Study of Frames in 2-Hilbert Spaces

Abstract: The definition of frame associated to a fixed element in 2-Hilbert spaces was introduced with example. Properties of frame operator were studied. We extended some results of frames in Hilbert spaces to 2-Hilbert spaces.

Key words: 2-norm, 2-inner product, frame, 2-Hilbert space.

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2. Introduction

The concept of frames in Hilbert spaces has been introduced by Duffin and Schaefer in 1952 to study some deep problems in nonharmonic Fourier series. D. Han and D.R. Larson [1] have developed a number of basic aspects of operator-theoretic approach to frame theory in Hilbert space. Peter G. Casazza [2] presented a tutorial on frame theory and he suggested the major directions of research in frame theory.

The concept of linear 2-normed spaces has been investigated by S.Gahler in 1965[3] and has been developed extensively in different subjects by many authors. A concept which is related to a 2-normed space is 2-inner product space which have been intensively studied by many mathematicians in the last three decades. The concept of 2-frames for 2-inner product spaces was introduced by Ali Akbar Arefijammaal and Ghadir Sadeghi [4] and described some fundamental properties of them. Y.J.Cho, S.S.Dragomir, A.White and S.S.Kim[5] are presented some inequalities in 2-inner product spaces. Some results on 2-inner product spaces are described by H.Mazaherl and R.Kazemi[6]. In [8] M. Eshaghi Gordji, A. Divandari, M.R. Safi, and Cho were introduced new concept of 2-Hilbert Spaces.

In this paper the definition of frame associated to a fixed element in 2-Hilbert spaces was introduced with example. Properties of frame operator were studied. We extended some results of frames in Hilbert spaces to 2-Hilbert spaces.

2. Preliminaries

The following basic definitions of 2-normed spaces and 2-inner product spaces from[3,6] are usefull in our discussion..

Definition2.1. Let H be a complex vector space of dimension greater than 1 and let $\|.,.\|$ be a real-valued function on HxH satisfying the following conditions: $forall\ \ a,b,c \in H\ \ and\ \ \ \alpha \in C$

a) $\|a,b\| \geq 0$ and $\|a,b\| = 0$ if and only if a and b are linearly dependent vectors.

b) $\|a,b\| = \|b,a\|$

c) $\|\alpha a,b\| = |\alpha|\|a,b\|$

d) $\|a+b,c\| \leq \|a,c\| + \|b,c\|$

Then $\|.,.\|$ is called 2-norm on H and $(H,\|.,.\|)$ called a linear 2-normed space.

We can consider a 2-norm on H defined by an inner product $\langle .,. \rangle$ on H as follows.

$\|a\ ,\ b\| = \begin{vmatrix} < a,a > & < a,b > \\ < b,a > & < b,b > \end{vmatrix}^{\frac{1}{2}}$ for all $a,b \in H$.

M. Eshaghi Gordji, A. Divandari, M.R. Safi, and Cho[8] were define a new definition of inner product space as follows.

Definition2.2 [8,9]. A Complex vector space H is called a 2-inner product space if there exists a complex-valued function $\langle (.,.),(.,.) \rangle$ on $H^2 \times H^2$ such that for all a,b,c,d,e ∈ H and $\lambda \in C$

(i) If a and b are linearly Independent in H, then $\langle (a,b),(a,b) \rangle > 0$

(ii) $\langle (a,b),(c,d) \rangle = \overline{\langle (c,d),(a,b) \rangle}$

(iii) $\langle (a,b),(c,d) \rangle = -\langle (b,a),(c,d) \rangle$

(iv) $\langle (\alpha a,b),(c,d) \rangle = \alpha \langle (a,b),(c,d) \rangle$

(v) $\langle (a+e,\ b),(c,d) \rangle = \langle (a,b),(c,d) \rangle + \langle (e,b),(c,d) \rangle$

Remerark. 2.3. By using the above we have the following axioms for a,b,c,d, $\bar{b},\bar{c},\bar{d} \in H$ and $\alpha, \beta \in C$

