) = e^x$$

$$E_{1,2}(x) = \sum_{k=0}^{+\infty} \frac{x^k}{(k+1)!} = \frac{e^x - 1}{x}$$

$$E_{2,1}(x^2) = cosh(x)$$

### 0.1.4 Fractional integration and derivation

Fractional derivation [16] [1] is a concept that uses derivatives of non-integer order.

#### 0.1.4.1 Fractional integral in the sense of Riemann Liouville

If $a$ is a real number and $\alpha$ a strictly positive real number, we denote by $f$ a locally integrable function defined on $[a;+\infty[$.
The fractional integral of order $\alpha$ of lower bound $a$ is :

$$({}_a^{RL}I_t^\alpha f)(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-\tau)^{\alpha-1} f(\tau) d\tau \tag{8}$$

If there is no ambiguity, we simply note $I_t^\alpha f(t)$ or $I^\alpha f(t)$.

#### 0.1.4.2 Fractional derivative in the sense of Riemann Liouville

Designate by $a$ a real number, by $\alpha$ a strictly positive réel number and by $n$ a non-zero natural number such that : $n-1<\alpha\leq n$.
The fractional derivative in the sense of Riemman Liouvielle is defined by :

$$ {}_{a}^{RL}D_{t}^{\alpha}f(t)=\frac{1}{\Gamma(n-\alpha)}\frac{d^{n}}{dt^{n}}\int_{a}^{t}(t-\tau)^{n-\alpha-1}f(\tau)d\tau \tag{9}$$

**properties :**
Let $\alpha$ and $\beta$ be two strictly positive real numbers, denote by $a$ a real number, $f$ a continuous and locally integrable function définite on [a,+∞[ and by $n$ a natural number such that : $n-1<\alpha\leq n$. We have :

$$ {}_{a}^{RL}D_{t}^{\alpha}f(t)=\frac{d^{n}}{dt^{n}}[{}_{a}I_{t}^{n-\alpha}f(t)] \tag{10}$$

$$ {}_{a}^{RL}D_{t}^{\alpha}[{}_{a}^{RL}I_{t}^{\beta}f(t)]={}_{a}^{RL}D_{t}^{\alpha-\beta}f(t) \tag{11}$$

with $\alpha\geq\beta\geq0$.

$$ {}_{a}^{RL}D_{t}^{\alpha}[{}_{a}^{RL}I_{t}^{\alpha}f(t)]=f(t) \tag{12}$$

$$ {}_{a}I_{t}^{\alpha}[{}_{a}^{RL}D_{t}^{\alpha}f(t)]=f(t)-\sum_{j=1}^{n}[{}_{a}D_{t}^{\alpha-j}f(t)]_{t=a}\frac{(t-a)^{\alpha-j}}{\Gamma(\alpha-j+1)} \tag{13}$$

Let $m$ be a non-zero natural number, and we have :

$$ \frac{d^{m}}{dt^{m}}[{}_{a}^{RL}D_{t}^{\alpha}f(t)]={}_{a}^{RL}D_{t}^{m+\alpha}f(t)] \tag{14}$$

and

$$ {}_{a}^{RL}D_{t}^{\alpha}f^{(m)}(t)]={}_{a}^{RL}D_{t}^{m+\alpha}f(t)-\sum_{j=1}^{m}f^{(j)}(a)\frac{(t-a)^{j-m-\alpha}}{\Gamma(j+1-m-\alpha)} \tag{15}$$

#### 0.1.4.3 Fractional derivative in Caputo's sense

The fractional derivative of order $\alpha$ of $f$ of lower bound $a$ in Caputo's sense is defined by :

$$ {}_{a}^{C}D_{t}^{\alpha}f(t)=\frac{1}{\Gamma(n-\alpha)}\int_{a}^{t}(t-\tau)^{n-\alpha-1}f^{(n)}(\tau)d\tau \tag{16}$$

$f^{(n)}$ denotes the derivative of order $n$ of $f$ and $n$ is a natural number such that
$n-1<\alpha<n$.
we also note ${}^{c}D_{t}^{\alpha}$ if $a=0$.
**Properties**

$$ {}_{a}^{C}D_{t}^{\alpha}f(t)={}_{a}I_{t}^{n-\alpha}[\frac{d^{n}}{dt^{n}}f(t)] \tag{17}$$

$$ {}_{a}^{C}D_{t}^{\alpha}[{}_{a}I_{t}^{\beta}f(t)]={}_{a}^{C}D_{t}^{\alpha-\beta}f(t) \tag{18}$$

in particular

$$ {}_{a}I_{t}^{\alpha}[{}_{a}^{C}D_{t}^{\alpha}f(t)]=f(t)-\sum_{k=0}^{n-1}\frac{t^{k}}{k!}f^{(k)}(0^{+}) \tag{19}$$

**Examples :**
${}^{c}D_{t}^{\alpha}(t-a)^{\beta}=\frac{\Gamma(\beta+1)}{\Gamma(-\alpha+\beta+1)}(t-a)^{\beta-\alpha}$.

${}^{c}D_t^{\alpha}C = 0.$
${}_0I_t^{\alpha}t^{\beta} = \frac{\Gamma(\beta+1)}{\Gamma(\alpha+\beta+1)}t^{\beta+\alpha}$
${}_0I_t^{\alpha}\sqrt{t} = 0.5\sqrt{\pi}t$
${}_aI_t^{\alpha}C = \frac{C}{\Gamma(1+\alpha)}(t-a)^{\alpha}$

### 0.1.5 Fredholm-type integro-differential equations :

The standard form of a Fredholm-type fractional-order integro-differential equation is given by :

$$^{c}Du(x) = f(x) + \frac{1}{\Gamma(\alpha)}\int_a^b (x-t)^{\alpha-1}K(x,t)u(t)dt \tag{20}$$

$K(x,t)$ is the kernel of the equation.

