UNDER PEER REVIEW

Original Research Article

# The Independent Edge Domination Topology Induced by Path Graphs

## Abstract

Let $G = (V(G), E(G))$ be a nonempty undirected graph. An independent edge dominating set is an independent set of edges which is also an edge dominating set of $G$. In the study of Ontulan and Balingit [12], they introduced a new approach of generating a unique topology from the family of independent edge dominating sets of $G$, this is called the *independent edge domination topology* on $E(G)$, herein denoted as $\tau_{ID}^{E}(G)$. This paper focuses on the independent edge domination topological space induced by the path graphs $P_n$, where $n > 1$. Moreover, some properties and characterizations of the independent edge dominating sets and the corresponding independent edge domination topology of path graphs are established.

*Keywords: Independent edge domination topology, independent edge dominating set, independent edge, edge dominating set, path graphs, python program*

2010 Mathematics Subject Classification: $54A05, 54A25, 05C38, 05C70, 05C75, 68R10$

# 1 Introduction

Graph theory and topology are two fundamental branches of mathematics that play a significant role in various applications, ranging from network analysis to computational topology. Graph theory studies the relationships between discrete objects, primarily focusing on vertices and edges, while topology examines properties of spaces that remain invariant under continuous deformations. The fusion of these disciplines has led to the development of topological graph theory, which explores how graph structures induce topological spaces and vice versa [7].

Several methodologies have been proposed to construct topological spaces from graphs. Some focus on vertex sets, while others utilize edge sets. For instance, Macaso and Balingit [9] introduced the block topology, generated by the family of vertex sets of the blocks of a graph. In the study of Hassan and Abed [8], they associated a new topology with graphs that permit isolated vertices. In 1983, Gervacio and Diesto presented another interesting way to constructing a topological space from a given undirected graph by using closed neighborhood subsets of the vertex set of a graph [5]. Canoy, Lemence, and Nianga also used techniques to investigate the construction of topologies induced by some special families of graphs and some unary and binary operations on graphs [3],[2]. Sari and Kapuzlo presented another interesting approach in the same field of study by collecting the minimal adjacency of vertices in a graph [13]. In 2018, Abdu and Kilicman [1] used the compatibility and incompatibility of edges on a directed graph to generate the topologies on the set of edges.

Ontulan and Balingit [12] recently introduced a novel method for generating a unique topology through the use of edges, known as the independent edge domination topology. The authors came up with properties and a way to describe a breakdown of independent edge dominating sets as a subbase for building an independent edge dominating topology. They also examine various characterizations of the independent edge domination topology as induced by specific families of graphs.

This study expanded upon the independent edge domination topology resulting from path graphs. The study established characterizations and properties for the specified topology in the path graph. This study also presents a Python program that generates a list of the independent edge dominating sets of the path graph $P_n$ and a corresponding list of the open sets of the independent edge dominating topology derived from those sets. The total count of independent edge dominating sets and the total count of open sets of $P_n$ for a given $n$ is also provided.

# 2 Preliminary Notes

The fundamental definitions of graphs, topological spaces, and independent edge domination are provided in this section. The definition of the fundamental concepts in graph theory and topology used in this paper are provided in this section, aided with some illustration and examples.

**Definition 2.0.1.** [4] A **graph** $G = (V(G), E(G))$ is a finite nonempty set $V(G)$ of objects called **vertices** (the singular is **vertex**) together with a possibly empty set $E(G)$ of 2-element subsets of $V(G)$ called **edges**. Here, $V(G)$ is the vertex set of a graph $G$ while $E(G)$ is the edge set of the graph $G$. The **order** of a graph $G$ refers to the number of vertices in $G$, while the **size** (or **length**) of a graph $G$ refers to the number of edges in $G$. Two distinct vertices $v_1$ and $v_2$ are **adjacent** if $v_1, v_2 \in G$ and two edges are **adjacent** if they have a common vertex. A graph of size $0$ is called an **empty graph**. In any empty graph, no two vertices are adjacent. A **nonempty graph** then has one or more edges. A graph $G$ is **connected** if every two vertices of $G$ are adjacent, that is, if $G$ contains a $u - v$ path for every pair $u, v$ of vertices of $G$. A graph $G$ that is not connected is called **disconnected**.

**Notation:** Let $G = (V(G), E(G))$ be a simple graph of order $n$, where $V(G) = \{v_1, v_2, \ldots, v_n\}$. Henceforth, as a convention, we denote $[n] = \{1, 2, \ldots, n\}$ and $e_{i,j} = v_i v_j$ such that $i < j$, with $i, j \in [n]$.

Observe that the two edges are adjacent if they have a common vertex. With the above notation, the following remark is immediate.

**Remark 2.0.2.** [12] Let $G = \{V(G), E(G)\}$ be a simple graph. Two edges $e_{i_1,j_1}$ and $e_{i_2,j_2}$ of $G$ are adjacent if and only if $\{i_1, j_1\} \cap \{i_2, j_2\} \neq \varnothing$.