(i) $\langle (a, \alpha b + \bar{b}), (c,d) \rangle = \alpha \langle (a,b),(c,d) \rangle + \langle (a,\bar{b}),(c,d) \rangle$

(ii) $\langle (a,b),(\beta c + \bar{c}, d) \rangle = \bar{\beta} \langle (a,b),(c,d) \rangle + \langle (a,b),(\bar{c},d) \rangle$

(iii) $\langle (a,b),(c, \beta d + \bar{d}) \rangle = \bar{\beta} \langle (a,b),(c,d) \rangle + \langle (a,b),(c,\bar{d}) \rangle$

(iv) $\langle (a,b),(c,d) \rangle = \langle (b,a),(d,c) \rangle$

(v) $\langle (a,b),(a,b) \rangle = 0 \Leftrightarrow a,b$ are linearly dependent

(vi) $\langle (a,b),(a,b) \rangle \geq 0$

Theorem. 2.4[8]. Let H be a 2- inner product space. Then the real valued function $\|.,.\| : H \times H \to R$ defined by $\|a,b\| = \langle (a,b),(a,b) \rangle^{1/2}$ is a 2-norm on H.

Proof: Given H is a 2-inner product space for $a, b, \bar{a} \in H, \alpha \in C$

(i) $\|\alpha a, b\| = \langle (\alpha a, b),(\alpha a, b) \rangle^{1/2} = (\alpha \bar{\alpha})^{1/2} \langle (a,b),(a,b) \rangle^{1/2} = |\alpha| \|a,b\|$

(ii) $\|b,a\| = \langle (b,a),(b,a) \rangle^{1/2} = \langle (a,b),(a,b) \rangle^{1/2} = \|a,b\|$

(iii) $\|a,b\| = 0 \Leftrightarrow \langle (a,b),(a,b) \rangle^{1/2} \Leftrightarrow \langle (a,b),(a,b) \rangle = 0 \Leftrightarrow a,b \, are \, lD$

(iv) $\|a+\bar{a},b\| = \langle (a+\bar{a},b),(a+\bar{a},b) \rangle$

$$= \langle (a,b),(a+\bar{a},b) \rangle + \langle (\bar{a},b),(a+\bar{a},b) \rangle$$

$$= \langle (a,b),(a,b) \rangle + \langle (a,b),(\bar{a},b) \rangle + \langle (\bar{a},b),(a,b) \rangle + \langle (\bar{a},b),(\bar{a},b) \rangle$$

$$= \langle (a,b),(a,b) \rangle + \langle (\bar{a},b),(\bar{a},b) \rangle + \langle (a,b),(\bar{a},b) \rangle + \langle (\bar{a},b),(a,b) \rangle$$

$$= \|a,b\|^2 + \|\bar{a},b\|^2 + \langle (a,b),(\bar{a},b) \rangle + \langle \overline{(a,b),(\bar{a},b)} \rangle$$

$$= \|a,b\|^2 + \|\bar{a},b\|^2 + 2\langle (a,b),(\bar{a},b) \rangle$$

$$\leq \|a,b\|^2 + \|\bar{a},b\|^2 + 2\|a,b\|\|\bar{a},b\| \qquad \text{[By Schwartz inequality)}$$

$$= (\|a,b\| + \|\bar{a},b\|)^2$$

$$\Rightarrow \|a+\bar{a},b\| \leq \|a,b\| + \|\bar{a},b\|$$

For all $a, \bar{a}, b \in H$

$\Rightarrow$ H is a 2- normad space.

Example 2.5. Let H be a complex vector space with inner product <.,.>, we define

$$\langle (a,b),(c,d)\rangle = \begin{vmatrix} <a,c> & <a,d.> \\ <b,c> & <b,d> \end{vmatrix}$$

For all $a,b,c,d \in H$. Then H is a 2-inner product space.

Theorem. 2.6[8]. [The Schwartz inequality] Let H be a 2 – inner product space. Then $\left|\langle (a,b),\ (c,b)\rangle\right|^2 \le \langle (a,b),(a,b)\rangle\langle (c,b),(c,b)\rangle$ for all $a,b,c \in H$.

