UNDER PEER REVIEW

Generalized Jacobsthal-Narayana Numbers and Generalized co-Jacobsthal-Narayana Numbers

Abstract. In this paper, we introduce and investigate two third order recurrence sequences so called generalized Jacobsthal-Narayana sequence and co-Jacobsthal-Narayana sequence and their two special subsequences which are related each other. There are close interrelations between recurrence equations of and roots of characteristic equations of generalized Jacobsthal-Narayana and generalized co-Jacobsthal-Narayana numbers. We present Binet's formulas, generating functions, some identities, Simson's formulas, recurrence properties, sum formulas and matrices related with these sequences.

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1. Introduction: Generalized Tribonacci Numbers

The generalized Tribonacci numbers

$$\{W_n(W_0, W_1, W_2; r, s, t)\}_{n\geq 0}$$

(or $\{W_n\}_{n\geq 0}$ or shortly $\{W_n\}_{n\geq 0}$) is defined as follows:

$$W_n = rW_{n-1} + sW_{n-2} + tW_{n-3}, \qquad W_0 = a, W_1 = b, W_2 = c, \quad n \geq 3 \tag{1.1}$$

where $W_0, W_1, W_2$are arbitrary complex (or real) numbers and r, s and t are real numbers with $t \neq 0$.

The sequence $\{W_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$W_{-n} = -\frac{s}{t}W_{-(n-1)} - \frac{r}{t}W_{-(n-2)} + \frac{1}{t}W_{-(n-3)}$$

for $n = 1, 2, 3, ...$ when $t \neq 0$. Therefore, recurrence (1.1) holds for all integers n.

For r, s, t satisfying Eq. (1.1), the generalized co-Tribonacci numbers

$$\{Y_n(Y_0, Y_1, Y_2; -s, -rt, t^2)\}_{n\geq 0}$$

(or shortly $\{Y_n\}_{n\geq 0}$) is defined as follows:

$$Y_n = -sY_{n-1} - rtY_{n-2} + t^2Y_{n-3}, \qquad Y_0 = d, Y_1 = e, Y_2 = f, \quad n \geq 3 \tag{1.2}$$

i.e.,

$$Y_n = r_1Y_{n-1} + s_1Y_{n-2} + t_1Y_{n-3}, \qquad Y_0 = d, Y_1 = e, Y_2 = f, \quad n \geq 3$$

where $Y_0, Y_1, Y_2$are arbitrary complex (or real) numbers and $r_1 = -s$, $s_1 = -rt$, $t_1 = t^2$.

The sequence $\{Y_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$\begin{aligned} Y_{-n} &= -\frac{-rt}{t^2}Y_{-(n-1)} - \frac{-s}{t^2}Y_{-(n-2)} + \frac{1}{t^2}Y_{-(n-3)} \\ &= -\frac{s_1}{t_1}Y_{-(n-1)} - \frac{r_1}{t_1}Y_{-(n-2)} + \frac{1}{t_1}Y_{-(n-3)} \end{aligned}$$

for $n = 1, 2, 3, ...$ when $t \neq 0$. Therefore, recurrence (1.2) holds for all integer n. For more information on generalized Tribonacci and co-Tribonacci numbers, see [3].

Note that we can easily use and modify the results given for r, s, t in [3] by substituting r_1, s_1, t_1 for r, s, t and we will do this in this paper.

There are close interrelations between roots of characteristic equations of generalized Tribonacci and generalized co-Tribonacci numbers, see [3, Lemma 17.]: If α, β, γ are the roots of characteristic equation of $\{W_n\}$ which is given as

$$z^3 - rz^2 - sz - t = 0,$$

and if $\theta_1, \theta_2, \theta_3$ are the roots of characteristic equation of $\{Y_n\}$ which is given as

$$y^3 - r_1y^2 - s_1y - t_1 = y^3 + sy^2 + rty - t^2 = 0,$$

then we get

$$\theta_1 = \beta\gamma,$$

$$\theta_2 = \alpha\beta,$$

$$\theta_3 = \alpha\gamma.$$

There are also close connections and relations between recurrence equations of generalized Tribonacci and generalized co-Tribonacci numbers, see, for example, Lemma 32 in [3].

2. Generalized Jacobsthal-Narayana Numbers

In this section, we consider the case $r = 1,\ s = 0,\ t = 2$.The generalized Jacobsthal-Narayana numbers

$$\{W_n(W_0, W_1, W_2; 1, 0, 2)\}_{n\geq 0}$$

(or $\{W_n\}_{n\geq 0}$ or shortly $\{W_n\}_{n\geq 0}$) is defined as follows:

$$W_n = W_{n-1} + 2W_{n-3}, \qquad W_0 = a, W_1 = b, W_2 = c, \quad n \geq 3 \tag{2.1}$$

where W_0, W_1, W_2 are arbitrary complex (or real) numbers.

The sequence $\{W_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$W_{-n} = -\frac{1}{2}W_{-(n-2)} + \frac{1}{2}W_{-(n-3)}$$

for $n = 1, 2, 3, ...$ when $t \neq 0$. Therefore, recurrence (2.1) holds for all integers n.

The first few generalized Jacobsthal-Narayana numbers with positive subscript and negative subscript are given in the following Table 1.