**Definition 2.0.3.** [10] A set $F$ of edges in a graph $G$ is an **edge dominating set** if every edge not in $F$ is adjacent to some edge in $F$. A set $F$ of edges is an **independent edge set** if no two edges in $F$ are adjacent. Consequently, an **independent edge dominating set (IEDS)** of $G$ is an independent set of edges which is also an edge dominating set. (The family of all IEDS of $G$ is denoted by $ID_G^E$).

**Example 2.0.4.** Let $G$ be a graph shown in Figure 1. This graph $G$ is labeled considering the notation convention for denoting the vertices and edges. Note that two edges are adjacent if they shared a common subscripts, that is, $e_{1,2}$ and $e_{2,4}$. Observably, the independent edge dominating set of $G$ are $\{e_{1,2}, e_{3,4}\}$, $\{e_{1,4}\}$ and $\{e_{2,4}\}$.

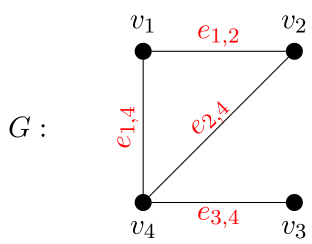


Figure 1: Graph $G$

**Definition 2.0.5.** [11] Let $X$ be a set. A **topology** on a point set $X$ is a collection $\tau$ of subsets of $X$ having the following properties:

$i.$ $\varnothing$ and $X$ are in $\tau$.

$ii.$ The union of the elements of any subcollection of $\tau$ is in $\tau$; that is, if $\{U_\alpha\}_{\alpha \in A} \subset \tau$ then $\bigcup_{\alpha \in A} U_\alpha \in \tau$.

$iii.$ The intersection of the elements of any finite subcollection of $\tau$ is in $\tau$; that is, if $U_1, U_2, \ldots, U_n \in \tau$ then $\bigcap_{i=1}^{n} U_i \in \tau$.

A set $X$ for which a topology $\tau$ has been specified is a **topological space**, denoted as the pair $(X, \tau)$. A subset of $X$ which is in $\tau$ is called a **$\tau$-open set**. If $X$ is any set and $\tau_1$ is the collection of all subsets of $X$ (that is, $\tau_1$ is the power set of $X$, $\tau_1 = \mathcal{P}(X)$) then this is a topological space. $\tau_1$ is called the ***discrete topology*** on $X$. At the other extreme is the topology $\tau_2 = \{\varnothing, X\}$, called the ***indiscrete topology*** or ***trivial topology*** on $X$.

**Theorem 2.0.6.** *[11] In a discrete topological space $(X, \mathcal{P}(X))$, every singleton set $\{x\}$ is both open and closed.*

**Definition 2.0.7.** [6] Let $(X, \tau)$ be a topological space. Given any family $\Sigma = \{A_\alpha : \alpha \in \mathcal{A}\}$ of subsets of $X$, there always exists a unique, smallest topology $\tau(\Sigma) \supset \Sigma$. The family $\tau(\Sigma)$ can be described as follows: It consists of $\varnothing, X$, all finite intersection of the $A_\alpha$, and all arbitrary unions of these finite intersections. $\Sigma$ is called a ***subbasis*** for $\tau(\Sigma)$, and $\tau(\Sigma)$ is said to be ***generated by*** $\Sigma$.

**Definition 2.0.8.** Let $G$ be a nonempty graph. The ***independent edge domination topology*** of $G$, denoted by $\tau_{ID}^{E}(G)$ is the topology generated by the family $ID_G^E$ of all independent edge dominating sets of $G$. The pair $(E(G), \tau_{ID}^{E})$ is called the ***independent edge domination topological space*** of $G$.

By direct application of Definition 2.0.7, the family of all independent edge dominating sets $ID_G^E$ is equivalently equal to $\Sigma$, that is, $\Sigma = ID_G^E$. Furthermore, the independent edge domination topology is equivalently equal to $\tau(\Sigma)$, that is, $\tau(\Sigma) = \tau_{ID}^{E}(G)$.

**Theorem 2.0.9.** *[12] Let $G$ be a nonempty graph. The topology $\tau_{ID}^{E}(G)$ is the indiscrete topology on $E(G)$ if and only if $G$ has $k \geq 1$ components, where each component is isomorphic to either $P_1$ or $P_2$, and at least one component isomorphic to $P_2$.*

**Example 2.0.10.** Consider the graph $G$ in Figure 1 and let $E(G) = \{e_{1,2}, e_{1,4}, e_{2,4}, e_{3,4}\}$. By Example 2.0.4 the $ID_G^E = \{\{e_{1,4}\}, \{e_{2,4}\}, \{e_{1,2}, e_{3,4}\}\}$. By Definition 2.0.7 and Definition 2.0.8, the $\tau_{ID}^{E}(G) = \{\varnothing, \{e_{1,4}\}, \{e_{2,4}\}, \{e_{1,2}, e_{3,4}\}, \{e_{1,4}, e_{2,4}\}, \{e_{1,2}, e_{1,4}, e_{3,4}\}, \{e_{1,2}, e_{2,4}, e_{3,4}\}, E(G)\}$.

UNDER PEER REVIEW

# 3 Main Results

This section presents the key findings of the study, focusing on the independent edge domination topology induced by path graphs.