Proof: For any Complex number λ, we have

$$0 \le \langle (\lambda a + c, b),(\lambda a + c, b)\rangle$$

$$= \langle (\lambda a, b),(\lambda a + c, b)\rangle + \langle (c,b),(\lambda a + c, b)\rangle$$

$$= \langle (\lambda a, b),(\lambda a, b)\rangle + \langle (\lambda a, b),(c,b)\rangle + \langle (c,b),(\lambda a, b)\rangle + \langle (c,b),(c,b)\rangle$$

$$= \lambda\bar{\lambda}\langle (a,b),(a,b)\rangle + \lambda\langle (a,b),(c,b)\rangle + \bar{\lambda}\langle (c,b),(a,b)\rangle + \langle (c,b),(c,b)\rangle$$

Putting $\lambda = -\dfrac{\langle (c,b),(a,b)\rangle}{\langle (a,b),(a,b)\rangle}$ *we get*

$$\le -\frac{\left|\langle (a,b),(c,b)\rangle\right|^2}{\langle (a,b),(a,b)\rangle} - \frac{\langle (c,b),(a,b)\rangle\langle (a,b),(c,b)\rangle}{\langle (a,b),(a,b)\rangle} - \frac{\langle (a,b),(c,b)\rangle\langle (c,b),(a,b)\rangle}{\langle (a,b),(a,b)\rangle} + \langle (c,b),(c,b)\rangle$$

$$\Rightarrow \frac{\left|(a,b),(c,b)\right|^2}{\langle (a,b),(a,b)\rangle} \le \langle (c,b),(c,b)\rangle$$

$$\Rightarrow \left|\langle (a,b),(c,b)\rangle\right|^2 \le \langle (a,b),(a,b)\rangle\langle (c,b),(c,b)\rangle$$

Mazahen and Kazemin [6,8] introduced concept of $\xi -$ cauchy sequence, where ξ is non-zero vector in 2-innerproduct space H.

Definition2.7[8]. Let $\left(H,\langle (.,.),(.,.)\rangle\right)$ be a 2-inner product space and $0 \ne \xi \in H$.

(i) A sequence $\{x_n\}$ in H is a $\xi -$ Cauchy sequence if for any $\in > 0$ there exists a positive integer N such that

$$0 \le \left\| x_m - x_n, \xi \right\| < \in \textit{for all } m > n \ge N$$

(ii) If every ξ – Cauchy sequence converges a point in a semi -2-normad space $(H, \|.,x\|)$ Then H is called a ξ – Hilbert space.
if H is a ξ - Hilbert space for any $\xi \in H$ then we say that H is called 2 – Hilbert space.

3. Frames

The following definitions from [1,2] are useful in our discussion.

Definition3.1. *A sequence $\{x_i\}_{i=1}^{\infty}$ of vectors in a Hilbert space H is called a frame if there exist constants $0 < A \leq B < \propto$ such that*

$$A \|x\|^2 \leq \sum_{i=1}^{\infty} \left|\langle x, x_i \rangle\right|^2 \leq B \|x\|^2 \quad \textit{for all } x \in \text{H}.$$

The above inequality is called the frame inequality. The numbers A and B are called lower and upper frame bounds respectively. If A=B then $\{x_i\}_{i=1}^{\infty}$ is called tight frame, if A=B=1 then $\{x_i\}_{i=1}^{\infty}$ is called normalized tight frame.

Definition3.2. Let $\{x_i\}_{i=1}^{\infty}$ be a frame for H, a synthesis operator T : $l_2 \rightarrow$H is defined as

$$T(c_i) = \sum_{i=1}^{\infty} c_i x_i \,.$$

Definition3.3. Let $\{x_i\}_{i=1}^{\infty}$ be a frame for H, the analysis operator T^* : H $\rightarrow l_2$ is the adjoint of synthesis operator T and is defined as $T^* x = \{\langle x, x_i \rangle\}_{i=1}^{\infty}$ for all x $\in$ H.