Table 1. A few generalized Jacobsthal-Narayana numbers

n	W_n	W_{-n}
0	W_0	W_0
1	W_1	$\frac{1}{2}W_2 - \frac{1}{2}W_1$
2	W_2	$\frac{1}{2}W_1 - \frac{1}{2}W_0$
3	$2W_0 + W_2$	$\frac{1}{2}W_0 + \frac{1}{4}W_1 - \frac{1}{4}W_2$
4	$2W_0 + 2W_1 + W_2$	$\frac{1}{4}W_0 - \frac{1}{2}W_1 + \frac{1}{4}W_2$
5	$2W_0 + 2W_1 + 3W_2$	$\frac{1}{8}W_1 - \frac{1}{2}W_0 + \frac{1}{8}W_2$
6	$6W_0 + 2W_1 + 5W_2$	$\frac{1}{8}W_0 + \frac{3}{8}W_1 - \frac{1}{4}W_2$
7	$10W_0 + 6W_1 + 7W_2$	$\frac{3}{8}W_0 - \frac{5}{16}W_1 + \frac{1}{16}W_2$
8	$14W_0 + 10W_1 + 13W_2$	$\frac{3}{16}W_2 - \frac{1}{8}W_1 - \frac{5}{16}W_0$
9	$26W_0 + 14W_1 + 23W_2$	$\frac{11}{32}W_1 - \frac{1}{8}W_0 - \frac{5}{32}W_2$
10	$46W_0 + 26W_1 + 37W_2$	$\frac{11}{32}W_0 - \frac{3}{32}W_1 - \frac{1}{16}W_2$
11	$74W_0 + 46W_1 + 63W_2$	$\frac{11}{64}W_2 - \frac{15}{64}W_1 - \frac{3}{32}W_0$
12	$126W_0 + 74W_1 + 109W_2$	$\frac{7}{32}W_1 - \frac{15}{64}W_0 - \frac{3}{64}W_2$
13	$218W_0 + 126W_1 + 183W_2$	$\frac{7}{32}W_0 + \frac{9}{128}W_1 - \frac{15}{128}W_2$

As $\{W_n\}$ is a third-order recurrence sequence (difference equation), it's characteristic equation (cubic equation) is

$$z^3 - z^2 - 2 = (z - \alpha)(z - \beta)(z - \gamma) = 0.$$

The roots α, β, γ of characteristic equation of $\{W_n\}$ are given as

$$\begin{aligned}
\alpha &= \frac{1}{3} + \left(\frac{28}{27} + \sqrt{\frac{29}{27}}\right)^{1/3} + \left(\frac{28}{27} - \sqrt{\frac{29}{27}}\right)^{1/3}, \\
\beta &= \frac{1}{3} + \omega\left(\frac{28}{27} + \sqrt{\frac{29}{27}}\right)^{1/3} + \omega^2\left(\frac{28}{27} - \sqrt{\frac{29}{27}}\right)^{1/3}, \\
\gamma &= \frac{1}{3} + \omega^2\left(\frac{28}{27} + \sqrt{\frac{29}{27}}\right)^{1/3} + \omega\left(\frac{28}{27} - \sqrt{\frac{29}{27}}\right)^{1/3},
\end{aligned}$$

where

$$\omega = \frac{-1 + i\sqrt{3}}{2} = \exp(2\pi i/3).$$

There are the following relations between the roots of characteristic equation:

$$\begin{cases} \alpha + \beta + \gamma = 1 \\ \alpha\beta + \alpha\gamma + \beta\gamma = 0 \\ \alpha\beta\gamma = 2 \end{cases}$$

The sequence $\{W_n\}$ can be expressed with Binet's formula. Using the roots of characteristic equation and the recurrence relation of W_n, Binet's formula of W_n can be given as follows:

THEOREM 1. *For all integers n, Binet's formula of generalized Jacobsthal-Narayana numbers is given as follows.*

$$\begin{aligned}
W_n &= \frac{p_1\alpha^n}{(\alpha-\beta)(\alpha-\gamma)} + \frac{p_2\beta^n}{(\beta-\alpha)(\beta-\gamma)} + \frac{p_3\gamma^n}{(\gamma-\alpha)(\gamma-\beta)} \\
&= A_1\alpha^n + A_2\beta^n + A_3\gamma^n,
\end{aligned}$$

where

$$\begin{aligned}
p_1 &= W_2 - (\beta+\gamma)W_1 + \beta\gamma W_0, \\
p_2 &= W_2 - (\alpha+\gamma)W_1 + \alpha\gamma W_0, \\
p_3 &= W_2 - (\alpha+\beta)W_1 + \alpha\beta W_0.
\end{aligned}$$

and

$$\begin{aligned}
A_1 &= \frac{p_1}{(\alpha-\beta)(\alpha-\gamma)} = \frac{W_2-(\beta+\gamma)W_1+\beta\gamma W_0}{(\alpha-\beta)(\alpha-\gamma)} \\
&= \frac{(\alpha W_2+\alpha(-1+\alpha)W_1+2W_0)}{\alpha^2+6}, \\
A_2 &= \frac{p_2}{(\beta-\alpha)(\beta-\gamma)} = \frac{W_2-(\alpha+\gamma)W_1+\alpha\gamma W_0}{(\beta-\alpha)(\beta-\gamma)} \\
&= \frac{(\beta W_2+\beta(-1+\beta)W_1+2W_0)}{\beta^2+6}, \\
A_3 &= \frac{p_3}{(\gamma-\alpha)(\gamma-\beta)} = \frac{W_2-(\alpha+\beta)W_1+\alpha\beta W_0}{(\gamma-\alpha)(\gamma-\beta)} \\
&= \frac{(\gamma W_2+\gamma(-1+\gamma)W_1+2W_0)}{\gamma^2+6}
\end{aligned}$$

Proof. Set $r=1$, $s=0$, $t=2$ in [3, Theorem 3 (a)]. □

Next, we give the ordinary generating function $\sum_{n=0}^{\infty} W_n z^n$ of the sequence W_n.