## 3.1 Independent Edge Domination Topology of Path Graphs

This subsection examines the nature of independent edge domination in path graphs. It formally defines path graphs, introduces relevant notations, and establishes the conditions under which a set of edges qualifies as an independent edge dominating set. The structure and size of its family of independent edge dominating sets are presented.

**Definition 3.1.1.** [4] For the integer $n \geq 1$, the ***path*** $P_n$ is a graph of order $n$ and size $n-1$ whose vertices can be labeled by $v_1, v_2, \ldots, v_n$ and whose edges are $v_i v_{i+1}$ for $i = 1, 2, \ldots, n-1$.

**Notation:** For the path graph $P_n$, we use the following notations: $V(P_n) = \{v_1, v_2, \ldots, v_n\}$, and $E(P_n) = \{v_1v_2, \ldots, v_{n-1}v_n\} = \{e_{1,2}, \ldots, e_{n-1,n}\}$.
**Illustration:** The complete graph $P_5$ in Figure 2 is labeled using the abovementioned notation convention.

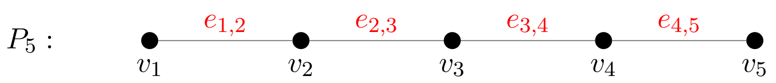


Figure 2: Path Graph $P_5$

Here, if $e_{i,j} \in E(P_n)$, then $j = i + 1$.

**Theorem 3.1.2.** *Let $P_n$ be a path graph and $S \subseteq E(P_n)$. $S \in ID^E_{P_n}$ if and only if $S$ is of the form $S = \{e_{i_1,j_1}, \ldots, e_{i_k,j_k}\}$ such that $i_1 = 1$ or $2$ and $i_k = n-2$ or $i_k = n-1$ where $2 \leq i_s - i_{s-1} = j_s - j_{s-1} \leq 3$ for all $1 < s \leq k$.*

*Proof.*
($\Leftarrow$) Let $S \subseteq E(P_n)$. Assume that $S = \{e_{i_1,j_1}, \ldots, e_{i_k,j_k}\}$. Observe that the condition, $2 \leq i_s - i_{s-1} = j_s - j_{s-1} \leq 3$ for all $1 < s \leq k$ and the fact that $j_s = i_s + 1$ and $j_{s-1} = i_{s-1} + 1$, imply that $S$ is independent, by Remark 2.0.2. Now consider an edge $e_{p,q} \in E(P_n) \setminus S$. If $e_{p,q} = e_{1,2}$, then $e_{i_1,j_1} = e_{2,3}$ is in $S$, and is adjacent to $e_{p,q}$. Similarly, if $e_{p,q} = e_{2,3}$, then $e_{i_1,j_1} = e_{1,2}$ is in $S$, and is adjacent to $e_{p,q}$. Now, if $e_{p,q} = e_{n-1,n}$, then $e_{i_k,j_k} = e_{p,p-1} \in S$, and, if $e_{p,q} = e_{n-2,n-1}$, then $e_{i_k,j_k} = e_{q+1,q} \in S$. Lastly, if $e_{p,q} \neq e_{i_1,j_1}$ or $e_{p,q} \neq e_{i_k,j_k}$, then this means that there exist $1 < s < k$ such that $e_{p,q}$ is between $e_{i_{s-1},j_{s-1}}$ and $e_{i_s,j_s}$. If $e_{p,q}$ is not adjacent to either $e_{i_{s-1},j_{s-1}}$ or $e_{i_s,j_s}$, then $p - i_{s-1} \geq 2$ and $i_s - p \geq 2$ which imply that $i_s - i_{s-1} = j_s - j_{s-1} \geq 4$, a contradiction to the definition of $S$. Thus, $_{pq}$ is adjacent to either $e_{i_{s-1},j_{s-1}}$ or $e_{i_s,j_s}$, and so $S$ is an edge dominating set. Therefore, $S \in ID^E_{P_n}$.
($\Rightarrow$) Suppose $S$ is not of the given form.

Case 1 : If $e_{i_1,j_1} \notin \{e_{1,2}, e_{2,3}\}$, then $e_{1,2} \notin S$ is not dominated by any edge in $S$. Symmetrically, if $i_k \neq n-2$ and $i_k \neq n-1$, then $e_{n-1,n} \notin S$ is not dominated by any edge in $S$. Thus, $S \notin ID^E_{P_n}$.

Case 2 : Suppose there exists $1 < s < k$ such that $i_s - i_{s-1} = j_s - j_{s-1} \neq 2, 3$. If $i_s - i_{s-1} = 1$, then there exists an edge in $S$ that shared a common vertex to one edge in $S$. By Remark 2.0.2, $S$ is not independent edge set. If there exists $1 < s < k$ such that $i_s - i_{s-1} = j_s - j_{s-1} > 3$, then there exists an edge $e_{p,q} \notin S$, such that $e_{p,q}$ is not dominated by any edge in $S$. Hence, $S$ is not an edge dominating set. Thus, $S \notin ID^E_{P_n}$.

