Original Research Article

On Propositions Pertaining to the Riemann Hypothesis IV

Abstract

In this paper, we enumerate certain hypotheses regarding the Riemann zeta function. The hypotheses are in the form of bounds on the norm of the tail of the sequence that determines the Riemann zeta function and also an optimization problem involving Diophantine approximation. We also relate these hypotheses with the Riemann and Lindelof hypotheses.

Keywords: Riemann hypothesis; Riemann zeta function; Lindelof hypothesis; Probability measures over the unit circle; Ω-phenomena

2010 Mathematics Subject Classification: 97I80; 51M04; 51M05

1 Introduction

The study of consistent patterns observed in specific sub-domains of the Riemann zeta function (Riemann [1859]) is one which has accumulated many different methods (Titchmarsh and Heath-Brown [1986], Edwards [1974]). One is interested in deriving bounds of norms of sums of vectors and also on a sum of norms, with it being possible that a combination of a wide number of associated hypotheses may be stated. A particular phenomenon that we are interested in is the occurence of concentration in the distribution over the unit circle, which corresponds to a complex variable and the series generating the zeta function. The theorems in the paper prove some implications and satisfiability of multiple hypotheses. Some results involve an optimization problem arising from a general Diophantine approximation problem and hypotheses about dependence of the optimal value of the program on a parameter. The prior literature on this subject of study, may be found in Mangoldt [1905], Hardy [1914], Weyl [1916], Weyl [1921], Hardy and Littlewood [1921], Littlewood [1922], Titchmarsh [1928], Conrey [2003], Lagarias [2002], Bump et al. [2000], Borwein et al. [2008], Platt and Trudgian [2021], Nicolas [2021], Johnston [2022], Basu [2022], Basu [2023a], Basu [2023b], Basu [2024], Maynard and Pratt [2024], Guth and Maynard [2024] and Tao et al. [2025].

2 Asymptotic considerations for the Riemann zeta function

In this section, we shall study the following topics.

1) A Diophantine approximation problem in the form of an optimization problem and lower bound the optimal value conditional on a hypothesis on it's limit. As one scales over the size m of the problem, we can show that there does not exist an upper bound that is the exponential of a power of logarithm i.e. $O(e^{\ln^k(m)})$ with $k \in (0,1)$. This will be related to an approximate functional equation for the Riemann zeta function for the line $\sigma = 1$.

2) We will consider the validity of certain hypotheses concerning the Riemann zeta function in the critical region i.e. $0 < \sigma < 1$ and also in the situation when $\sigma = 1$. We will relate these hypothesis with the Riemann hypothesis (Riemann [1859],Riemann [1859]), the Lindelof hypothesis (Lindelof [1908]) and a strong version of the Lindelof hypothesis involving the tail of the absolutely convergent series corresponding to the Riemann zeta function.

3) Finally, we will derive certain polynomial inequalities (of order four) in integers, which would then be related to the Diophantine approximation problem.

4) All of the above problems will then related to the idea of searching for discrepancies creating in series, such as those corresponding to the law of iterated expectations such as concentration (Basu [2022], Basu [2023a],Basu [2023b], Basu [2024]) and Ω-phenomena (see chapter 8 of Titchmarsh and Heath-Brown [1986]).

We will introduce some notation for asymptotics. Given a subset of the Euclidean space $X \subseteq \mathbb{R}^{d_1}$ and functions $f, g : X \to \mathbb{R}^{d_2}$ we say $f = O(g)$ if there exists a constant $C > 0$ such that $||f(x)|| \leq C||g(x)||$ for each $x \in X$. We say that $f = \Omega(g)$ if there exists an unbounded subset $X' \subseteq X$ and a constant $C > 0$ such that $||f(x)|| \geq C||g(x)||$ for each $x \in X'$.