Lemma 2. *Suppose that $f_{W_n}(z) = \sum_{n=0}^{\infty} W_n z^n$ is the ordinary generating function of the generalized Jacobsthal-Narayana numbers $\{W_n\}_{n\geq 0}$. Then, $\sum_{n=0}^{\infty} W_n z^n$ is given by*

$$\sum_{n=0}^{\infty} W_n z^n = \frac{W_0+(W_1-W_0)z+(W_2-W_1)z^2}{1-z-2z^3}.$$

Proof. Set $r=1$, $s=0$, $t=2$ in [3, Lemma 9.]. □

Two special cases of the sequence $\{W_n\}$ are the well known Jacobsthal-Narayana sequence $\{B_n\}_{n\geq 0}$ and Jacobsthal-Narayana-Lucas sequence $\{C_n\}_{n\geq 0}$. Jacobsthal-Narayana sequence$\{B_n\}_{n\geq 0}$, Jacobsthal-Narayana-Lucas sequence $\{C_n\}_{n\geq 0}$ are defined, respectively, by the third-order recurrence relations

$$B_n = B_{n-1}+2B_{n-3}, \quad B_0=0, B_1=1, B_2=1, \tag{2.2}$$

$$C_n = C_{n-1}+2C_{n-3}, \quad C_0=3, C_1=1, C_2=1. \tag{2.3}$$

The sequences $\{B_n\}_{n\geq 0}$, $\{C_n\}_{n\geq 0}$, can be extended to negative subscripts by defining

$$\begin{aligned}
B_{-n} &= -\frac{1}{2}B_{-(n-2)}+\frac{1}{2}B_{-(n-3)}, \\
C_{-n} &= -\frac{1}{2}B_{-(n-2)}+\frac{1}{2}B_{-(n-3)},
\end{aligned}$$

for $n=1,2,3,...$ respectively. Therefore, recurrences (2.2)-(2.3) hold for all integer n.

Next, we present the first few values of the Jacobsthal-Narayana and Jacobsthal-Narayana-Lucas numbers with positive and negative subscripts:

Table 2. The first few values of the special third-order numbers with positive and negative subscripts.

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13
B_n	0	1	1	1	3	5	7	13	23	37	63	109	183	309
B_{-n}	0	0	$\frac{1}{2}$	0	$-\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{8}$	$-\frac{1}{4}$	$\frac{1}{16}$	$\frac{3}{16}$	$-\frac{5}{32}$	$-\frac{1}{16}$	$\frac{11}{64}$	$-\frac{3}{64}$
C_n	3	1	1	7	9	11	25	43	65	115	201	331	561	963
C_{-n}	3	0	-1	$\frac{3}{2}$	$\frac{1}{2}$	$-\frac{5}{4}$	$\frac{1}{2}$	$\frac{7}{8}$	$-\frac{7}{8}$	$-\frac{3}{16}$	$\frac{7}{8}$	$-\frac{11}{32}$	$-\frac{17}{32}$	$\frac{39}{64}$

The sequence $\{B_n\}$ is labelled in [2] as A077949 with the expansion of $\dfrac{1}{1-z-2z^3}$. In [1], authors defined Jacobsthal-Narayana numbers and then investigated the Binet's formula, generating functions and some identities of the sequence $\{B_n\}$.

For all integers n, Binet's formula of Jacobsthal-Narayana and Jacobsthal-Narayana-Lucas numbers (using initial conditions ((2.2) and (2.3)) in Theorem 1) can be expressed as follows:

Theorem 3. *For all integers n, Binet's formulas of Jacobsthal-Narayana and Jacobsthal-Narayana-Lucas numbers are*

$$\begin{aligned} B_n &= \frac{\alpha^{n+1}}{(\alpha-\beta)(\alpha-\gamma)} + \frac{\beta^{n+1}}{(\beta-\alpha)(\beta-\gamma)} + \frac{\gamma^{n+1}}{(\gamma-\alpha)(\gamma-\beta)} \\ &= \frac{\alpha^{n+2}}{\alpha^2+6} + \frac{\beta^{n+2}}{\beta^2+6} + \frac{\gamma^{n+2}}{\gamma^2+6}, \\ C_n &= \alpha^n + \beta^n + \gamma^n, \end{aligned}$$

respectively.

Lemma 2 gives the following results as particular examples (generating functions of Jacobsthal-Narayana and Jacobsthal-Narayana-Lucas numbers).

Corollary 4. *Generating functions of Jacobsthal-Narayana and Jacobsthal-Narayana-Lucas numbers are*

$$\begin{aligned} \sum_{n=0}^{\infty} B_n z^n &= \frac{z}{1-z-2z^3}, \\ \sum_{n=0}^{\infty} C_n z^n &= \frac{3-2z}{1-z-2z^3}, \end{aligned}$$

respectively.

2.1. Some Identities of Generalized Jacobsthal-Narayana Numbers. Now, we present some identities of generalized Jacobsthal-Narayana, Jacobsthal-Narayana and Jacobsthal-Narayana-Lucas numbers. First, we can give a few basic relations between $\{B_n\}$ and $\{C_n\}$.

Lemma 5. *The following equalities are true:*

- **(a):** $2C_n = 3B_{n+4} - 5B_{n+3} + 2B_{n+2}$.
- **(b):** $C_n = -B_{n+3} + B_{n+2} + 3B_{n+1}$.
- **(c):** $2C_n = 6B_{n+1} - 4B_n$.