UNDER PEER REVIEW

By the cases, if $S$ is not of the given form, then $S \notin ID^E_{P_n}$. □

**Corollary 3.1.3.** *If $S \in ID^E_{P_n}$, then $l \leq |S| \leq \lfloor \frac{n}{2} \rfloor$, where*

$$l = \begin{cases} \frac{n}{3} & \text{, if } n \equiv 0 \mod 3 \\ \lfloor \frac{n}{3} \rfloor & \text{, if } n \equiv 1 \mod 3 \\ \lceil \frac{n}{3} \rceil & \text{, if } n \equiv 2 \mod 3 \end{cases}.$$

*Proof.* Assume the $S \in ID^E_{P_n}$. In view of Theorem 3.1.2, to maximize $|S|$, the distance of each element in $S$ must be 2, such that $i_s - i_{s-1} = j_s - j_{s-1} = 2$ for all $s = 2, \ldots, n$. Hence, if $n$ is even, then $|S| = \dfrac{n}{2}$, and so, if $n$ is odd, then $|S| = \dfrac{n-1}{2}$. Thus, to generalize, $|S| = \left\lfloor \frac{n}{2} \right\rfloor$. Consequently, to minimize $|S|$, the distance of each element in $S$, is 3 and so, $i_s - i_{s-1} = j_s - j_{s-1} = 3$ for all $s = 2, \ldots, n$. Hence, the following cases are observed:

Case $1$ : If $n \equiv 0 \mod 3$, then $|S| = \dfrac{n}{3}$.

Case $2$ : If $n \equiv 1 \mod 3$, then $|S| = \dfrac{n-1}{3}$.

Case $3$ : If $n \equiv 2 \mod 3$, then $|S| = \dfrac{n+1}{3}$.

Thus, to generalize, $l \leq |S| \leq \lfloor \frac{n}{2} \rfloor$ such that

$$l = \begin{cases} \frac{n}{3} & \text{, if } n \equiv 0 \mod 3 \\ \lfloor \frac{n}{3} \rfloor & \text{, if } n \equiv 1 \mod 3 \\ \lceil \frac{n}{3} \rceil & \text{, if } n \equiv 2 \mod 3 \end{cases}.$$

□

**Theorem 3.1.4.** *For path graph $P_n$ with $n \geq 2$,*

$$|ID^E_{P_n}| = \sum_{k=l}^{\lfloor \frac{n}{2} \rfloor} \left[ \binom{k-1}{n-2k} + 2\binom{k-1}{n-2k-1} + \binom{k-1}{n-2k-2} \right]$$

*where* $l = \begin{cases} \frac{n}{3} & \text{, if } n \equiv 0 \mod 3 \\ \left\lfloor \frac{n}{3} \right\rfloor & \text{, if } n \equiv 1 \mod 3 \\ \left\lceil \frac{n}{3} \right\rceil & \text{, if } n \equiv 2 \mod 3 \end{cases}$.

*Proof.* The number of independent edge dominating set or order $|S| = k$ can be counted using the stars and bars method. $P_n$ has $n-1$ edges and need to select $k$ non-adjacent edges, where $2$ consecutively selected edges have distance of either $2$ or $3$.

Let bars represent the elements of $S$ and stars represent the edges outside of $S$. The problem reduces to counting the numbers of ways to arrange $k$ bars between $n-k-1$ stars such that no two bars are adjacent, no three consecutive stars are adjacent, and there could only be $0$ or $1$ star before the first bar and after the last bar.

Let $x$ be the total number of stars before the first bar and after the last bar. Then $x = 0, 1, 2$. If $x_1$ is the number of stars between the first and second bars, $x_2$ is the number of stars between second and third bars, ..., $x_{k-1}$ is the number of stars between the $(k-1)^{\text{th}}$ and $k^{\text{th}}$ bars, then $x + x_1 + x_2 + \cdots + x_{k-1} = n - k - 1$ stars. Note that $1 \leq x_i \leq 2$. Assign $1$ star for each $x_i$ and choose which among the $x_i$'s will receive the remaining $n-k-1-(k-1)-x = n-2k-x$ stars.

Case 1 : If $x = 0$, then there are $n-2k$ stars to distribute among the $k-1$ positions. This is equivalent to $\binom{k-1}{n-2k}$ ways.

Case 2 : If $x = 1$, then there are $n - 2k - 1$ stars to distribute among the $k - 1$ positions which is equivalent to $\binom{k-1}{n-2k-1}$ ways. Note that if $x = 1$, either there is one star before the first bar or one star after the last bar. Hence, there are a total of $2\binom{k-1}{n-2k-1}$ ways for this arrangement.

Case 3 : If $x = 2$, then there are $n - 2k - 2$ stars to distribute among the $k - 1$ positions which is equivalent to $\binom{k-1}{n-2k-2}$ ways.