#### 0.1.5.1 Volterra-type integro-differential equation

The standard form of a Volterra-type fractional-order integro-differential equation is given by :

$$^{c}Du(x) = f(x) + \frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1}K(x,t)u(t)dt \tag{21}$$

$K(x,t)$ is the kernel of the equation.

## 0.2 Resolution method

The MCA method is a combination of the constant method and the Adomian method[5].

### 0.2.1 Description of the method on a Fredholm fractional-order integro-differential equation

Consider the following problem :

$$\begin{cases} \frac{\partial^{\alpha}\varphi(x)}{\partial^{\alpha}x} = f(x) + \lambda\int_a^b K(x,t)\varphi(t)dt \\ \varphi(0) = A \end{cases} \tag{22}$$

with $0 \leq a \leq b$.

#### 0.2.1.1 Description

Applying the fractional integral to 22, we obtain :

$$\varphi(x) = A + I_x^{\alpha}(f(x)) + \lambda I_x^{\alpha}(\int_a^b K(x,t)\varphi(t)dt) \tag{23}$$

We introduce the function $\phi(x)$ defined by :

$$\phi(x) = I_x^{\alpha}(\int_a^b K(x,t)\varphi(t)dt) \tag{24}$$

we get

$$\varphi(x) = A + I_x^{\alpha}(f(x)) + \lambda\phi(x) \tag{25}$$

Replacing $\varphi(t)$ by its expression in ( 24), we obtain :

$$\phi(x) = \lambda I_x^\alpha \int_a^b K(x,t)[A + I_t^\alpha(f(t)) + \lambda\phi(t)]dt \tag{26}$$

we get

$$\phi(x) = I_x^\alpha \int_a^b K(x,t)Adt + I_x^\alpha(\int_a^b K(x,t)I_t^\alpha(f(t)))dt$$
$$+ \lambda I_x^\alpha(\int_a^b K(x,t)\phi(t))dt \tag{27}$$

Posing
$F(x) = I_x^\alpha \int_a^b K(x,t)Adt + I_x^\alpha \int_a^b K(x,t)I_t^\alpha(f(t))dt$,
we get

$$\phi(x) = F(x) + \lambda I_x^\alpha(\int_a^b K(x,t)\phi(t))dt \tag{28}$$

The solution $\phi(x)$ is found in the form

$$\phi(x) = \sum_{n=0}^{+\infty} \phi_n(x) \tag{29}$$

We derive the Adomian algorithm below :

$$\begin{cases} \phi_0(x) = F(x) \\ \phi_n(x) = \lambda I_x^\alpha(\int_a^b K(x,t)\phi_{n-1}(t))dt \;\; n \geq 1 \end{cases} \tag{30}$$

If the algorithm 30 converges, then we obtain $\phi(x)$ in the form :

$$\phi(x) = \sum_{n=0}^{+\infty} \phi_n(x) \tag{31}$$

The solution of 22 is deduced :

$$\varphi(x) = A + I_x^\alpha(f(x)) + \lambda\phi(x) \tag{32}$$

**Note**
In some cases, for fast convergence, we can use **the modified Adomian algorithm** [5], which consists in decomposing $F(x)$ into the form

$$F(x) = F_1(x) + F_2(x) \tag{33}$$

we get the **modified Adomian algorithm** :

$$\begin{cases} \phi_0(x) = F_1(x) \\ \phi_1(x) = F_2(x) + \lambda I_x^\alpha \int_a^b K(x,t)\phi_0(t)dt \;\; n \geq 1 \\ \phi_n(x) = \lambda I_x^\alpha(\int_a^b K(x,t)\phi_{n-1}(t))dt \;\; n \geq 2 \end{cases} \tag{34}$$

#### 0.2.1.2 Method convergence

Let's go back to the algorithm 30 :

$$\begin{cases} \phi_0(x) = F(x) \\ \phi_n(x) = \lambda I_x^\alpha(\int_a^b K(x,t)\phi_{n-1}(t))dt \;\; n \geq 1 \end{cases} \tag{35}$$

with

$$F(x) = I_x^\alpha \int_a^b K(x,t)Adt + I_x^\alpha \int_a^b K(x,t)I_t^\alpha(f(t))dt$$

#### 0.2.1.3 Proposition

Under the assumptions, $f \in C([0,T]$, $K \in C([a,T]^2)$, $t \in [0,T]$ and $|\frac{M(b-a)T^\alpha}{\Gamma(\alpha+1)}| < 1$ algorithm 30 converges.

**Proof** :

$f \in C([0,T]$ and $K \in C([0,T]^2)$ so there are two real numbers $m$ and $M$ such that $|f(x)| \leq m$ and $|K(x,t)| \leq M$

$$\begin{aligned}
|\phi_0(x)| = |F(x)| &= |I_x^\alpha \int_a^b K(x,t)Adt + I_x^\alpha \int_a^b K(x,t)I_t^\alpha(f(t))dt| \\
&= |I_x^\alpha[\int_a^b K(x,t)(A + I_t^\alpha(f(t))dt]| \\
&\leq |I_x^\alpha[\int_a^b K(x,t)(A + \frac{mt^\alpha}{\Gamma(\alpha+1)})dt]| \\
&\leq I_x^\alpha[\int_a^b M|A + \frac{mT^\alpha}{\Gamma(\alpha+1)}|dt]
\end{aligned}$$

let : $q = |A + \frac{mT^\alpha}{\Gamma(\alpha+1)}|$.