Definition3.4. Let $\{x_i\}_{i=1}^{\infty}$ be a frame for the Hilbert space H. A frame operator $S = TT^*$: H$\rightarrow$ H is defined as $Sx = \sum_{i=1}^{\infty} \langle x, x_i \rangle x_i$ for all x $\in$ H.

4. Frames in 2-Hilbert spaces

Definition4.1. Let H be a 2-Hilbert space and $\xi \in H$. A Sequence $\{x_i\}_{i=1}^{\propto}$ of elements of H is called a frame associated to ξ for 2-Hilbert pace H if there exist 0<A<B<$\propto$ such that

$$A\|x,\xi\|^2 \leq \sum_{i=1}^{\propto} \left|\langle (x,\xi),(x_i,\xi)\rangle\right|^2 \leq B\|x,\xi\|^2 \ \textit{for all } x \in H \,.$$

The above inequality is called the frame inequality. The numbers A and B are called lower and upper frame bounds respectively. If A=B then $\{x_i\}_{i=1}^{\infty}$ is called tight frame associated to ξ, if A=B=1 then $\{x_i\}_{i=1}^{\infty}$ is called normalized tight frame associated to ξ.

Example: 4.2: Let $H = R^2$ be a 2-Hilbert space. Consider the set of vectors $\left\{(0,1),\left(\frac{-\sqrt{3}}{2},\frac{-1}{2}\right),\left(\frac{\sqrt{3}}{2},\frac{-1}{2}\right)\right\}$ *in* R^2 associated to $\xi = (1,0) \in R^2$ is a tight frame for R^2 with frame bound 3.

Solution: Let $x = (x_1, x_2) \in R^2$

Consider

$$\sum_{i=1}^{3} \left|\langle (x,\xi),\ (x_i,\xi)\rangle\right|^2 =$$

$$\left|\langle ((x_1,x_2),(1,0)),\ ((0,1),(1,0))\rangle\right|^2 + \left|\left\langle ((x_1,x_2),(1,0)),\ \left((\frac{-\sqrt{3}}{2},\frac{-1}{2}),(1,0)\right)\right\rangle\right|^2 + \left|\left\langle ((x_1,x_2),(1,0))\ \left((\frac{\sqrt{3}}{2},\frac{-1}{2})(1,0)\right)\right\rangle\right|^2$$

$$= \begin{vmatrix} \langle (x_1,x_2),(0,1)\rangle & \langle (x_1,x_2),(1,0)\rangle \\ \langle (1,0),(0,1)\rangle & \langle (1,0),(1,0)\rangle \end{vmatrix}^2 + \begin{vmatrix} \left\langle (x_1,x_2),(\frac{-\sqrt{3}}{2},\frac{-1}{2})\right\rangle & \langle (x_1,x_2),(1,0)\rangle \\ \left\langle (1,0),(\frac{-\sqrt{3}}{2},\frac{-1}{2})\right\rangle & \langle (1,0),(1,0)\rangle \end{vmatrix}^2 +$$

$$\begin{vmatrix} \left\langle (x_1,x_2),(\frac{\sqrt{3}}{2},\frac{-1}{2})\right\rangle & \langle (x_1,x_2),(1,0)\rangle \\ \left\langle (1,0),(\frac{\sqrt{3}}{2},\frac{-1}{2})\right\rangle & \langle (1,0),(1,0)\rangle \end{vmatrix}$$

$$= \begin{vmatrix} x_2 & x_1 \\ 0 & 1 \end{vmatrix}^2 + \begin{vmatrix} \frac{-\sqrt{3}}{2}x_1 - \frac{x_2}{2} & x_1 \\ \frac{-\sqrt{3}}{2} & 1 \end{vmatrix}^2 + \begin{vmatrix} \frac{\sqrt{3}}{2}x_1 - \frac{x_2}{2} & x_1 \\ \frac{\sqrt{3}}{2} & 1 \end{vmatrix}^2 = x_2^2 + x_2^2 + x_2^2 = 3x_2^2 = 3\left\|(x_1,x_2),(1,0)\right\|^2 = 3\|x,\ \xi\|^2$$

$$\Rightarrow \sum_{i=1}^{3} \left|\langle (x,\xi),\ (x_i,\xi)\rangle\right|^2 = 3\|x,\ \xi\|^2 \qquad \forall x \in H(=R^2)$$

$$\Rightarrow \{x_i\}_{i=1}^{3}\ \textit{is tight frame associated to}\ \xi = (1,0)\ \textit{with frame bound}\ 3.$$