Thus, there are a total of $\binom{k-1}{n-2k}+2\binom{k-1}{n-2k-1}+\binom{k-1}{n-2k-2}$ ways to achieve the prescribed arrangement. This is equivalent to the numbers of independent edge dominating sets of $P_n$ having $k$ elements. Since $l \leq k \leq \lfloor \frac{n}{2} \rfloor$ with

$$l = \begin{cases} \frac{n}{3} & \text{, if } n \equiv 0 \mod 3 \\ \left\lfloor \frac{n}{3} \right\rfloor & \text{, if } n \equiv 1 \mod 3 \\ \left\lfloor \frac{n}{3} \right\rfloor & \text{, if } n \equiv 2 \mod 3 \end{cases},$$

by Corollary 3.1.3, one has

$$|ID^E_{P_n}| = \sum_{k=l}^{\lfloor \frac{n}{2} \rfloor} \left[ \binom{k-1}{n-2k} + 2\binom{k-1}{n-2k-1} + \binom{k-1}{n-2k-2} \right].$$

□

**Example 3.1.5.** Consider the path graph $P_5$ in Figure 2 and let $E(P_5) = \{e_{1,2}, e_{2,3}, e_{3,4}, e_{4,5}\}$. By Theorem 3.1.2, notice that $e_{1,2}, e_{3,4} \in E(P_5)$, with a distance of $|e_{1,2} - e_{3,4}| = 2$, that can dominate the entire graph and they are independent edge, and so $S = \{e_{1,2}, e_{3,4}\}$ is an independent edge dominating set. Furthermore, here are the other independent edge dominating set, $S_2 = \{e_{1,2}, e_{4,5}\}$, and $S_3 = \{e_{2,3}, e_{4,5}\}$. In this case, $P_5$ has 3 independent edge dominating sets.

By the Theorem 2.0.6, Theorem 2.0.9 and Definition 2.0.7, the following remarks are observed.

**Remark 3.1.6.** Let $P_n$ be the path graph and $A \subseteq \tau^E_{ID}(P_n)$.

$i.$ If $n = 2$, then $\tau^E_{ID}(P_2)$ is the indiscrete topology on $E(P_2)$, given by $\tau^E_{ID}(P_2) = \{\varnothing, E(P_2)\}$ such that $E(P_2) = \{e_{1,2}\}$.

$ii.$ If $n = 3$, then $\tau^E_{ID}(P_3)$ is the discrete topology on $E(P_3)$, given by $\tau^E_{ID}(P_3) = \{\varnothing, \{e_{1,2}\}, \{e_{2,3}\}, E(P_3)\}$ such that $E(P_3) = \{\{e_{1,2}\}, \{e_{2,3}\}\}$.

$iii.$ If $n \geq 4$, then $\tau^E_{ID}(P_n)$ is neither the indiscrete nor discrete topology on $E(P_4)$, given by $\tau^E_{ID}(P_4) = \{\varnothing, \{e_{2,3}\}, \{e_{1,2}, e_{3,4}\}, E(P_4)\}$ such that $E(P_4) = \{\{e_{1,2}\}, \{e_{2,3}\}, \{e_{3,4}\}\}$. To see this, if $S \in ID^E_{P_n}$ such that $e_{3,4} \in S$, then $e_{1,2} \in S$. For if $e_{1,2} \notin S$, then $e_{2,3} \in S$ since $e_{1,2}$ is dominated by $e_{2,3}$ only, and $S$ is an independent edge dominating set, which is a contradiction since $e_{2,3}$ and $e_{3,4}$ are adjacent. Thus, every independent edge dominating set containing $e_{3,4}$ must also contain $e_{1,2}$, and so every open set containing $e_{3,4}$ must also contain $e_{1,2}$. Hence $\{e_{3,4}\}$ is not $\tau^E_{ID}(P_n)$-open, and so, by Theorem 2.0.6, $\tau^E_{ID}(P_n)$ is not the discrete topology on $E(P_n)$.

By the observation of the Remark 3.1.6 $(iii)$ and by the reflectional symmetry property of $P_n$, the first edge $E_{1,2}$ and the last edge $e_{n-1,n}$ of a path graph $P_n$ are with the same feature. In this case, the following remark is observed.

**Remark 3.1.7.** Every open set containing $e_{3,4}$ must contain $e_{1,2}$. Moreover, every open set containing $e_{n-3,n-2}$ must also contain $e_{n-1,n}$.

With the Remark 3.1.7, one has the following generalization:

**Lemma 3.1.8.** *Let $P_n$ be a path graph with $n \geq 5$. If for all $e_{i,j} \in E(P_n)$ such that $i = \{1, 2, \ldots, n-1\} \setminus \{3, n-3\}$, then $\{e_{i,j}\}$ is $\tau_{ID}^E(P_n)$-open.*

*Proof.* Let $n \geq 5$ and $e_{p,q} \in E(P_n)$. By Theorem 3.1.2, since $2 \leq i_{s+1} - i_s = j_{s+1} - j_s \leq 3$ for all $1 < s \leq k$, $S_1 = \{e_{s,t}, \ldots, e_{p-3,q-3}, e_{p,q}, e_{p+3,q+3}, \ldots, e_{s',t'}\}$, where $e_{s,t} = e_{1,2}$ or $e_{2,3}$ and $e_{s',t'} = e_{n-2,n-1}$ or $e_{n-1,n}$, is an independent edge dominating set of $P_n$. Observe that $S_2 = \{e_{f,g}, \ldots, e_{p-2,q-2}, e_{p,q}, e_{p+2,q+2}, \ldots, e_{f',g'}\}$, where $e_{f,g} = e_{1,2}$ or $e_{2,3}$ and $e_{f',g'} = e_{n-2,n-1}$ or $e_{n-1,n}$, is an independent edge dominating set of $P_n$ such that $e_{s,t} \neq e_{f,g}$ and $e_{s',t'} \neq e_{f',g'}$. Thus, $S_1 \cap S_2 = \{e_{p,q}\} \in \tau_{ID}^E(P_n)$. Hence, $\{e_{p,q}\}$ is an arbitrary, and so every $\{e_{i,j}\} \in E(P_n)$ is $\tau_{ID}^E(P_n)$-open. However, by Remark 3.1.7, $\{e_{3,4}\}, \{e_{n-3,n-2}\}$ are not $\tau_{ID}^E(P_n)$-open. □