The Riemann zeta function corresponds to series that involves complex numbers of the form n^{-s}, in which $s = (\sigma, t)$. The Riemann zeta function, for $0 < \sigma \leq 1$, is defined by the function

$$\zeta(s) = (\frac{1}{1 - 2^{1-s}}) \times \sum_{n \geq 0} Z_n(s) \tag{2.1}$$

in which $Z_0(s) = (1, 0)$; $Z_n(s) = \frac{1}{(2n+1)^s} - \frac{1}{(2n)^s}$, for each $n \in \mathbb{Z}^+$. We are also interested in properties of the following function.

$$\zeta^*(s) = \sum_{n \geq 0} Z_n(s). \tag{2.2}$$

The above series is absolutely convergent and one may derive the polar form of the individual elements in the sequence $Z_n(s)$.

$$Z^*(\sigma, t, m) := \sum_{n \geq m+1} ||Z_n(s)||. \tag{2.3}$$

It will be useful to prove a theorem involving lower bounds for a sum of vectors in the complex plane. For any $z \in \mathbb{R}^2$, denote as $\theta(z) \in [0, 2\pi)$, the angle corresponding to the complex number z.

Proposition 2.1. *Suppose that $\theta', \theta'' \in [0, 2\pi)$ are such that $0 \leq \theta'' - \theta' \leq \frac{\pi}{2}$. Suppose that $z_1, z_2, ..., z_n$ are vectors in $\mathbb{R}^2$ such that $\theta(z_m) \in [\theta', \theta'']$ for each $m \in \{1, 2, .., n\}$. Then,*

$$|| \sum_{m=1}^{n} z_m|| \geq \cos(\theta'' - \theta')(\sum_{m=1}^{n} ||z_m||). \tag{2.4}$$

Proof. We prove the theorem by induction. We first prove the result for two vectors. Suppose we have z_1, z_2, all within an angle of $\theta'' - \theta'$. One notes that the set $\{z : \theta(z) \in [\theta', \theta'']\}$ is a convex cone.

Then, by dropping a perpendicular from $z_1 + z_2$ on the line joining 0 and z_2, say at point z', then we have that $\{0, z_1 + z_2, z'\}$ form a right angled triangle. Note that the angle between z_1 and z_2 is equal to the angle from z_2 to the altitude $[z_1 + z_2, z']$ of the triangle (denoting $[z', z'']$ to be the line segment joining z' and z''). Define this angle to be θ'''. Then, we have that $\theta''' \leq \theta'' - \theta'$. The length of the hypotenuse is the length $||z_1 + z_2||$. By Pythagoras's theorem,

$$\begin{aligned} ||z_1 + z_2||^2 &= (||z_2|| + \cos(\theta''')||z_1||)^2 + \sin^2(\theta''')||z_1||^2 && (2.5)\\ &\geq \cos^2(\theta'' - \theta')(||z_1|| + ||z_2||)^2. && (2.6) \end{aligned}$$

For the inductive step, we have $z_1, ..., z_{n-1}$ such that $\theta(z_m) \in [\theta', \theta'']$ and $m \in \{1, ..., n-1\}$. Then, we may apply the above observation for two vectors and then the result follows. ☐

Of course, by the symmetry of the unit circle, the above theorem also holds in the situation where we have $\theta', \theta'' \in [0, 2\pi)$ such that $0 \leq \theta' + (2\pi - \theta'') \leq \frac{\pi}{2}$ and $\theta(z_m) \in [0, \pi] \cup [\pi'', 2\pi)$.

In Basu [2024], a formula was derived for the norm $||Z_n(s)||$ by a half-angle formula for the sine and cosine trigonometric functions.

$$||Z_n(s)|| = (2n)^{-\sigma} - (2n+1)^{-\sigma} + 2(2n)^{-\sigma}\Big|\sin\Big(0.5|t|\ln\Big(1 + \frac{1}{2n}\Big)\Big|. \tag{2.7}$$

This allow us to obtain the upper bound $Z^*(\sigma, t, m) = O(h(\sigma, t, m))$, where

$$h(\sigma, t, m) = \int_{(0, \frac{1}{m+1})} \omega^{\sigma-2}\ln(1 + \omega|t|)d\nu(\omega). \tag{2.8}$$

The measure ν denotes the Lebesgue measure on $[0, 1]$. The next theorem is a relatively simple approximate formula for the Riemann zeta function.