(d): $C_n = 3B_{n+1} - 2B_n$.

(e): $C_n = B_n + 6B_{n-2}$.

(f): $116B_n = -3C_{n+4} + C_{n+3} + 20C_{n+2}$.

(g): $58B_n = -C_{n+3} + 10C_{n+2} - 3C_{n+1}$.

(h): $58B_n = 9C_{n+2} - 3C_{n+1} - 2C_n$.

(i): $58B_n = 6C_{n+1} - 2C_n + 18C_{n-1}$.

(j): $29B_n = 2C_n + 9C_{n-1} + 6C_{n-2}$.

Proof. Set $G_n = B_n$, $H_n = C_n$ and $r = 1$, $s = 0$, $t = 2$ in [3, Lemma 36.]. □

Note that all the identities in the above lemma can be proved by induction as well.

Next, we give a few basic relations between $\{B_n\}$ and $\{W_n\}$.

Lemma 6. *The following equalities are true:*

(a): $(W_2^3 + 2W_1^3 + 4W_0^3 + W_1^2W_2 - 2W_1W_2^2 + 2W_0^2W_2 + 2W_0W_1^2 - 6W_0W_1W_2)\ B_n = (W_1^2 + 2W_0^2 - W_1W_2)W_{n+2} + (W_2^2 - W_1W_2 - 2W_0W_1)W_{n+1} + (2W_1^2 - 2W_0W_2)W_n$

(b): $(W_2^3 + 2W_1^3 + 4W_0^3 + W_1^2W_2 - 2W_1W_2^2 + 2W_0^2W_2 + 2W_0W_1^2 - 6W_0W_1W_2)\ B_n = (W_2^2 + W_1^2 + 2W_0^2 - 2W_1W_2 - 2W_0W_1)W_{n+1} + (2W_1^2 - 2W_0W_2)W_n + (2W_1^2 + 4W_0^2 - 2W_1W_2)W_{n-1}$

(c): $(W_2^3 + 2W_1^3 + 4W_0^3 + W_1^2W_2 - 2W_1W_2^2 + 2W_0^2W_2 + 2W_0W_1^2 - 6W_0W_1W_2)\ B_n = (W_2^2 + 3W_1^2 + 2W_0^2 - 2W_1W_2 - 2W_0W_2 - 2W_0W_1)W_n + (2W_1^2 + 4W_0^2 - 2W_1W_2)W_{n-1} + (2W_2^2 + 2W_1^2 + 4W_0^2 - 4W_1W_2 - 4W_0W_1)W_{n-2}$

(d): $2W_n = (W_2 - W_1)B_{n+2} + (-W_2 + W_1 + 2W_0)B_{n+1} + (2W_1 - 2W_0)B_n$.

(e): $W_n = W_0B_{n+1} + (W_1 - W_0)B_n + (W_2 - W_1)B_{n-1}$.

(f): $W_n = W_1B_n + (W_2 - W_1)B_{n-1} + 2W_0B_{n-2}$.

Proof. Set $G_n = B_n$ and $r = 1$, $s = 0$, $t = 2$ in 3, Lemma 37.]. □

Now, we present a few basic relations between $\{C_n\}$ and $\{W_n\}$.

Lemma 7. *The following equalities are true:*

(a): $(W_2^3 + 2W_1^3 + 4W_0^3 + W_1^2W_2 - 2W_1W_2^2 + 2W_0^2W_2 + 2W_0W_1^2 - 6W_0W_1W_2)C_n = (3W_2^2 + W_1^2 + 2W_0^2 - 4W_1W_2 - 6W_0W_1)W_{n+2} + (-2W_2^2 + 6W_1^2 - 6W_0W_2 + 2W_1W_2 + 4W_0W_1)W_{n+1} + (2W_1^2 + 12W_0^2 - 6W_1W_2 + 4W_0W_2)W_n$.

(b): $(W_2^3 + 2W_1^3 + 4W_0^3 + W_1^2W_2 - 2W_1W_2^2 + 2W_0^2W_2 + 2W_0W_1^2 - 6W_0W_1W_2)C_n = (W_1^2 + W_2^2 + 6W_1^2 + 2W_0^2 - 6W_0W_2 - 2W_1W_2 - 2W_0W_1)W_{n+1} + (12W_0^2 - 6W_1W_2 + 2W_1^2 + 4W_0W_2)W_n + (6W_2^2 + 2W_1^2 + 4W_0^2 - 8W_1W_2 - 12W_0W_1)W_{n-1}$.

(c): $(W_2^3 + 2W_1^3 + 4W_0^3 + W_1^2W_2 - 2W_1W_2^2 + 2W_0^2W_2 + 2W_0W_1^2 - 6W_0W_1W_2)C_n = (W_2^2 + 9W_1^2 + 14W_0^2 - 8W_1W_2 - 2W_0W_2 - 2W_0W_1)W_n + (6W_2^2 + 2W_1^2 + 4W_0^2 - 8W_1W_2 - 12W_0W_1)W_{n-1} + (2W_2^2 + 14W_1^2 + 4W_0^2 - 4W_1W_2 - 12W_0W_2 - 4W_0W_1)W_{n-2}$.

(d): $116W_n = (-2W_2 + 20W_1 - 6W_0)C_{n+2} + (20W_2 - 26W_1 + 2W_0)C_{n+1} + (-6W_2 + 2W_1 + 40W_0)C_n$.

(e): $116W_n = (18W_2 - 6W_1 - 4W_0)C_{n+1} + (-6W_2 + 2W_1 + 40W_0)C_n + (-4W_2 + 40W_1 - 12W_0)C_{n-1}$.