**Theorem 3.1.9.** *Let $P_n$ be a path graph with $n \geq 5$. A set $A \subseteq E(P_n)$ is $\tau_{ID}^E(P_n)$-open if and only if $A$ satisfies any of the following forms:*

- $i.$ $A = A' \subseteq E(P_n) \setminus \{e_{3,4}, e_{n-3,n-2}\}$;
- $ii.$ $A = \{e_{1,2}, e_{3,4}\} \cup A'$;
- $iii.$ $A = \{e_{n-3,n-2}, e_{n-1,n}\} \cup A'$; *and*
- $iv.$ $A = \{e_{1,2}, e_{3,4}, e_{n-3,n-2}, e_{n-1,n}\} \cup A'$,

*where $A' \subseteq E(P_n) \setminus \{e_{3,4}, e_{n-3,n-2}\}$*

*Proof.*
$(\Rightarrow)$ Let $P_n$ be a path graph with $n \geq 5$ and $A \subseteq E(P_n)$.

- $i.$ This immediately follows by Lemma 3.1.8 and Remark 3.1.7, and so $A$ is $\tau_{ID}^E(P_n)$-open.
- $ii.$ This immediately follows by $(i)$ and Remark 3.1.7, and so $A$ is $\tau_{ID}^E(P_n)$-open.
- $iii.$ Similarly, immediately follows by $(i)$ and Remark 3.1.7, and so $A$ is $\tau_{ID}^E(P_n)$-open.
- $iv.$ Consequently, by Remark 3.1.7 and immediately follows by $(i)$, $(ii)$, and $(iii)$. So, $A$ is $\tau_{ID}^E(P_n)$-open.

Therefore, the following forms are satisfied.
$(\Leftarrow)$ If $A$ contains $e_{3,4}$ but not $e_{1,2}$ or $A$ contains $e_{n-3,n-2}$ but not $e_{n-1,n}$, then, by Remark 3.1.7, $A$ is not a $\tau_{ID}^E(P_n)$-open set. □

**Theorem 3.1.10.** *For a path graph $P_n$ with $n \geq 2$,*

$$|\tau_{ID}^E(P_n)| = \begin{cases} 2 & \text{, if } n = 2 \\ 4 & \text{, if } n = 3, 4 \\ 2^{n-2} + 2^{n-5} & \text{, if } n \geq 5 \end{cases}.$$

*Proof.* Let $P_n$ be a path graph of order $n \geq 2$.

Case $1:$ If $n = 2$, then, by Remark 3.1.6 $(i)$, $|\tau_{ID}^E(P_2)| = 2$.

Case $2:$ If $n = 3, 4$, then, by Remark 3.1.6 $(ii)$ and $(iii)$, $|\tau_{ID}^E(P_3)| = 4$ and $|\tau_{ID}^E(P_4)| = 4$.

Case $3:$ If $n \geq 5$, then, by Theorem 3.1.9, consider the following counting:

UNDER PEER REVIEW

$i.$ there are $2^{n-3}$ choices of subsets of $E(P_n)$ for $A'$;

$ii.$ there are $2^{n-4}$ choices of which do not contain $e_{1,2}$ union of $A'$;

$iii.$ there are $2^{n-4}$ choices of which do not contain $e_{n-3,n-2}$ union of $A'$; and

$iv.$ there are $2^{n-5}$ choices of which do not contain both $e_{1,2}$ and $e_{n-3,n-2}$ union of $A'$.

Hence, $|\tau^E_{ID}(P_n)| = 2^{n-3} + 2(2^{n-4}) + 2^{n-5} = 2^{n-2} + +2^{n-5}$. □

**Example 3.1.11.** Consider the path graph $P_5$ in Figure 2 and let $E(P_5) = \{e_{1,2}, e_{2,3}, e_{3,4}, e_{4,5}\}$ of the Example 3.1.5. Here are the independent edge dominating sets of $P_5$:

$S_1 = \{e_{1,2}, e_{3,4}\}$;

$S_2 = \{e_{1,2}, e_{4,5}\}$; and

$S_3 = \{e_{2,3}, e_{4,5}\}$.

By Definition 2.0.7, the generated topology of $P_5$ is $\tau^E_{ID}(P_n) = \{\varnothing, \{e_{1,2}\}, \{e_{4,5}\}, \{e_{1,2}, e_{3,4}\}, \{e_{1,2}, e_{4,5}\}, \{e_{2,3}, e_{4,5}\}, \{e_{1,2}, e_{2,3}e_{3,4}\}, \{e_{1,2}, e_{3,4}, e_{4,5}\}, E(P_5)\}$. Observe that by Remark 3.1.7 is satisfied.