Proposition 2.2. *For each $\sigma \in (0, 1]$, we have that*

$$\zeta(s) = \sum_{m=1}^{n} \frac{1}{m^s} + O(n^{1-\sigma}) + O(h(\sigma, t, n)). \tag{2.9}$$

Proof. The result follows because of an equality that involves multiplication by the term $(1 - 2^{1-s})$. Note that

$$(1 - 2^{1-s})(\sum_{m=1}^{n} \frac{1}{m^s}) = \sum_{m=1}^{n} \frac{(-1)^{m+1}}{m^s} + O(n^{1-\sigma}). \tag{2.10}$$

The $O(n^{1-\sigma})$ term follows from a comparison with the integral $\int_n^{2n} x^{-\sigma}dx$. Then, we prove the result by applying the tail bound (Basu [2024]), the last term being $O(h(\sigma, t, n))$, which is also $O(\frac{|t|}{n^\sigma})$ for the tail (see, for example Basu [2023a] for a geometric proof), ☐

Before proceeding, we will introduce another notion in order to study sufficient conditions for Ω-phenomena. For a complex variable s, we define the distribution μ_s on the unit circle $\mathbb{S}^1$ as follows. For each measurable subset $A \subseteq \mathbb{S}^1$, we define

$$\mu_s(A) = \frac{\sum_{m\geq 0:\theta(Z_n(s))\in A} ||Z_n(s))||}{\sum_{m\geq 0} ||Z_n(s))||}. \tag{2.11}$$

We say that μ_s is concentrated if either there exist $\theta', \theta'' \in [0, 2\pi)$ such that $0 \leq \theta'' - \theta' \leq \frac{\pi}{2}$ and

$$\mu_s(\{z : \theta' \leq \theta(z) \leq \theta''\}) > \frac{1}{1 + \cos\left(\frac{\theta''-\theta'}{2}\right)} \tag{2.12}$$

or there exist $\theta', \theta'' \in [0, 2\pi)$ such that $0 \le \theta' + (2\pi - \theta'') \le \frac{\pi}{2}$ and

$$\mu(\{z : \theta(z) \in [0, \theta'] \cup [\theta'', 2\pi]\}) > \frac{1}{1 + \cos\left(\frac{\theta' + (2\pi - \theta'')}{2}\right)}. \tag{2.13}$$

The following optimization problem is related to the notion of concentration above i.e. situations when μ_s is concentrated. This is needed to get an upper bound on the sum of norms $\sum_{n=1}^{m} ||Z_n(s)||$ by applying the formula for the norm of $Z_n(s)$.

$$\min_{(t,(q_n)_{n=1}^m) \in \mathbb{R} \times \mathbb{Z}^{m-1}} t$$

$$\text{subject to : } 2(2n)^{-\sigma}\left|\sin\left(0.5t\ln\left(1 + \frac{1}{2n}\right)\right)\right| \le \frac{\sqrt{2}-1}{\sqrt{2}m} \text{ for all } 1 \le n \le m. \tag{2.14}$$

$$t \ge 100. \tag{2.15}$$

Define $t^{**}(m)$ to be the optimal value of the above optimization problem.

The following integer program related to Diophantine approximation (Schmidt [1996], Schrijver [1998], Conforti et al. [2014], Boyd and Vandenberghe [2004]) allows one to deduce periodicities of concentrated points and large norms in terms of an Ω-phenomenon.

$$\min_{(t,(q_n)_{n=2}^m) \in \mathbb{R} \times \mathbb{Z}^{m-1}} t$$

$$\text{subject to : } \left|\Big(\frac{\ln(n)}{2\pi}\Big)t - q_n\right| \le \frac{r}{2\pi} \text{ for all } 2 \le n \le m. \tag{2.16}$$

$$q_n \ge 0 \text{ for all } 2 \le n \le m. \tag{2.17}$$

$$\sum_{n=2}^{m} q_n \ge 1. \tag{2.18}$$

$$t \ge 0. \tag{2.19}$$

Suppose that $t^*(m, r)$ is the optimal value of the program. In this paper, we will consider the following hypothesis concerning ζ, ζ^* and the sequence $Z_n(s)$. Then, theorems are proved concerning the implications and possibilities of situations when multiple hypotheses are satisfied.