(f): $116W_n = (12W_2 - 4W_1 + 36W_0)C_n + (-4W_2 + 40W_1 - 12W_0)C_{n-1} + (36W_2 - 12W_1 - 8W_0)C_{n-2}$.

Proof. Set $H_n = C_n$, and $r = 1$, $s = 0$, $t = 2$ in [3, Lemma 38.]. □

Now, we present some identities of generalized Jacobsthal-Narayana numbers and its special cases.

Lemma 8. *Suppose that* $\{Z_n\}_{n\geq 0} = \{Z_n(Z_0, Z_1, Z_2)\}_{n\geq 0}$ *is also defined by the third-order recurrence relations*

$$Z_n = Z_{n-1} + 2Z_{n-3} \tag{2.4}$$

i.e.,

$$Z_{n+3} = Z_{n+2} + 2Z_n$$

with the initial values Z_0, Z_1, Z_2 *not all being zero and*

$$Z_{-n} = -\frac{1}{2}Z_{-(n-2)} + \frac{1}{2}Z_{-(n-3)}$$

so that (2.4) is true for all integer n.

Then the following equalities are true:

(a): $(Z_0Z_3^2 + Z_1^2Z_4 + Z_2^3 - Z_0Z_2Z_4 - 2Z_1Z_2Z_3)W_n = ((Z_1^2 - Z_0Z_2)W_2 + (Z_0Z_3 - Z_1Z_2)W_1 + (Z_2^2 - Z_1Z_3)W_0)Z_{n+2} + ((Z_0Z_3 - Z_1Z_2)W_2 + (Z_2^2 - Z_0Z_4)W_1 + (Z_1Z_4 - Z_2Z_3)W_0)Z_{n+1} + ((Z_2^2 - Z_1Z_3)W_2 + (Z_1Z_4 - Z_2Z_3)W_1 + (Z_3^2 - Z_2Z_4)W_0)Z_n$.

(b): $(W_0W_3^2 + W_1^2W_4 + W_2^3 - W_0W_2W_4 - 2W_1W_2W_3)B_n = (W_1^2 + 2W_0^2 - W_1W_2)W_{n+2} + (W_2^2 - W_1W_2 - 2W_0W_1)W_{n+1} + 2(W_1^2 - W_0W_2)W_n$.

(c): $2W_n = (W_2 - W_1)B_{n+2} + (-W_2 + W_1 + 2W_0)B_{n+1} + 2(W_1 - W_0)B_n$.

(d): $(W_0W_3^2 + W_1^2W_4 + W_2^3 - W_0W_2W_4 - 2W_1W_2W_3)C_n = (3W_2^2 + W_1^2 + 2W_0^2 - 4W_1W_2 - 6W_0 W_1)W_{n+2} + (-2W_2^2 + 6W_1^2 + 2W_1W_2 - 6W_0W_2 + 2(s^2 + tr)W_0W_1)W_{n+1} + 2(W_1^2 + 6W_0^2 - 3W_1W_2 + 2W_0W_2)W_n$.

(e): $116W_n = 2(-W_2 + 10W_1 - 3W_0)C_{n+2} + 2(10W_2 - 13W_1 + W_0)C_{n+1} + 2(-3W_2 + W_1 + 20W_0)C_n$.

Proof.

(a): Writing

$$W_n = p_1 \times Z_{n+2} + p_2 \times Z_{n+1} + p_3 \times Z_n$$

and solving the system of equations

$$\begin{aligned} W_0 &= p_1 \times Z_2 + p_2 \times Z_1 + p_3 \times Z_0 \\ W_1 &= p_1 \times Z_3 + p_2 \times Z_2 + p_3 \times Z_1 \\ W_2 &= p_1 \times Z_4 + p_2 \times Z_3 + p_3 \times Z_2 \end{aligned}$$

we find the required identity.

(b): Replace W_n and Z_n with B_n and W_n, respectively in (a).

(c): Replace Z_n with B_n in (a).

(d): Replace W_n and Z_n with C_n and W_n, respectively in (a).

(e): Replace Z_n with C_n in (a). □

2.2. Simson's Formulas of Generalized Jacobsthal-Narayana Numbers. The following theorem gives Simson's formula of the generalized Jacobsthal-Narayana numbers $\{W_n\}$.

THEOREM 9 (Simson's Formula of Generalized Jacobsthal-Narayana Numbers). *For all integers n, we have*

$$\begin{vmatrix} W_{n+2} & W_{n+1} & W_n \\ W_{n+1} & W_n & W_{n-1} \\ W_n & W_{n-1} & W_{n-2} \end{vmatrix} = 2^n \begin{vmatrix} W_2 & W_1 & W_0 \\ W_1 & W_0 & W_{-1} \\ W_0 & W_{-1} & W_{-2} \end{vmatrix}$$

$$= 2^n \begin{vmatrix} W_2 & W_1 & W_0 \\ W_1 & W_0 & \frac{1}{2}(W_2 - W_1) \\ W_0 & \frac{1}{2}(W_2 - W_1) & \frac{1}{2}(W_1 - W_0) \end{vmatrix}.$$

Proof. Set $r = 1$, $s = 0$, $t = 2$ in [3, Theorem 33.]. □

The previous theorem gives the following results as particular examples.