## 3.2 Python Program on Independent Edge Domination Topology of Path Graphs

This subsection presents a Python program designed to construct independent edge dominating sets and their associated topologies for path graphs, hence supporting the theoretical findings. The method offers computational verification of the theoretical findings, facilitating practical investigation of independent edge dominationtopology. The program illustrates the construction of independent edge dominating sets, their enumeration, and the generating of topology derived from these sets.

```python
#Generating the Independent Edge Domination Topology on Path Graph using Python
from intertools import combinations

print("Proponent: JHON NECEIR S. ONTULAN")
print("Adviser: CHERRY MAE R. BALINGIT, PhD")

def is_independent_edge_set(edge_set, edges):
    for i in range(len(edge_set)):
        for j in range(i + 1, len(edge_set)):
            if set(edge_set[i]) & set(edge_set[j]):
                return False
    return True

def is_edge_dominating_set(edge_set, edges):
    dominated_set = set()
    for edge in edge_set:
        u, v = edge
        for e in edges:
            e_u, e_v = e
            if u in (e_u, e_v) or v in (e_u, e_v):
                dominated_edges.add(tuple(sorted([e_u, e_v])))
    return len(dominated_edges) == len(edges)

def generate_independent_edge_dominating_sets(n):
```

```
    edges = [(i, i + 1) for i in range(1, n)]
    independent_edge_dominating_sets = []
    for size in range(1, len(edges) + 1):
        for edge_set in combinations(edges, size):
            if is_independent_edge_set(edge_set, edges) and is_edge_dominating_set(edge_set, edges):
                independent_edge_dominating_sets.append(tuple(sorted (edge_set)))
    return sorted(independent_edge_dominating_sets, key=lambda x: (len(x), x))

def generate_topology_from_subbasis(subbasis, universal_set):
    topology = set()
    finite_intersection = set()
    for r in range(1, len(subbasis) + 1):
        for subset in combinations(subbasis, r):
            intersection_set = set(universal_set)
            for s in subset:
                intersection_set &= s
            finite_intersections.add(frozenset(intersection_set))
    for r in range(len(finite_intersections) + 1):
        for subset in combinations(finite_intersections, r):
            union_set = frozenset().union(*subset)
            topology.add(union_set)
    topology.add(frozenset())
    topology.add(frozenset(universal_set))
    return sorted(topology, key=lambda x: (len(x), sorted(x)))

try:
    n = int(input("Enter the number of vertices for the path graph P_n (n ≥ 2): "))
    if n < 2:
        raise ValueError("n must be at least 2.")

    independent_edge_dominating_sets = generate_independent_edge_dominating_sets(n)

    print(f"\nIndependent Edge Dominating Sets for P_{n}:")
    universal_set = set()
    formatted_subbasis = []

    for idx, edge_set in enumerate(independent_edge_dominating_sets, start=1):
        formatted_set = "{ " + ", ".join(f"e_{{{i}{j}}}" for i, j in edge_set) + " }"
        formatted_subbasis.append(frozenset(edge_set))
        universal_set.update(edge_set)
        print(f"{idx}: {formatted_set}")

    print(f"\nTotal number of Independent Edge Dominating Sets for P_{n}: {len(independent_edge_dominating_sets)}")

    topology = generate_topology_from_subbasis(formatted_subbasis, universal_set)
    print(f"\nIndependent Edge Domination Topology on E(P_{n}):")

```

```python
        for idx, subset in enumerate(topology, start=1):
            formatted_topology_set = "{ " + ", ".join(f"e_{{{i}{j}}}" for i, j in sorted(subset)) + " }"
            print(f"{idx}: {formatted_topology_set}")

        print(f"\nTotal number of Open Sets: {len(topology)}")

    except ValueError as e:
        print(f"Invalid input: {e}")
```

List 1 : Generating the Independent Edge Domination Topology on Path Graph using Python