Hypothesis 2.1. *Suppose that* $0.5 \le \sigma < 1$. *Then, there exist* $\alpha \in (0, 1)$, $k \in (0, 1)$ *such that* $Z^*(\sigma, t, m) = O((e^{\ln^k(|t|)})m^{-\alpha})$.

Hypothesis 2.2. *Suppose that* $0.5 \le \sigma < 1$. *Then, there exist* $\alpha > 1$ *such that* $Z^*(\sigma, t, m) = O(\frac{|t|}{e^{\ln^{\alpha}(m)}})$.

Hypothesis 2.3. *There exist* $\alpha > 1$ *such that* $Z^*(1, t, m) = O(\frac{|t|}{e^{\ln^{\alpha}(m)}})$.

Hypothesis 2.4. *Suppose that* $0.5 \le \sigma < 1$. *Then, for each* $\epsilon > 0$, *there exists* $\alpha \in (0, 1)$, *such that* $Z^*(\sigma, t, m) = O(|t|^{\epsilon} m^{-\alpha})$.

Hypothesis 2.5. *There exists* $\bar{r} > 0$ *such that for each* $0 < r \le \bar{r}$, *there exists* $Q > 0$, *for which* $t^*(m, r) = O(m^Q)$.

Hypothesis 2.6. *There exists* $\bar{r} > 0$ *such that for each* $0 < r \le \bar{r}$, *there exists* $Q > 1$, *for which* $t^*(m, r) = O(e^{\ln^Q(m)})$.

Hypothesis 2.7. *For every $0 < r < 1$, we have that $\lim_{m\to\infty} t^*(m,r) = +\infty$.*

Hypothesis 2.8. *There exists $Q > 0$ such that $t^{**}(m) = O(m^Q)$.*

Hypothesis 2.9. *There exists $Q > 1$ such that $t^{**}(m) = O(e^{\ln^Q(m)})$.*

Hypothesis 2.10. *We have that $\lim_{m\to\infty} t^{**}(m) = +\infty$.*

Hypothesis 2.11. *(Riemann) Suppose that $s = (\sigma, t)$ such that $0 < \sigma < 1$. If $\zeta(s) = 0$, then $\sigma = 1/2$.*

Hypothesis 2.12. *(Lindelof) Suppose that $0.5 \leq \sigma < 1$. Then, for each $\epsilon > 0$, we have that $\zeta(s) = O(|t|^\epsilon)$.*

We prove the following theorems.

Proposition 2.3. *Suppose that $0.5 \leq \sigma < 1$. Suppose that Hypothesis 2.1, Hypothesis 2.8 and Hypothesis 2.10 are satisfied. Then, there exist countably many pairwise disjoint intervals $\{[\underline{t}_k, \bar{t}_k]\}_{k\in\mathbb{Z}^+}$, such that $\lim_{k\to\infty} \underline{t}_k \to +\infty$ and for each k and $t \in [\underline{t}_k, \bar{t}_k]$, we have that $\mu_{(\sigma,t)}$ is concentrated.*

Proof. In this situation, as $t^{**}(m)$ varies over m, we upper bound the sum of norms $\sum_{n=1}^m ||Z_n(s)|| < 1$ and is bounded away from 1, while the tail $Z^*(\sigma, t^{**}(m), m) = O((e^{\ln^k(m)})m^{-\alpha})$ converges to zero. Since $\lim_{m\to+\infty} t^{**}(m) = +\infty$, we may find. Hence, we may find an unbounded set of t-values such that $\mu_{(\sigma,t)}(\{(1,0)\}) > 0.5$. □

Proposition 2.4. *Suppose that $0.5 \leq \sigma < 1$. Suppose that Hypothesis 2.2, Hypothesis 2.9 and Hypothesis 2.10 are satisfied, in which $Q < \alpha$. Then, there exist countably many pairwise disjoint intervals $\{[\underline{t}_k, \bar{t}_k]\}_{k\in\mathbb{Z}^+}$, such that $\lim_{k\to\infty} \underline{t}_k \to +\infty$ and for each k and $t \in [\underline{t}_k, \bar{t}_k]$, we have that $\mu_{(\sigma,t)}$ is concentrated.*