COROLLARY 10. *For all integers n, Simson's formula of Jacobsthal-Narayana and Jacobsthal-Narayana-Lucas numbers are given as*

$$\begin{vmatrix} B_{n+2} & B_{n+1} & B_n \\ B_{n+1} & B_n & B_{n-1} \\ B_n & B_{n-1} & B_{n-2} \end{vmatrix} = -2^{n-1},$$

$$\begin{vmatrix} C_{n+2} & C_{n+1} & C_n \\ C_{n+1} & C_n & C_{n-1} \\ C_n & C_{n-1} & C_{n-2} \end{vmatrix} = -29 \times 2^n,$$

respectively.

Proof. Set $W_n = B_n$ and $W_n = C_n$ in Theorem 9, respectively. □

2.3. Recurrence Properties of Generalized Jacobsthal-Narayana Numbers. The generalized Jacobsthal-Narayana numbers W_n at negative indices can be expressed by the sequence itself at positive indices.

THEOREM 11. *For $n \in \mathbb{Z}$, we have*

$$W_{-n} = 2^{-n}(W_{2n} - C_n W_n + \frac{1}{2}(C_n^2 - C_{2n})W_0).$$

Proof. Set $r=1$, $s=0$, $t=2$ and $H_n=C_n$ in [3, Theorem 39.]. □

As special cases of Theorem 11, we have the following corollary.

COROLLARY 12. *For $n\in\mathbb{Z}$, we have*

(a): $B_{-n}=\dfrac{1}{2^n}(2B_n^2+B_{2n}-3B_nB_{n+1})$

(b): $C_{-n}=\dfrac{1}{2^{n+1}}(C_n^2-C_{2n})$.

Proof. Set $r=1$, $s=0$, $t=2$ and $G_n=B_n$ and $H_n=C_n$,respectively, in [3, Corollary 42.] or take $W_n=B_n$ and $W_n=C_n$,respectively, in Theorem 11. □

2.4. Sum Formulas $\sum\limits_{k=0}^{n}W_k$, $\sum\limits_{k=0}^{n}W_{2k}$, $\sum\limits_{k=0}^{n}W_{2k+1}$, $\sum\limits_{k=0}^{n}W_{-k}$, $\sum\limits_{k=0}^{n}W_{-2k}$, $\sum\limits_{k=0}^{n}W_{-2k+1}$ and Generating Functions $\sum_{n=0}^{\infty}W_nz^n$, $\sum_{n=0}^{\infty}W_{2n}z^n$, $\sum_{n=0}^{\infty}W_{2n+1}z^n$, $\sum_{n=0}^{\infty}W_{-n}z^n$, $\sum_{n=0}^{\infty}W_{-2n}z^n$, $\sum_{n=0}^{\infty}W_{-2n+1}z^n$ of Generalized Jacobsthal-Narayana Numbers. Next, we present sum formulas of generalized Jacobsthal-Narayana numbers

THEOREM 13. *For $n\geq 0$, we have the following sum formulas for generalized Jacobsthal-Narayana numbers:*

(a): $\sum\limits_{k=0}^{n}W_k=\dfrac{1}{2}(W_{n+2}+2W_n-W_2)$.

(b): $\sum\limits_{k=0}^{n}W_{2k}=\dfrac{1}{8}(W_{2n+2}+2W_{2n+1}+6W_{2n}-W_2-2W_1+2W_0)$.

(c): $\sum\limits_{k=0}^{n}W_{2k+1}=\dfrac{1}{8}(3W_{2n+2}+6W_{2n+1}+2W_{2n}-3W_2+2W_1-2W_0)$.

(d): $\sum\limits_{k=0}^{n}W_{-k}=\dfrac{1}{2}(-W_{-n+2}+W_2+2W_0)$.

(e): $\sum\limits_{k=0}^{n}W_{-2k}=\dfrac{1}{8}(-W_{-2n}-2W_{-2n-1}-6W_{-2n-2}+W_2+2W_1+6W_0)$.

(f): $\sum\limits_{k=0}^{n}W_{-2k+1}=\dfrac{1}{8}(-3W_{-2n}-6W_{-2n-1}-2W_{-2n-2}+3W_2+6W_1+2W_0)$.

Proof.

(a): Set $r=1,s=0,t=2$ and $z=1$ in [3, Theorem 62 (a) (i)].

(b): Set $r=1,s=0,t=2$ and $z=1$ in [3, Theorem 62 (b) (i)].

(c): Set $r=1,s=0,t=2$ and $z=1$ in [3, Theorem 62 (c) (i)].

(d): Set $r=1,s=0,t=2$ and $z=1$ in [3, Theorem 62 (d) (i)].

(e): Set $r=1,s=0,t=2$ and $z=1$ in [3, Theorem 62 (e) (i)].

(f): Set $r=1,s=0,t=2$ and $z=1$ in [3, Theorem 62 (f) (i)]. □

From the last Theorem, we have the following Corollary which gives sum formulas of Jacobsthal-Narayana numbers (take $W_n=B_n$ with $B_0=0,B_1=1,B_2=1$).

COROLLARY 14. *For $n\geq 0$, Jacobsthal-Narayana numbers have the following properties.*

(a): $\sum\limits_{k=0}^{n}B_k=\dfrac{1}{2}(B_{n+2}+2B_n-1)$.

(b): $\sum\limits_{k=0}^{n} B_{2k} = \frac{1}{8}(B_{2n+2} + 2B_{2n+1} + 6B_{2n} - 3)$.
(c): $\sum\limits_{k=0}^{n} B_{2k+1} = \frac{1}{8}(3B_{2n+2} + 6B_{2n+1} + 2B_{2n} - 1)$.
(d): $\sum\limits_{k=0}^{n} B_{-k} = \frac{1}{2}(-B_{-n+2} + 1)$.
(e): $\sum\limits_{k=0}^{n} B_{-2k} = \frac{1}{8}(-B_{-2n} - 2B_{-2n-1} - 6B_{-2n-2} + 3)$.
(f): $\sum\limits_{k=0}^{n} B_{-2k+1} = \frac{1}{8}(-3B_{-2n} - 6B_{-2n-1} - 2B_{-2n-2} + 9)$.