**INPUT and OUTPUT:** Let $n = 7$, then

```
C:\Users\User\AppData\Local\Microsoft\WindowsApps\python3.11.exe "C:\Users\User\Documents\
Thesis Algorithm\Path Graph with topology (main).py"
Proponent: JHON NECEIR S. ONTULAN
Adviser: CHERRY MAE R. BALINGIT, PhD
Enter the number of vertices for the path graph P_n (n ≥ 2): 7

Independent Edge Dominating Sets for P_7:
1: { e_{23}, e_{56} }
2: { e_{12}, e_{34}, e_{56} }
3: { e_{12}, e_{34}, e_{67} }
4: { e_{12}, e_{45}, e_{67} }
5: { e_{23}, e_{45}, e_{67} }

Total number of Independent Edge Dominating Sets for P_7: 5

Independent Edge Domination Topology on E(P_7):
1: {  }
2: { e_{12} }
3: { e_{23} }
4: { e_{56} }
5: { e_{67} }
6: { e_{12}, e_{23} }
7: { e_{12}, e_{34} }
8: { e_{12}, e_{56} }
9: { e_{12}, e_{67} }
10: { e_{23}, e_{56} }
11: { e_{23}, e_{67} }
12: { e_{45}, e_{67} }
13: { e_{56}, e_{67} }
14: { e_{12}, e_{23}, e_{34} }
15: { e_{12}, e_{23}, e_{56} }
16: { e_{12}, e_{23}, e_{67} }
17: { e_{12}, e_{34}, e_{56} }
18: { e_{12}, e_{34}, e_{67} }
19: { e_{12}, e_{45}, e_{67} }
20: { e_{12}, e_{56}, e_{67} }
21: { e_{23}, e_{45}, e_{67} }
22: { e_{23}, e_{56}, e_{67} }
23: { e_{45}, e_{56}, e_{67} }
24: { e_{12}, e_{23}, e_{34}, e_{56} }
25: { e_{12}, e_{23}, e_{34}, e_{67} }
26: { e_{12}, e_{23}, e_{45}, e_{67} }
27: { e_{12}, e_{23}, e_{56}, e_{67} }
28: { e_{12}, e_{34}, e_{45}, e_{67} }
29: { e_{12}, e_{34}, e_{56}, e_{67} }
30: { e_{12}, e_{45}, e_{56}, e_{67} }
31: { e_{23}, e_{45}, e_{56}, e_{67} }
32: { e_{12}, e_{23}, e_{34}, e_{45}, e_{67} }
33: { e_{12}, e_{23}, e_{34}, e_{56}, e_{67} }
34: { e_{12}, e_{23}, e_{45}, e_{56}, e_{67} }
35: { e_{12}, e_{34}, e_{45}, e_{56}, e_{67} }
36: { e_{12}, e_{23}, e_{34}, e_{45}, e_{56}, e_{67} }

Total number of Open Sets: 36

Process finished with exit code 0
```

# 4 CONCLUSIONS

This paper investigates the independent edge domination topology of path graphs consequently offering a formal mathematical foundation for understanding the structure of independent edge dominating sets and their induced topologies. The results show the distinctive characteristics and significance of independent edge domination in path graphs, hence extending the existing knowledge of topological structures in graph theory. This paper provides a substantial contribution through a Python program that systematically produces independent edge dominating sets and develops their associated topologies. This computational tool validates theoretical conclusions and offers an efficient method for examining larger network topologies that may be difficult to investigate manually. The program offers an automated method for analyzing and displaying the topological features of path graphs, serving as an essential tool for academics and practitioners involved in independent edge domination.

## Motivation and Novelty of the Study

The increasing need to understand the connection between graph theory and topology drives this effort particularly in situations where structural connectivity and dominantioncharacteristics are crucial. In network theory, optimization, and communication systems, independent edge dominationis essential; nevertheless, its topological implications are still not sufficiently investigated. This work presents the independent edge dominationtopology thereby linking combinatorial graph structures with topological spaces and offers a fresh perspective on the properties and behaviors of these graphs under topological constraints.

The formalization of independent edge dominance as a topological notion is novelty of this work. This study advances the concept of dominationcharacteristics beyond traditional graph theory into topology therefore facilitating a more thorough and methodical research. A significant progress has been made in the inclusion of a Python program to computationally build and analyze independent edge dominationtopologies because it both supports theoretical conclusions and facilitates actual study and application of the suggested framework.

## References

[1] Abdu, K. and Kilicman, A. (2018). Topologies on the edges set of directed graph. *International Journal of Mathematical Analysis*, 12(2):71–84.

[2] Canoy Jr, S., Gamorez, A., and Nianga, C. (2019). Topologies induced bu neighborhood of a graph under some binary operations. *European Journal of Pure and applied Mathematics*, 12(3):749–755.

[3] Canoy Jr, S. and Lemence, R. (1999). Topologies induced by some special graphs. *Journal of Mathematics*, 2(2):45–50.

[4] Chartrand, G., Lesniak, L., and Zhang, P. (2015). *Graphs and digraphs*. CRC Press, 6th edition.

[5] Diesto, S. and Gervacio, S. (1983). Finite topological graphs. *Journal Research and Development*, 1(1):76–81.

[6] Dugundji, J. (1966). *Topology*. Allyn and Bacon, Inc., 1st edition.

[7] Gross, J. and Yellen, J. (2005). *Graph theory and its applications*. CRC Press, 2nd edition.

[8] Hassan, A. and Abed, Z. (2020). Independent (non-adjacent vertices) topological spaces associated with undirected graphs, with some applications in biomathematics. *Journal of Physics: Conference Series*, 1591:1–10.

[9] Macaso, J. and Balingit, C. (2024). The block topological space and block topological graph induced by undirected graphs. *European Journal of Pure and Applied Mathematics*, 17(2):663–675.

[10] Mojdeh, D. and Sadeghi, R. (2007). Indepedent edge dominating set of certain graphs. *International Mathematical Forum*, 2(7):315–320.

[11] Munkres, J. (1966). *Topology*. Prentice Hall, Inc., 2nd edition.

[12] Ontulan, J. and Balingit, C. Independent edge domination topolohy of some graph families. *(Submitted for publication, under peer-review)*.

[13] Sari, H. and Kopuzlu, A. (2020). On topological spaces generated by simple undirected graphs. *Aims Mathematics*, 5(6):5541–5550.