we get :

$$\begin{aligned}
|\Phi_0(x)| &\leq MI_x^\alpha \int_a^b qdt \\
&\leq Mq(b-a)\frac{x^\alpha}{\Gamma(\alpha+1)} \\
&\leq Mq(b-a)\frac{T^\alpha}{\Gamma(\alpha+1)} \\
&\leq Mq\frac{T^\alpha}{\Gamma(\alpha+1)}(b-a)
\end{aligned}$$

we obtain

$$\begin{aligned}
|\phi_n(x)| &= |\lambda I_x^\alpha \int_a^b K(x,t)\phi_{n-1}(t)dt| \\
&\leq \lambda|I_x^\alpha \int_a^b M\phi_{n-1}(t)dt|
\end{aligned}$$

we get :

$$\begin{aligned}
|\phi_1(x)| &\leq \lambda|I_x^\alpha \int_a^b M\phi_0(t)dt| \\
&\leq \lambda|I_x^\alpha \int_a^b M^2q\frac{T^\alpha}{\Gamma(\alpha+1)}(b-a)dy \\
&\leq q\lambda|\frac{M(b-a)T^\alpha}{\Gamma(\alpha+1)}|^2
\end{aligned}$$

Similarly, we have :

$$
\begin{aligned}
|\phi_2(x)| &= \lambda | I_x^\alpha \int_a^b K(x,t)\phi_1(t)dt| \\
&\leq \lambda | I_x^\alpha \int_a^b Mq\lambda |\frac{M(b-a)T^\alpha}{\Gamma(\alpha+1)}|^2 dt \\
&\leq q\lambda^2 |\frac{M(b-a)T^\alpha}{\Gamma(\alpha+1)}|^3
\end{aligned}
$$

recurrently :

$$|\phi_{n-1}(x)| \leq \frac{q}{\lambda} |\frac{\lambda M(b-a)T^\alpha}{\Gamma(\alpha+1)}|^n \tag{36}$$

we obtain

$$\sum_{i=0}^{n-1} |\phi_i(x)| \leq \frac{q}{\lambda} \frac{1-(\frac{M(b-a)T^\alpha}{\Gamma(\alpha+1)})^n}{1-\frac{M(b-a)T^\alpha}{\Gamma(\alpha+1)}} \tag{37}$$

Under the conditions $|\frac{M(b-a)T^\alpha}{\Gamma(\alpha+1)}| < 1$, we get

$$\sum_{n=0}^{+\infty} |\phi_n(x)| \leq \frac{q}{\lambda} \frac{1}{1-\frac{M(b-a)T^\alpha}{\Gamma(\alpha+1)}} \tag{38}$$

The series

$$\sum_{n=0}^{+\infty} \phi_n(x) \tag{39}$$

is therefore absolutely convergent.
We deduce that the algorithm 30 is convergent.

### 0.2.2 Description of the method on a Volterra fractional-order integro-differential equation

consider the following problem :

$$\begin{cases} \frac{\partial^\alpha \varphi(x)}{\partial^\alpha x} = f(x) + \lambda \int_0^x K(x,t)\varphi(t)dt \\ \varphi(0) = a \end{cases} \tag{40}$$

#### 0.2.2.1 Description

Let's apply the fractional integral to 40 we obtain :

$$\varphi(x) = a + I_x^\alpha(f(x)) + \lambda I_x^\alpha(\int_0^x K(x,t)\varphi(t)dt) \tag{41}$$

We introduce the function $\phi(x)$ defined by :

$$\phi(x) = I_x^\alpha(\int_0^x K(x,t)\varphi(t)dt) \tag{42}$$

We get

$$\varphi(x) = a + I_x^\alpha(f(x)) + \lambda\phi(x) \tag{43}$$

If we replace $\varphi(t)$ by its expression in ( 42), we obtain :

$$\phi(x) = I_x^\alpha[\int_0^x K(x,t)(a + I_t^\alpha(f(t)) + \lambda\phi(t))]dt. \tag{44}$$

We get

$$\phi(x) = I_x^\alpha \int_0^x K(x,t)adt + I_x^\alpha(\int_0^x K(x,t)I_t^\alpha(f(t)))dt$$
$$+ \lambda I_x^\alpha(\int_0^x K(x,t)\phi(t))dt. \tag{45}$$

Let
$F(x) = I_x^\alpha(\int_0^x K(x,t)a)dt + I_x^\alpha \int_0^x K(x,t)I_t^\alpha(f(t))dt$,
we obtain

$$\phi(x) = F(x) + \lambda I_x^\alpha(\int_0^x K(x,t)\phi(t))dt. \tag{46}$$

We are looking for the solution $\phi(x)$ in the form

$$\phi(x) = \sum_{n=0}^{+\infty} \phi_n(x) \tag{47}$$

The algorithm below can be deduced from this :

$$\begin{cases} \phi_0(x) = F(x) \\ \phi_n(x) = \lambda I_x^\alpha(\int_0^x K(x,t)\phi_{n-1}(t))dt \;\; n \geq 1 \end{cases} \tag{48}$$

if the series

$$\sum_0^{+\infty} \phi_n(x)$$

converge the we obtain $\phi_(x)$ such that :

$$\phi(x) = \sum_0^{+\infty} \phi_n(x) \tag{49}$$

This leads to the solution of 40 :

$$\varphi(x) = a + I_x^\alpha(f(x)) + \lambda\phi(x) \tag{50}$$

#### 0.2.2.2 convergence study

Let's go back to the algorithm 48 :

$$\begin{cases} \phi_0(x) = F(x) \\ \phi_n(x) = \lambda I_x^\alpha(\int_0^x K(x,t)\phi_{n-1}(t))dt \;\; n \geq 1 \end{cases}$$

with $F(x) = I_x^\alpha \int_0^x K(x,t)a)dt + I_x^\alpha \int_0^x K(x,t)I_t^\alpha(f(t))dt$