Taking $W_n = C_n$ with $C_0 = 3, C_1 = 1, C_2 = 1$ in the last Theorem, we have the following Corollary which gives sum formulas of Jacobsthal-Narayana-Lucas numbers.

Corollary 15. *For $n \geq 0$, Jacobsthal-Narayana-Lucas numbers have the following properties:*
(a): $\sum\limits_{k=0}^{n} C_k = \frac{1}{2}(C_{n+2} + 2C_n - 1)$.
(b): $\sum\limits_{k=0}^{n} C_{2k} = \frac{1}{8}(C_{2n+2} + 2C_{2n+1} + 6C_{2n} + 3)$.
(c): $\sum\limits_{k=0}^{n} C_{2k+1} = \frac{1}{8}(3C_{2n+2} + 6C_{2n+1} + 2C_{2n} - 7)$.
(d): $\sum\limits_{k=0}^{n} C_{-k} = \frac{1}{2}(-C_{-n+2} + 7)$.
(e): $\sum\limits_{k=0}^{n} C_{-2k} = \frac{1}{8}(-C_{-2n} - 2C_{-2n-1} - 6C_{-2n-2} + 21)$.
(f): $\sum\limits_{k=0}^{n} C_{-2k+1} = \frac{1}{8}(-3C_{-2n} - 6C_{-2n-1} - 2C_{-2n-2} + 15)$.

Next, we give the ordinary generating function of special cases of the generalized Jacobsthal-Narayana numbers $\{W_{mn+j}\}$.

Corollary 16. *The ordinary generating functions of the sequences W_n, W_{2n}, W_{2n+1}, W_{-n}, W_{-2n}, W_{-2n+1} are given as follows:*

(a): $(|z| < \min\{|\alpha|^{-1}, |\beta|^{-1}, |\gamma|^{-1}\} = |\alpha|^{-1} \simeq 0.589754)$.
$$\sum_{n=0}^{\infty} W_n z^n = \frac{-W_0 + (W_0 - W_1)z + (W_1 - W_2)z^2}{2z^3 + z - 1}.$$
(b): $(|z| < \min\{|\alpha|^{-2}, |\beta|^{-2}, |\gamma|^{-2}\} = |\alpha|^{-2} \simeq 0.34781)$.
$$\sum_{n=0}^{\infty} W_{2n} z^n = \frac{-W_0 + (W_0 - W_2)z + (2W_0 - 2W_1)z^2}{4z^3 + 4z^2 + z - 1}.$$
(c): $(|z| < \min\{|\alpha|^{-2}, |\beta|^{-2}, |\gamma|^{-2}\} = |\alpha|^{-2} \simeq 0.34781)$.
$$\sum_{n=0}^{\infty} W_{2n+1} z^n = \frac{-W_1 - (2W_0 - W_1 + W_2)z + 2\left(W_1 - W_2\right)z^2}{4z^3 + 4z^2 + z - 1}.$$
(d): $(|z| < \min\{|\alpha|, |\beta|, |\gamma|\} = |\beta| = |\gamma| \simeq 1.086052)$.
$$\sum_{n=0}^{\infty} W_{-n} z^n = \frac{2W_0 + (W_2 - W_1)z + W_1 z^2}{-z^3 + z^2 + 2}.$$

(e): $(|z|<\min\{|\alpha|^2,|\beta|^2,|\gamma|^2\}=|\beta|^2=|\gamma|^2\simeq 1.179509)$.

$$\sum_{n=0}^{\infty}W_{-2n}z^n=\frac{4W_0+(2W_0+2W_1)z+W_2z^2}{-z^3+z^2+4z+4}.$$

(f): $(|z|<\min\{|\alpha|^2,|\beta|^2,|\gamma|^2\}=|\beta|^2=|\gamma|^2\simeq 1.179509)$.

$$\sum_{n=0}^{\infty}W_{-2n+1}z^n=\frac{4W_1+(2W_1+2W_2)z+(2W_0+W_2)z^2}{-z^3+z^2+4z+4}.$$

Proof. Take $r=1,s=0,t=2$ in [3, Corollary 67.]. □

Now, we consider special cases of the last corollary.

Corollary 17. *The ordinary generating functions of special cases of the generalized Jacobsthal-Narayana numbers are given as follows:*

(a): $(|z|<\min\{|\alpha|^{-1},|\beta|^{-1},|\gamma|^{-1}\}=|\alpha|^{-1}\simeq 0.589754)$.

$$\sum_{n=0}^{\infty}B_nz^n=\frac{-z}{2z^3+z-1},$$

$$\sum_{n=0}^{\infty}C_nz^n=\frac{2z-3}{2z^3+z-1}.$$

(b): $(|z|<\min\{|\alpha|^{-2},|\beta|^{-2},|\gamma|^{-2}\}=|\alpha|^{-2}\simeq 0.34781)$.

$$\sum_{n=0}^{\infty}B_{2n}z^n=\frac{-2z^2-z}{4z^3+4z^2+z-1},$$

$$\sum_{n=0}^{\infty}C_{2n}z^n=\frac{4z^2+2z-3}{4z^3+4z^2+z-1}.$$

(c): $(|z|<\min\{|\alpha|^{-2},|\beta|^{-2},|\gamma|^{-2}\}=|\alpha|^{-2}\simeq 0.34781)$.