Example: 4.3: Let $H = R^2$ be a 2-Hilbert space. Consider the set of vectors $\left\{(0,1), \left(\frac{-\sqrt{3}}{2}, \frac{-1}{2}\right), \left(\frac{\sqrt{3}}{2}, \frac{-1}{2}\right)\right\} in\ R^2$ associated to $\xi = (0,1) \in R^2$ is a tight frame for R^2 with frame bound $\frac{3}{2}$.

Solution: Let $x = (x_1, x_2) \in R^2$

Consider

$$\sum_{i=1}^{3} \left|\langle (x,\xi),\ (x_i,\xi)\rangle\right|^2 =$$

$$\left|\langle ((x_1,x_2),(0,1)),\ ((0,1),(,1))\rangle\right|^2 + \left|\left\langle ((x_1,x_2),(0,1)),\ \left((\frac{-\sqrt{3}}{2},\frac{-1}{2}),(0,1)\right)\right\rangle\right|^2 + \left|\left\langle ((x_1,x_2),(0,1))\ \left((\frac{\sqrt{3}}{2},\frac{-1}{2})(0,1)\right)\right\rangle\right|^2$$

$$= \begin{vmatrix} \langle (x_1,x_2),(0,1)\rangle & \langle (x_1,x_2),(0,1)\rangle \\ \langle (0,1),(0,1)\rangle & \langle (0,1),(0,1)\rangle \end{vmatrix}^2 + \begin{vmatrix} \left\langle (x_1,x_2),(\frac{-\sqrt{3}}{2},\frac{-1}{2})\right\rangle & \langle (x_1,x_2),(0,1)\rangle \\ \left\langle (0,1),(\frac{-\sqrt{3}}{2},\frac{-1}{2})\right\rangle & \langle (0,1),(0,1)\rangle \end{vmatrix}^2 + \begin{vmatrix} \left\langle (x_1,x_2),(\frac{\sqrt{3}}{2},\frac{-1}{2})\right\rangle & \langle (x_1,x_2),(0,1)\rangle \\ \left\langle (0,1),(\frac{\sqrt{3}}{2},\frac{-1}{2})\right\rangle & \langle (0,1),(0,1)\rangle \end{vmatrix}^2$$

$$= \begin{vmatrix} x_2 & x_2 \\ 1 & 1 \end{vmatrix}^2 + \begin{vmatrix} \frac{-\sqrt{3}}{2}x_1 - \frac{x_2}{2} & x_2 \\ \frac{-1}{2} & 1 \end{vmatrix}^2 + \begin{vmatrix} \frac{\sqrt{3}}{2}x_1 - \frac{x_2}{2} & x_2 \\ \frac{-1}{2} & 1 \end{vmatrix}^2 = 0 + \frac{3}{4}x_1^2 + \frac{3}{4}x_1^2 = \frac{3}{2}x_1^2 = \frac{3}{2}\left\|(x_1,x_2),(0,1)\right\|^2 = \frac{3}{2}\|x,\ \xi\|^2$$

$$\Rightarrow \sum_{i=1}^{3} \left|\langle (x,\xi),\ (x_i,\xi)\rangle\right|^2 = \frac{3}{2}\|x,\ \xi\|^2 \qquad \forall x \in H(=R^2)$$

$$\Rightarrow \{x_i\}_{i=1}^{3}\ is\ a\ tight\ frame\ associated\ to\ \xi = (0,1)\ with\ frame\ bound\ \frac{3}{2}.$$