#### 0.2.2.3 Proposition

It is assumed that $f \in C([0,T]$ and $K \in C([0,T]^2)$. The algorithm 48 Converges.
**Proof** :
$f \in C([0,T]$ and $K \in C([0,T]^2)$ so there are two real numbers $m$ and $M$ such that $|f(x)| \leq m$ and $|K(x,t)| \leq M$

$$\begin{aligned}|\phi_0(x)| &= |I_x^\alpha \int_0^x K(x,t)adt + I_x^\alpha \int_0^x K(x,t)I_t^\alpha(f(t))dt| \\ &= |I_x^\alpha[\int_0^x K(x,t)(a + I_t^\alpha(f(t))dt]| \\ &\leq |I_x^\alpha[\int_0^x K(x,t)(a + \frac{mt^\alpha}{\Gamma(\alpha+1)})dt]| \\ &\leq I_x^\alpha[\int_0^x K(x,t)|a + \frac{mT^\alpha}{\Gamma(\alpha+1)}|dt]\end{aligned}$$

There is a real $L$ such that : $|a + \frac{mT^\alpha}{\Gamma(\alpha+1)}| < L$. We obtain

$$\begin{aligned}|\Phi_0(x) &\leq I_x^\alpha[\int_0^x K(x,t)Ldt] \\ &\leq I_x^\alpha[\int_0^x MLdt] \\ &\leq I_x^\alpha xMLdt] \\ &\leq \frac{MLx^{\alpha+1}}{\Gamma(\alpha+2)}\end{aligned}$$

We get

$$\begin{aligned}|\phi_1(x)| &= |\lambda I_x^\alpha \int_0^x K(x,t)\phi_0(t)dt| \\ &\leq |\lambda I_x^\alpha(\frac{M^2Lx^{\alpha+2}}{\Gamma(\alpha+3)})| = |\lambda(\frac{M^2Lx^{2\alpha+2}}{\Gamma(2\alpha+3)})|\end{aligned}$$

Similarly, we have

$$\begin{aligned}|\phi_2(x)| &= |\lambda I_x^\alpha \int_0^x K(x,t)\phi_1(t)dt| \\ &\leq |\lambda^2 I_x^\alpha(\frac{M^3Lt^{2\alpha+3}}{\Gamma(2\alpha+4)})dt| \\ &\leq |\lambda^2(\frac{M^3Lx^{3\alpha+3}}{\Gamma(3\alpha+4)})|\end{aligned}$$

Recurrently :

$$|\phi_{n-1}(x)| \leq |\frac{L}{\lambda}(\frac{\lambda^n M^n x^{n\alpha+n}}{\Gamma(n\alpha+n+1)})| \tag{51}$$

and

$$|\phi_{n-1}(x)| \leq \frac{L}{\lambda}|\frac{(\lambda M x^{(\alpha+1)})^n}{\Gamma(n(\alpha+1)+1)}| \tag{52}$$

indeed

$$\begin{aligned}
|\phi_n(x)| &= |\lambda I_x^\alpha(\int_0^x K(x,t)\phi_{n-1}(t))dt| \\
&\leq \lambda|I_x^\alpha(\int_0^x K(x,t)(\frac{L}{\lambda}\frac{(\lambda M t^{(\alpha+1)})^n}{\Gamma(n(\alpha+1)+1)})|dt \\
|\phi_n(x)| &\leq I_x^\alpha(\frac{L\lambda^n M^{n+1} x^{n\alpha+n+1}}{\Gamma(n\alpha+n+2)}) \\
|\phi_n(x)| &\leq \frac{L}{\lambda}(\frac{\lambda^{n+1} M^{n+1} x^{n\alpha+\alpha+n+1}}{\Gamma(n\alpha+\alpha+n+2)})
\end{aligned}$$

From this we deduce

$$|\phi_n(x)| \leq \frac{L}{\lambda}\frac{(\lambda M x^{\alpha+1})^{n+1}}{\Gamma((n+1)(\alpha+1)+1)} \tag{53}$$

We obtain :

$$\begin{aligned}
\sum_{n=0}^{+\infty}|\phi_n(x)| &\leq \sum_{n=1}^{+\infty}\frac{L}{\lambda}|\frac{(\lambda M t^{(\alpha+1)})^n}{\Gamma(n(\alpha+1)+1)}| \\
\sum_{n=0}^{+\infty}|\phi_n(x)| &\leq \frac{L}{\lambda}E_{\alpha+1}(\lambda M t^{\alpha+1}) - \frac{L}{\lambda}
\end{aligned}$$

The series

$$\sum_{n=0}^{+\infty}\phi_n(x) \tag{54}$$

is therefore absolutely convergent.
We deduce that the algorithm 48 is convergent.