Proof. The argument and conclusion are the same as in the previous theorem. Except, now we have the tail bound $Z^*(\sigma, t^{**}(m), m) = O(\frac{e^{\ln^Q(m)}}{e^{\ln^\alpha(m)}})$. □

In the next three theorems, we will be interested in lower bounding the norm of the term $\sum_{m=1}^n m^{-s}$ rather than upper bounding the sum of norms $\sum_{n=1}^m ||Z_n(s)||$ as in the context involving concentration phenomena on points along the imaginary axis.

Proposition 2.5. *Suppose that Hypothesis 2.7 is satisfied. Then, there exists $0 < \bar{r} < 1$ such that for every $0 < r \leq \bar{r}$, there does not exist any $k \in (0,1)$ such that $t^*(m,r) = O(e^{\ln^k(m)})$.*

Proof. The result follows from the previous propositions 1.1 and 1.2 along with the behaviour of the Riemann zeta function on the line $\sigma = 1$. The $O(n^{1-\sigma})$ term is bounded. We may choose θ', θ'' such that $0 < \theta' + (2\pi - \theta'') < \frac{\pi}{2}$ and the value $\theta' + (2\pi - \theta'')$ will determine the value of $\bar{r}$.

Then, by the integer program, we may find a t-value such that the norm $||\sum_{m=1}^n \frac{1}{m^s}||$ is $\Omega(\ln(m))$, by Diophantine approximation, with $t^*(m,r)$ varying over m and $t^*(m,r) = O(e^{\ln^k(m)})$. The tail term $O(\frac{t^*(n,r)}{n^\sigma})$ converges to zero. This implies that $\zeta(1,t) = \Omega(\ln(|t|))$. However, this contradicts the fact that $\zeta(1,t) = O(\frac{\ln(|t|)}{\ln(\ln(|t|))})$ (see Weyl [1921], Littlewood [1922] and Theorem 5.16 in Titchmarsh and Heath-Brown [1986]). □

Proposition 2.6. *It is impossible that Hypothesis 2.4, Hypothesis 2.5, Hypothesis 2.7 and Hypothesis 2.12 are satisfied.*

Proof. We prove the theorem by contradiction. Suppose that the four hypotheses can be satisfied. Suppose that $0.5 < \sigma < 1$. Note that we have the lower bound on the norm of the term $\sum_{m=1}^{n} m^{-s} + O(n^{1-\sigma})$ in the approximate formula, which is $\Omega(n^{1-\sigma})$. This is because, by choosing $\theta' < \theta''$ such that $2^{1-2\sigma} < \theta' + (2\pi - \pi'') < \frac{\pi}{4}$, we lower bound the norm $||\sum_{m=1}^{n} m^{-s}||$ as $\cos(\theta' + (2\pi - \pi''))(\sum_{m=1}^{n} m^{-\sigma})$ by Proposition 1.1, while the $O(n^{1-\sigma})$ is in fact less than $2^{1-2\sigma}n^{1-\sigma}$. Then, by Hypothesis 1.5 and Hypothesis 1.7, we have the $\sum_{m=1}^{n} m^{-s} + O(n^{1-\sigma})$ is $\Omega((|t|^{\frac{1-\sigma}{Q}})$ and by Hypothesis 1.4, there exists small $\epsilon > 0$ such that $Q\epsilon < \alpha$ and the tail $Z^*(\sigma, t^{**}(m), m) = O(m^{Q\epsilon}m^{-\alpha})$ converges to zero. Then, by Hypothesis 1.12, by choosing $\epsilon < \frac{1-\sigma}{Q}$, we obtain a contradiction. $\square$

Proposition 2.7. *It is impossible that Hypothesis 2.3, Hypothesis 2.6, Hypothesis 2.7 and Hypothesis 2.11 are satisfied, in which $Q < \alpha$.*