Definition4.4. Let $\{x_i\}_{i=1}^{\infty}$ be a frame associated to ξ for 2-Hilbert space H. Then the Synthesis operator associated to ξ is $T_\xi : l_2 \to H$ is defined as $T_\xi\{c_i\} = \sum_{i=1}^{\infty} c_i x_i$

Definition4.5. Let $\{x_i\}_{i=1}^{\infty}$ be a frame associated to ξ for 2-Hilbert space H. Then the 2-Anaiysis operaor $T_\xi : H \to l^2$ is defined as $T_\xi(x) = \{\langle (x,\xi),(x_i,\xi)\rangle\}_{i=1}^{\infty}$

Definition4.6. Let $\{x_i\}_{i=1}^{\infty}$ be a frame associated to ξ for 2-Hilbert space H. Then the frame Operator

$S_\xi = T_\xi T_\xi^* : H \to H$ is defined as $S_\xi x = \sum_{i=1}^{\infty} \langle (x,\xi),(x_i,\xi)\rangle x_i$

Result 4.7. Consider $\langle (S_\xi x,\xi),(x,\xi)\rangle = \left\langle (\sum_i \langle (x,\xi),(x_i,\xi\rangle x_i,\xi),(x,\xi)\right\rangle$

$$= \sum_i \langle (x,\xi),(x_i,\xi\rangle\langle (x_i,\xi),(x,\xi\rangle$$

$$= \sum_i \langle (x,\xi),(x_i,\xi)\rangle\overline{\langle (x,\xi),(x_i,\xi)\rangle}$$

$$= \sum_i \left|\langle (x,\xi),(x_i,\xi)\rangle\right|^2$$

$$\Rightarrow \langle (S_\xi x,\xi),(x,\xi)\rangle = \sum_i \left|\langle (x,\xi),(x_i,\xi)\rangle\right|^2$$

Result 4.8. Consider $\langle (x,\xi),(S_\xi x,\xi)\rangle = \left\langle (x,\xi),(\sum_i \langle (x,\xi),(x_i,\xi)\rangle x_i,\xi\right\rangle$

$$= \sum_i \langle \overline{(x,\xi),(x_i,\xi)}\rangle\langle (x,\xi),(x_i,\xi)\rangle$$

$$= \sum_i \left|\langle (x,\xi),(x_i,\xi)\rangle\right|^2$$

$$= \langle (S_\xi x,\xi),(x,\xi)\rangle$$

$$\Rightarrow \quad \langle (x,\xi),(S_\xi x,\xi)\rangle = \langle (S_\xi x,\xi),(x,\xi)\rangle$$

Which shows that S_ξ is self – adjoint operator

Theorem 4.9. Suppose $\{x_i\}$ is a frame associated to ξ for 2-Hilbert space $H \Leftrightarrow AI \le S_\xi \le BI$

Proof: Suppose $\{x_i\}$ is a frame associated to ξ for 2-Hilbert space H, so we have

$$A\|x,\xi\|^2 \le \sum_i \left|\langle (x,\xi),(x_i,\xi)\rangle\right|^2 \le B\|x,\xi\|^2 \ \ for\, all\ x \in H$$

Consider $\langle (AIx,\xi),(x,\xi)\rangle = A\langle (x,\xi),(x,\xi)\rangle = A\|x,\xi\|^2$

$$\le \sum_i \left|\langle (x,\xi),(x_i,\xi)\rangle\right|^2 \le B\|x,\xi\|^2 = \langle (BIx,\xi),(x,\xi)\rangle$$

$$\Rightarrow \langle (AIx,\xi),(x,\xi)\rangle \le \langle (S_{\xi}x,\xi),(x,\xi)\rangle \le \langle (BIx,\xi),(x,\xi)\rangle$$

$$\Rightarrow AI \le S_{\xi} \le BI$$

Conversly suppose $AI \le S_{\xi} \le BI$

$$\Rightarrow \langle (AIx,\xi),(x,\xi)\rangle \le \langle (S_{\xi}x,\xi),(x,\xi)\rangle \le \langle (BIx,\xi),(x,\xi)\rangle$$

$$\Rightarrow A\|x,\xi\|^2 \le \sum_i |\langle (x,\xi),(x,\xi)\rangle|^2 \le B\|x,\xi\|^2 \; \forall x \in H$$

$\Rightarrow \{x_i\}$ i sequence is a frame associated to ξ for 2-Hilbert space H.