## 0.3 Applications

### 0.3.1 Example 1 :Application to a fractional order integro-differential equation of Volterra type in dimension 1

We consider the following problem :

$$^cDu(x) = \frac{6}{\Gamma(4-\alpha)}x^{3-\alpha} - \beta(\alpha;5)x^{\alpha+5} + \int_0^x (x-t)^{\alpha-1} x t u(t)dt \tag{55}$$

Let us apply the Riemann fractional integral. We obtain :

$$u(x) = x^3 - \frac{\beta(\alpha;5)\Gamma(\alpha+6)}{\Gamma(6+2\alpha)}x^{2\alpha+6} +_x I^\alpha(\int_0^x (x-t)^{\alpha-1} x t u(t)dt) \tag{56}$$

Let

$$\Phi(x) = I_x^\alpha(\int_0^x (x-t)^{\alpha-1} x t u(t)dt). \tag{57}$$

The equation 55 deviates :

$$u(x) = x^3 - \frac{\beta(\alpha;5)\Gamma(\alpha+6)}{\Gamma(6+2\alpha)} x^{2\alpha+6} + \Phi(x) \tag{58}$$

Let us replace in 57 the expression of $u(x)$ obtained in 58. We obtain :

$$\begin{aligned}
\Phi(x) &= I_x^\alpha\left(\int_0^x (x-t)^{\alpha-1} xtu(t)dt\right) \\
&= I_x^\alpha[(\int_0^x (x-t)^{\alpha-1} xt(t^3 - \frac{\beta(\alpha;5)\Gamma(\alpha+6)}{\Gamma(6+2\alpha)} t^{2\alpha+6} + \Phi(t))dt)] \\
&= I_x^\alpha[(\int_0^x (x-t)^{\alpha-1} xt^4 dt - \frac{\beta(\alpha;5)\Gamma(\alpha+6)}{\Gamma(6+2\alpha)} (\int_0^x (x-t)^{\alpha-1} xt^{2\alpha+7} dt] + I_x^\alpha(\int_0^x (x-t)^{\alpha-1} xt\Phi(t)dt) \\
&= I_x^\alpha[\beta(\alpha,5)x^{\alpha+5}) - \frac{\beta(\alpha;5)\Gamma(\alpha+6)}{\Gamma(6+2\alpha)} \int_0^x xt^{7+2\alpha}(x-t)^{\alpha-1}dt] + I_x^\alpha[\int_0^x (x-t)^{\alpha-1} xt\Phi(t)dt] \\
&= \frac{\beta(\alpha+5)\Gamma(\alpha+6)}{\Gamma(2\alpha+6)} x^{2\alpha+6} - \frac{\beta(\alpha+5)\Gamma(\alpha+6)\beta(\alpha;8+2\alpha)}{\Gamma(2\alpha+6)} I_x^\alpha(x^{3\alpha+8}) + I_x^\alpha[\int_0^x (x-t)^{\alpha-1} xt\Phi(t)dt)]
\end{aligned}$$

We get :
$\Phi(x) = \frac{\beta(\alpha+5)\Gamma(\alpha+6)}{\Gamma(2\alpha+6)} x^{2\alpha+6} - \frac{\beta(\alpha+5)\Gamma(\alpha+6)\beta(\alpha;8+2\alpha)\Gamma(3\alpha+9)}{\Gamma(2\alpha+6)\Gamma(4\alpha+9)} x^{4\alpha+8} + I_x^\alpha \int_0^x (x-t)^{\alpha-1} xt\Phi(t)dt$

**Adomian Algorithm**

$$\begin{cases}
\Phi_0(x) &= \frac{\beta(\alpha+5)\Gamma(\alpha+6)}{\Gamma(2\alpha+6)} x^{2\alpha+6} \\
\Phi_1(x) &= -\frac{\beta(\alpha+5)\Gamma(\alpha+6)\beta(\alpha;8+2\alpha)\Gamma(3\alpha+9)}{\Gamma(2\alpha+6)\Gamma(4\alpha+9)} x^{4\alpha+8} + I_x^\alpha[\int_0^x (x-t)^{\alpha-1} xt\Phi_0(t)dt] \\
\Phi_n(x) &= I_x^\alpha[\int_0^x (x-t)^{\alpha-1} xt\Phi_{n-1}(t)dt]
\end{cases} \tag{59}$$

**Calculation of $\Phi_1(x)$**

$$\begin{aligned}
\Phi_1(x) &= -\frac{\beta(\alpha+5)\Gamma(\alpha+6)\beta(\alpha;8+2\alpha)\Gamma(3\alpha+9)}{\Gamma(2\alpha+6)\Gamma(4\alpha+9)} x^{4\alpha+8} + I_x^\alpha[\int_0^x (x-t)^{\alpha-1} xt\Phi_0(t)dt] \\
&= -\frac{\beta(\alpha+5)\Gamma(\alpha+6)\beta(\alpha;8+2\alpha)\Gamma(3\alpha+9)}{\Gamma(2\alpha+6)\Gamma(4\alpha+9)} x^{4\alpha+8} \\
&+ I_x^\alpha[\int_0^x (x-t)^{\alpha-1} xt(\frac{\beta(\alpha+5)\Gamma(\alpha+6)}{\Gamma(2\alpha+6)} t^{2\alpha+6})dt] \\
&= -\frac{\beta(\alpha+5)\Gamma(\alpha+6)\beta(\alpha;8+2\alpha)\Gamma(3\alpha+9)}{\Gamma(2\alpha+6)\Gamma(4\alpha+9)} x^{4\alpha+8} \\
&+ (\frac{\beta(\alpha+5)\Gamma(\alpha+6)}{\Gamma(2\alpha+6)} I_x^\alpha[\int_0^x (x-t)^{\alpha-1} xt^{2\alpha+7}]dt \\
&= -\frac{\beta(\alpha+5)\Gamma(\alpha+6)\beta(\alpha;8+2\alpha)\Gamma(3\alpha+9)}{\Gamma(2\alpha+6)\Gamma(4\alpha+9)} x^{4\alpha+8} \\
&+ \frac{\beta(\alpha+5)\Gamma(\alpha+6)\beta(\alpha;8+2\alpha)\Gamma(3\alpha+9)}{\Gamma(2\alpha+6)\Gamma(4\alpha+9)} x^{4\alpha+8} \\
&= 0
\end{aligned}$$

$$\begin{cases}
\Phi_0(x) &= \frac{\beta(\alpha+5)\Gamma(\alpha+6)}{\Gamma(2\alpha+6)} x^{2\alpha+6} \\
\Phi_1(x) &= -\frac{\beta(\alpha+5)\Gamma(\alpha+6)\beta(\alpha;8+2\alpha)\Gamma(3\alpha+9)}{\Gamma(2\alpha+6)\Gamma(4\alpha+9)} x^{4\alpha+8} + I_x^\alpha[\int_0^x (x-t)^{\alpha-1} xt\Phi_0(t)dt] = 0 \\
\vdots \\
\Phi_n(x) &= 0; n = 2; 3...
\end{cases} \tag{60}$$