Proof. The theorem is proved by the same argument as the previous theorem, by contradiction. However, we consider complex variables (σ, t) on the line $\sigma = 1$. Suppose that four hypotheses are satisfied. In the approximate formula, the lower bound on the term $\sum_{m=1}^{n} m^{-s} + O(n^{1-\sigma})$ is $\Omega(|\ln^{\frac{1}{Q}}(|t|)|)$. However, by Hypothesis 1.11, we have that $\zeta(s) = O(\ln(\ln(|t|)))$ (see Littlewood [1925] and Theorem 14.8 in Titchmarsh and Heath-Brown [1986]), which leads to a contradiction. $\square$

One may consider the following, more general version of the optimization problem related to Diophantine approximation studied above. Suppose that $a = (a_1, a_2, ..., a_n)$ is a vector of positive real numbers $a_1, a_2, ..., a_n > 0$.

$$\min_{(t,(q_j)_{j=1}^n)\in\mathbb{R}\times\mathbb{Z}^{m-1}} t$$

$$\text{subject to : } \Big|a_j t - q_j\Big| \leq \frac{r}{2\pi} \text{ for all } 1 \leq j \leq n. \tag{2.20}$$

$$q_j \geq 0 \text{ for all } 1 \leq j \leq n. \tag{2.21}$$

$$\sum_{j=1}^{n} q_j \geq 1. \tag{2.22}$$

$$t \geq 0. \tag{2.23}$$

Then, for a vector of integers $q = (q_1, q_2, ..., q_n) \in \mathbb{Z}^n \backslash \{0\}$, we define the function

$$f(t; q) = \sum_{j=1}^{n} (a_j t - q_j)^2. \tag{2.24}$$

The optimal value of $f(t; q)$ at q is defined as

$$F(q) = \min_{t\in\mathbb{R}} f(t; q). \tag{2.25}$$

One may then explicitly derive the function F is closed form as a polynomial in q. This is done as follows. Suppose that we have an optimizer $\hat{t}(q)$, then

$$f'(\hat{t}(q)) = \sum_{j=1}^{n} 2a_j(a_j\hat{t}(z) - q_j) = 0. \tag{2.26}$$

Hence,

$$\hat{t}(q) = \frac{< a, q >}{||a||_2^2}, \tag{2.27}$$

in which $< a, q >= \sum_{j=1}^{n} a_j q_j$ is the inner product and $||a||_2 = \sqrt{\sum_{j=1}^{n} a_j^2}$ is the Euclidean norm. Hence,

$$F(q) = ||q||_2^2 - \frac{< a, q >^2}{||a||_2^2}. \tag{2.28}$$

Note that $F(q)$ is quadratic in a and q. By Dirichlet's approximation theorem, we have that

$$\inf_{q \in \mathbb{Z}^n \setminus \{0\}} F(q) = 0 \tag{2.29}$$

One may then perform continuous Diophantine approximation in t, by means of the following polynomial inequalities for $\varepsilon, \delta > 0$ i.e.

$$F(q) < \varepsilon \tag{2.30}$$

and

$$(< a, q > a_j - q_i ||a||_2^2)^2 < \delta ||a||_2^4. \tag{2.31}$$

3 Conclusion

This paper allows us investigate the properties of various hypotheses that may be stated in the context of the Riemann zeta function. The behaviour in the region $0 < \sigma < 1$ and also the region $\sigma = 1$ both play a role in proving the results in this situation and other related situations. Perhaps interestingly, a connection emerges with occurence of concentration phenomena infinitely often along the imaginary axis. This would also be an example of a discrepancy created in the zeta series following from the law of iterated expectations. Another connection is formed with asymptotics and bounds on specific objects of interest.

References

Pathikrit Basu. On the riemann hypothesis. *Asian Journal of Probability and Statistics*, 20(4):51–56, 2022.

Pathikrit Basu. On propositions pertaining to the riemann hypothesis. *Asian Journal of Probability and Statistics*, 21(2):16–21, 2023a.

Pathikrit Basu. On propositions pertaining to the riemann hypothesis ii. *Asian Journal of Probability and Statistics*, 25(2):63–74, 2023b.