Theorem 4.10. Suppose $\{x_i\}_{i=1}^{\infty}$ is a sequence in 2-Hilbert space H, with $x = \sum_{i=1}^{\infty}\langle (x,\xi),\ (x_i,\ \xi)\rangle x_i$ holds for all $x \in H$ if and only if $\{x_i\}_{i=1}^{\infty}$ is a normalized tight frame associated to ξ for 2-Hilbert space H.

Proof: Suppose that $\{x_i\}_{i=1}^{\infty}$ is a normalized tight frame associated to ξ for 2-Hilbert space H,

for all $x \in H$ $\quad \Leftrightarrow \|x,\xi\|^2 = \sum_{i=1}^{\infty}|\langle (x,\xi),\ (x_i,\xi)\rangle|^2$

$$\Leftrightarrow \|x,\xi\|^2 = \sum_{i=1}^{\infty}\langle (x,\xi),(x_i,\xi)\rangle\overline{\langle (x,\xi),(x_i,\xi)\rangle}$$

$$\Leftrightarrow \|x,\xi\|^2 = \sum_{i=1}^{\infty}\langle (x,\xi),(x_i,\xi)\rangle\langle (x_i,\xi),(x,\xi)\rangle$$

$$\Leftrightarrow \langle (x,\xi),(x,\xi)\rangle = \left\langle \sum_{i=1}^{\infty}\langle (x,\xi),(x_i,\xi)\rangle (x_i,\xi),\quad (x,\xi)\right\rangle$$

$$\Leftrightarrow (x,\xi) = \sum_{i-1}^{\infty}\langle (x,\xi),(x_i,\xi)\rangle (x_i,\xi)$$

$$\Leftrightarrow (x,\xi) = \left(\sum_{i=1}^{\infty}\langle (x,\xi),(x_i,\xi)\rangle x_i,\ \xi\right)$$

$$\Leftrightarrow x = \sum_{i=1}^{\infty}\langle (x,\xi),\ (x_i,\ \xi)\rangle x_i$$

Theorem4.11. Suppose $\{x_i\}_{i=1}^{\infty}$ is a frame associated to ξ for 2-Hilbert space H and T is co-isometry. then $\{Tx_i\}_{i=1}^{\infty}$ is a frame associated to ξ for 2-Hilbert space H.

Proof: Given $\{x_i\}_{i=1}^{\infty}$ is a normalized tight frame associated to ξ for 2-Hilbert space H, by definition we have $A\|x,\xi\|^2 \le \sum_i |\langle (x,\xi),(x_i,\xi)\rangle|^2 \le B\|x,\xi\|^2$ *for all* $x \in H$ (1)

Since $T^*: H \to H$ is an operator for all $x \in H$, we have $T^*x \in H$ therefore equation (1) is true for $T^*x \in H$

$$A\|T^*x,\xi\|^2 \le \sum_i |\langle (T^*x,\xi),(x_i,\xi)\rangle|^2 \le B\|T^*x,\xi\|^2 \text{ for all } x \in H$$

$$\Rightarrow \quad A\|T^*x,\xi\|^2 \le \sum_i |\langle (x,\xi),(Tx_i,\xi)\rangle|^2 \le B\|T^*x,\xi\|^2 \text{ for all } x \in H$$

Since T is co-isometry so we have

$$\Rightarrow \quad A\|x,\xi\|^2 \le \sum_i |\langle (x,\xi),(Tx_i,\xi)\rangle|^2 \le B\|x,\xi\|^2 \text{ for all } x \in H$$

Which shows that $\{Tx_i\}_{i=1}^{\infty}$ is a frame associated to ξ for 2-Hilbert space H.

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