We obtain :

$$\Phi(x) = \sum_{n=0}^{+\infty} \Phi_n(x) = \frac{\beta(\alpha+5)\Gamma(\alpha+6)}{\Gamma(2\alpha+6)} x^{2\alpha+6} \tag{61}$$

According to 58, we have :

$$\begin{aligned} u(x) &= x^3 - \frac{\beta(\alpha;5)\Gamma(\alpha+6)}{\Gamma(6+2\alpha)} x^{2\alpha+6} + \Phi(x) \\ &= x^3 - \frac{\beta(\alpha;5)\Gamma(\alpha+6)}{\Gamma(6+2\alpha)} x^{2\alpha+6} + \frac{\beta(\alpha+5)\Gamma(\alpha+6)}{\Gamma(2\alpha+6)} x^{2\alpha+6} \\ &= x^3. \end{aligned}$$

We deduce the general solution to the problem :

$$u(x) = x^3 \tag{62}$$

### 0.3.2 Example 2 : Application to a fractional order integro-differential equation of volterra type in dimension 2

We consider the following problem :

$$\begin{cases} {}^cDu(x,t) = \frac{1}{\Gamma(2-\alpha)} t^{1-\alpha} e^x - \frac{x^{2+\alpha} e^x}{\Gamma(\alpha)} \beta(\alpha, 2) + \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} x u(x,t) dt \\ u(x,0) = 0 \end{cases} \tag{63}$$

Let's apply $I_t^\alpha$.

$$u(x,t) = \frac{e^x \Gamma(2-\alpha)}{\Gamma(2)\Gamma(2-\alpha)} t - \frac{x^{2+\alpha} e^x}{\Gamma(\alpha)\Gamma(1+\alpha)} \beta(\alpha,2) t^\alpha + I_t^\alpha [\frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} x u(x,t) dt]. \tag{64}$$

We obtain :

$$u(x,t) = t - \frac{x^{2+\alpha} e^x \beta(\alpha,2)}{\Gamma(\alpha)\Gamma(1+\alpha)} t^\alpha + I_t^\alpha [\frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} x u(x,t) dt]. \tag{65}$$

Let

$$\Phi(x,t) = I_t^\alpha [\frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} x u(x,t) dt]. \tag{66}$$

The equation 65 becomes :

$$u(x,t) = e^x t - \frac{x^{2+\alpha} e^x \beta(\alpha,2)}{\Gamma(\alpha)\Gamma(1+\alpha)} t^\alpha + \Phi(x,t) \tag{67}$$

Let us replace $u(x,t)$ with its expression in 66.
We obtain :

$$\begin{aligned}
\Phi(x,t) &= I_t^{\alpha}[\frac{1}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1}x[te^x - \frac{x^{2+\alpha}e^x\beta(\alpha,2)}{\Gamma(\alpha)\Gamma(1+\alpha)}t^{\alpha} + \Phi(x,t)]]dt \\
&= I_t^{\alpha}[\frac{1}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1}xe^x t dt] - I_t^{\alpha}[\frac{1}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1}\frac{x^{3+\alpha}e^x\beta(\alpha,2)}{\Gamma(\alpha)\Gamma(1+\alpha)}t^{\alpha} \\
&+ I_t^{\alpha}[\frac{1}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1}x\Phi(x,t)]dt \\
&= I_t^{\alpha}[\frac{x^{\alpha+2}e^x}{\Gamma(\alpha)}\beta(\alpha,2)] - I_t^{\alpha}[\frac{x^{3+\alpha}e^x\beta(\alpha,2)}{\Gamma^2(\alpha)\Gamma(1+\alpha)}\int_0^x (x-t)^{\alpha-1}t^{\alpha}dt] \\
&+ I_t^{\alpha}[\frac{1}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1}x\Phi(x,t)dt] \\
&= I_t^{\alpha}[\frac{x^{\alpha+2}e^x}{\Gamma(\alpha)}\beta(\alpha,2)] - I_t^{\alpha}[\frac{x^{3+3\alpha}e^x\beta(\alpha,2)}{\Gamma^2(\alpha)\Gamma(1+\alpha)}\beta(\alpha;\alpha+1) + I_t^{\alpha}[\frac{1}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1}x\Phi(x,t)dt] \\
&= \frac{x^{\alpha+2}\beta(\alpha,2)e^x}{\Gamma(\alpha)\Gamma(\alpha+1)}t^{\alpha} - \frac{x^{3+3\alpha}e^x\beta(\alpha,2)\beta(\alpha;\alpha+1)}{\Gamma^2(\alpha)\Gamma^2(1+\alpha)}t^{\alpha} + I_t^{\alpha}[\frac{1}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1}x\Phi(x,t)dt]
\end{aligned}$$