Pathikrit Basu. On propositions pertaining to the riemann hypothesis iii. *Asian Journal of Probability and Statistics*, 26(8):1–8, July 2024.

Peter Borwein, Stephen Choi, Brendan Rooney, and Andrea Weirathmueller. *The Riemann hypothesis: a resource for the afficionado and virtuoso alike*. Springer, 2008.

Stephen Boyd and Lieven Vandenberghe. *Convex optimization*. Cambridge university press, 2004.

Daniel Bump, Kwok-Kwong Choi, Pär Kurlberg, and Jeffrey Vaaler. A local riemann hypothesis, i. *Mathematische Zeitschrift*, 233(1):1–18, 2000.

Michele Conforti, Gérard Cornuéjols, and Giacomo Zambelli. *Integer programming models*. Springer, 2014.

J Brian Conrey. The riemann hypothesis. *Notices of the AMS*, 50(3):341–353, 2003.

Harold M. Edwards. *Riemann's zeta function*, volume 58. Academic press, 1974.

Larry Guth and James Maynard. New large value estimates for dirichlet polynomials. *arXiv preprint arXiv:2405.20552*, 2024.

Godfrey H Hardy and John E Littlewood. The zeros of riemann's zeta-function on the critical line. *Mathematische Zeitschrift*, 10(3):283–317, 1921.

Godfrey Harold Hardy. Sur les zéros de la fonction ζ (s) de riemann. *CR Acad. Sci. Paris*, 158(1914): 1012, 1914.

Daniel R Johnston. Improving bounds on prime counting functions by partial verification of the riemann hypothesis. *The Ramanujan Journal*, 59(4):1307–1321, 2022.

Jeffrey C Lagarias. An elementary problem equivalent to the riemann hypothesis. *The American Mathematical Monthly*, 109(6):534–543, 2002.

Ernst Lindelof. *Quelques remarques sur la croissance de la fonction [zeta](s)*. Gauthier-Villars, 1908.

J. E. Littlewood. On the riemann zeta-function. *Proceedings of the London Mathematical Society*, s2-24(1):175–201, 1925.

John E Littlewood. Researches in the theory of the riemann ζ-function. In *Proc. London Math. Soc.(2)*, volume 20, pages 22–28, 1922.

H von Mangoldt. Zur verteilung der nullstellen der riemannschen funktion........ *Mathematische Annalen*, 61:1–19, 1905.

James Maynard and Kyle Pratt. Half-isolated zeros and zero-density estimates. *International Mathematics Research Notices*, 2024(19):12978–13014, September 2024.

Jean-Louis Nicolas. The sum of divisors function and the riemann hypothesis. *The Ramanujan Journal*, pages 1–45, 2021.

Dave Platt and Tim Trudgian. The riemann hypothesis is true up to 3· 10 12. *Bulletin of the London Mathematical Society*, 53(3):792–797, 2021.

Bernhard Riemann. Ueber die anzahl der primzahlen unter einer gegebenen grosse. *Ges. Math. Werke und Wissenschaftlicher Nachlaß*, 2:145–155, 1859.

Wolfgang M Schmidt. *Diophantine approximation*. Springer Science & Business Media, 1996.

Alexander Schrijver. *Theory of linear and integer programming*. John Wiley & Sons, 1998.

Terence Tao, Tim Trudgian, and Andrew Yang. New exponent pairs, zero density estimates, and zero additive energy estimates: a systematic approach. *arXiv preprint arXiv:2501.16779*, 2025.

E. C. Titchmarsh. On an inequality satisfied by the zeta-function of riemann. *Proceedings of the London Mathematical Society*, s2-28(1):70–80, 1928.

Edward Charles Titchmarsh and David Rodney Heath-Brown. *The theory of the Riemann zeta-function*. Oxford university press, 1986.

H Weyl. Zur abschätzung von?(1+ ti). *Mathematische Zeitschrift*, 10:88–101, 1921.

Hermann Weyl. Über die gleichverteilung von zahlen mod. eins. *Mathematische Annalen*, 77(3): 313–352, 1916.