We obtain the canonical form of Adomian :

$$\Phi(x,t) = \frac{x^{\alpha+2}\beta(\alpha,2)e^x}{\Gamma(\alpha)\Gamma(\alpha+1)}t^{\alpha} - \frac{x^{3+3\alpha}e^x\beta(\alpha,2)\beta(\alpha;\alpha+1)}{\Gamma^2(\alpha)\Gamma^2(1+\alpha)}t^{\alpha} + I_t^{\alpha}[\frac{1}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1}x\Phi(x,t)dt \tag{68}$$

We get :

$$\begin{cases}
\Phi_0(x,t) &= \frac{x^{\alpha+2}\beta(\alpha,2)e^x}{\Gamma(\alpha)\Gamma(\alpha+1)}t^{\alpha} \\
\Phi_1(x,t) &= -\frac{x^{3+3\alpha}e^x\beta(\alpha,2)\beta(\alpha;\alpha+1)}{\Gamma^2(\alpha)\Gamma^2(1+\alpha)}t^{\alpha} + I_t^{\alpha}[\frac{1}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1}x\Phi_0(x,t)dt \\
\Phi_n(x,t) &= I_t^{\alpha}[\frac{1}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1}x\Phi_{n-1}(x,t)dt
\end{cases} \tag{69}$$

Calculation of $\Phi_1(x,t)$ :

$$\begin{aligned}
\Phi_1(x,t) &= -\frac{x^{3+3\alpha}e^x\beta(\alpha,2)\beta(\alpha;\alpha+1)}{\Gamma^2(\alpha)\Gamma^2(1+\alpha)}t^{\alpha} +_t I^{\alpha}[\frac{1}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1}x\Phi_0(x,t)dt \\
\Phi_1(x,t) &= -[\frac{x^{3+3\alpha}e^x\beta(\alpha,2)\beta(\alpha;\alpha+1)}{\Gamma^2(\alpha)\Gamma^2(1+\alpha)}t^{\alpha} +_t I^{\alpha}[\frac{1}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1}x(\frac{x^{\alpha+2}\beta(\alpha,2)e^x}{\Gamma(\alpha)\Gamma(\alpha+1)}t^{\alpha})dt] \\
&= -[\frac{x^{3+3\alpha}e^x\beta(\alpha,2)\beta(\alpha;\alpha+1)}{\Gamma^2(\alpha)\Gamma^2(1+\alpha)}t^{\alpha} +_t I^{\alpha}[\frac{x^{\alpha+2}\beta(\alpha,2)e^x}{\Gamma^2(\alpha)\Gamma(\alpha+1)}\int_0^x (x-t)^{\alpha-1}xt^{\alpha}dt] \\
&= -[\frac{x^{3+3\alpha}e^x\beta(\alpha,2)\beta(\alpha;\alpha+1)}{\Gamma^2(\alpha)\Gamma^2(1+\alpha)}t^{\alpha} +_t I^{\alpha}[\frac{x^{3\alpha+3}\beta(\alpha,2)e^x}{\Gamma^2(\alpha)\Gamma(\alpha+1)}\beta(\alpha;\alpha+1) \\
&= -[\frac{x^{3+3\alpha}e^x\beta(\alpha,2)\beta(\alpha;\alpha+1)}{\Gamma^2(\alpha)\Gamma^2(1+\alpha)}t^{\alpha} + \frac{x^{3+3\alpha}e^x\beta(\alpha,2)\beta(\alpha;\alpha+1)}{\Gamma^2(\alpha)\Gamma^2(1+\alpha)}t^{\alpha} \\
&= 0
\end{aligned}$$

We obtain

$$\begin{cases} \Phi_0(x,t) & = \frac{x^{\alpha+2}\beta(\alpha,2)e^x}{\Gamma(\alpha)\Gamma(\alpha+1)}t^\alpha \\ \Phi_1(x,t) & = -\frac{x^{3+3\alpha}e^x\beta(\alpha,2)\beta(\alpha;\alpha+1)}{\Gamma^2(\alpha)\Gamma^2(1+\alpha)}t^\alpha + I_t^\alpha[\frac{1}{\Gamma(\alpha)}\int_0^x(x-t)^{\alpha-1}x\Phi_0(x,t)dt = 0 \\ \vdots \\ \Phi_n(x,t) & = I_t^\alpha[\frac{1}{\Gamma(\alpha)}\int_0^x(x-t)^{\alpha-1}x\Phi_{n-1}(x,t)dt = 0 \end{cases} \quad (70)$$

Hence

$$\Phi(x,t) = \sum_0^{+\infty}\Phi_n(x,t) = \frac{x^{\alpha+2}\beta(\alpha,2)e^x}{\Gamma(\alpha)\Gamma(\alpha+1)}t^\alpha \quad (71)$$

We deduce from this

$$\begin{aligned} u(x,t) &= e^x t - \frac{x^{2+\alpha}e^x\beta(\alpha,2)}{\Gamma(\alpha)\Gamma(1+\alpha)}t^\alpha + \Phi(x,t) \\ &= e^x t \end{aligned}$$

We obtain the general solution to the problem :

$$u(x,t) = te^x \quad (72)$$

### 0.3.3 Exemple 3 : Application to a fractional integro-differential equation of Fredholm type in dimension 2

Consider the following problem :

$$\begin{cases} D_t^\alpha u(x,t) = \frac{t^{1-\alpha}}{\Gamma(2-\alpha)}sinx - \frac{1}{3}\lambda x sinx + \lambda\int_0^1 xtu(x,t)dt \\ u(x,0) = 0 \end{cases} \quad (73)$$

Let's apply $I_t^\alpha$. we get :

$$u(x,t) = sinxt - \frac{\lambda x sinx}{3}\frac{1}{\Gamma(\alpha+1)}t^\alpha + \frac{\lambda t^\alpha}{\Gamma(\alpha+1)}\int_0^1 xtu(x,t)dt \quad (74)$$

Let's ask

$$\phi(x,t) = \frac{t^\alpha}{\Gamma(\alpha+1)}\int_0^1 xtu(x,t)dt. \quad (75)$$

We obtain

$$u(x,t) = tsinx - \frac{\lambda x sinx}{3}\frac{1}{\Gamma(\alpha+1)}t^\alpha + \lambda\phi(x). \quad (76)$$

We get :

$$\begin{aligned} \phi(x,t) &= \frac{t^\alpha}{\Gamma(\alpha+1)}\int_0^1 xt(tsinx - \frac{\lambda x sinx}{3}\frac{1}{\Gamma(\alpha+1)}t^\alpha + \lambda\phi(x))dt \\ &= \frac{t^\alpha}{\Gamma(\alpha+1)}\int_0^1 xt^2 sinxdt - \frac{\lambda x sinx}{3}\frac{1}{\Gamma(\alpha+1)}\frac{t^\alpha}{\Gamma(\alpha+1)}\int_0^1 xt^{\alpha+1} + \lambda\frac{t^\alpha}{\Gamma(\alpha+1)}\int_0^1 xt\phi(x))dt \\ &= \frac{t^\alpha x sinx}{\Gamma(\alpha+1)}\int_0^1 t^2 dt - \frac{\lambda x^2 sinxt^\alpha}{3((\Gamma(\alpha+1))^2}\int_0^1 t^{\alpha+1}dt + \lambda\frac{xt^\alpha}{\Gamma(\alpha+1)}\int_0^1 t\phi(x,t))dt \\ &= \frac{t^\alpha x sinx}{3\Gamma(\alpha+1)} - \frac{\lambda x^2 sinxt^\alpha}{3(\alpha+2)(\Gamma(\alpha+1))^2} + \frac{\lambda x t^\alpha}{\Gamma(\alpha+1)}\int_0^1 t\phi(x,t))dt \end{aligned}$$

We deduce from this

$$\phi(x,t)=\frac{t^{\alpha}xsinx}{3\Gamma(\alpha+1)}-\frac{\lambda x^2sinxt^{\alpha}}{3(\alpha+2)(\Gamma(\alpha+1))^2}+\frac{\lambda xt^{\alpha}}{\Gamma(\alpha+1)}\int_0^1 t\phi(x,t))dt. \tag{77}$$

We seek the solution in the form :

$$\phi(x,t)=\sum_{n=0}^{+\infty}\phi_n(x,t) \tag{78}$$

By injecting 78 into 77 and proceeding with an identification, we obtain the following algorithm :

$$\begin{cases}\phi_0(x,t)=\frac{t^{\alpha}xsinx}{3\Gamma(\alpha+1)}\\ \phi_1(x,t)=-\frac{\lambda x^2sinxt^{\alpha}}{3(\alpha+2)(\Gamma(\alpha+1))^2}+\frac{\lambda xt^{\alpha}}{\Gamma(\alpha+1)}\int_0^1 t\phi_0(x,t))dt\\ \phi_n(x,t)=\frac{\lambda xt^{\alpha}}{\Gamma(\alpha+1)}\int_0^1 t\phi_{n-1}(x,t))dt;\ n\geq 1\end{cases} \tag{79}$$

$$\begin{aligned}\phi_1(x,t)&=-\frac{\lambda x^2sinxt^{\alpha}}{3(\alpha+2)(\Gamma(\alpha+1))^2}+\frac{\lambda xt^{\alpha}}{\Gamma(\alpha+1)}\int_0^1 t\phi_0(x,t))dt\\ &=-\frac{\lambda x^2sinxt^{\alpha}}{3(\alpha+2)(\Gamma(\alpha+1))^2}+\frac{\lambda xt^{\alpha}}{\Gamma(\alpha+1)}\int_0^1 t(\frac{t^{\alpha}xsinx}{3\Gamma(\alpha+1)})dt\\ &=-\frac{\lambda x^2sinxt^{\alpha}}{3(\alpha+2)(\Gamma(\alpha+1))^2}+\frac{\lambda x^2sinxt^{\alpha}}{3(\Gamma(\alpha+1))^2}\int_0^1 t^{\alpha+1}dt\\ &=-\frac{\lambda x^2sinxt^{\alpha}}{3(\alpha+2)(\Gamma(\alpha+1))^2}+\frac{\lambda x^2sinxt^{\alpha}}{3(\alpha+2)(\Gamma(\alpha+1))^2}=0\end{aligned}$$

We deduce :
$\phi_1(x,t)=\phi_2(x,t)=...=\phi_n(x,t)=0.$
We obtain

$$\phi(x,t)=\sum_{n=0}^{+\infty}\phi_n(x,t)=\frac{t^{\alpha}xsinx}{3\Gamma(\alpha+1)}. \tag{80}$$

The solution to the problem is :

$$\begin{aligned}u(x,t)&=tsinx-\frac{\lambda xsinx}{3}\frac{1}{\Gamma(\alpha+1)}t^{\alpha}+\lambda\phi(x)\\ &=tsinx-\frac{\lambda xsinx}{3\Gamma(\alpha+1)}t^{\alpha}+\lambda\frac{t^{\alpha}xsinx}{3\Gamma(\alpha+1)}\\ u(x,t)&=tsinx.\end{aligned}$$

# Conclusion

In this article, we describe a new approach for solving linear fractional integro-differential equations. The advantage of this method in linear fractional integro-differential equations is that it converges more quickly to the solution if one exists.

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