$$\sum_{n=0}^{\infty}B_{2n+1}z^n=\frac{-1}{4z^3+4z^2+z-1},$$

$$\sum_{n=0}^{\infty}C_{2n+1}z^n=\frac{-6z-1}{4z^3+4z^2+z-1}.$$

(d): $(|z|<\min\{|\alpha|,|\beta|,|\gamma|\}=|\beta|=|\gamma|\simeq 1.086052)$.

$$\sum_{n=0}^{\infty}B_{-n}z^n=\frac{z^2}{-z^3+z^2+2},$$

$$\sum_{n=0}^{\infty}C_{-n}z^n=\frac{z^2+6}{-z^3+z^2+2}.$$

(e): $(|z|<\min\{|\alpha|^2,|\beta|^2,|\gamma|^2\}=|\beta|^2=|\gamma|^2\simeq 1.179509)$.

$$\sum_{n=0}^{\infty}B_{-2n}z^n=\frac{z^2+2z}{-z^3+z^2+4z+4},$$

$$\sum_{n=0}^{\infty}C_{-2n}z^n=\frac{z^2+8z+12}{-z^3+z^2+4z+4}.$$

(f): $(|z| < \min\{|\alpha|^2, |\beta|^2, |\gamma|^2\} = |\beta|^2 = |\gamma|^2 \simeq 1.179509)$.

$$\sum_{n=0}^{\infty} B_{-2n+1}z^n = \frac{z^2+4z+4}{-z^3+z^2+4z+4},$$
$$\sum_{n=0}^{\infty} C_{-2n+1}z^n = \frac{7z^2+4z+4}{-z^3+z^2+4z+4}.$$

From the last corollary, we obtain the following results for special cases of z.

Corollary 18. *We have the following infinite sums .*

(a): $z = \frac{1}{2}$

$$\sum_{n=0}^{\infty} \frac{B_n}{2^n} = 2,$$
$$\sum_{n=0}^{\infty} \frac{C_n}{2^n} = 8.$$

(b): $z = \frac{1}{4}$

$$\sum_{n=0}^{\infty} \frac{B_{2n}}{4^n} = \frac{6}{7},$$
$$\sum_{n=0}^{\infty} \frac{C_{2n}}{4^n} = \frac{36}{7}.$$

(c): $z = \frac{1}{4}$

$$\sum_{n=0}^{\infty} \frac{B_{2n+1}}{4^n} = \frac{16}{7},$$
$$\sum_{n=0}^{\infty} \frac{C_{2n+1}}{4^n} = \frac{40}{7}.$$

(d): $z = 1$

$$\sum_{n=0}^{\infty} B_{-n} = \frac{1}{2},$$
$$\sum_{n=0}^{\infty} C_{-n} = \frac{7}{2}.$$

(e): $z = 1$

$$\sum_{n=0}^{\infty} B_{-2n} = \frac{3}{8},$$
$$\sum_{n=0}^{\infty} C_{-2n} = \frac{21}{8}.$$

(f): $z=1$

$$\sum_{n=0}^{\infty} B_{-2n+1} = \frac{9}{8},$$
$$\sum_{n=0}^{\infty} C_{-2n+1} = \frac{15}{8}.$$

2.5. Sum Formulas $\sum_{k=0}^{n} z^k W_k^2$, $\sum_{k=0}^{n} z^k W_{k+1}W_k$, $\sum_{k=0}^{n} z^k W_{k+2}W_k$ and Generating Functions $\sum_{n=0}^{\infty} W_n^2 z^n$, $\sum_{n=0}^{\infty} W_{n+1}W_n z^n$, $\sum_{n=0}^{\infty} W_{n+2}W_n z^n$ of Generalized Jacobsthal-Narayana Numbers. Next, we present sum formulas of generalized Jacobsthal-Narayana numbers.

Theorem 19. *For $n \geq 0$, we have the following sum formulas for generalized Jacobsthal-Narayana numbers:*

(a): $\sum_{k=0}^{n} W_k^2 = \frac{1}{8}(5W_{n+3}^2 + 12W_{n+2}^2 + 12W_{n+1}^2 - 12W_{n+2}W_{n+3} - 4W_{n+1}W_{n+3} - 8W_{n+1}W_{n+2} - 5W_2^2 - 12W_1^2 - 12W_0^2 + 12W_1W_2 + 4W_0W_2 + 8W_0W_1).$

(b): $\sum_{k=0}^{n} W_{k+1}W_k = \frac{1}{8}(-W_{n+3}^2 - 4W_{n+2}^2 - 4W_{n+1}^2 + 4W_{n+2}W_{n+3} + 4W_{n+1}W_{n+3} + W_2^2 + 4W_1^2 + 4W_0^2 - 4W_1W_2 - 4W_0W_2).$

(c): $\sum_{k=0}^{n} W_{k+2}W_k = \frac{1}{8}(-3W_{n+3}^2 - 12W_{n+2}^2 - 12W_{n+1}^2 + 12W_{n+2}W_{n+3} + 4W_{n+1}W_{n+3} + 8W_{n+1}W_{n+2} + 3W_2^2 + 12W_1^2 + 12W_0^2 - 12W_1W_2 - 4W_0W_2 - 8W_0W_1).$

Proof.

(a): Set $r=1, s=0, t=2$, and $z=1$ in [4, Theorem 2.1 (a) (i)].

(b): Set $r=1, s=0, t=2$, and $z=1$ in [4, Theorem 2.1 (b) (i)].

(c): Set $r=1, s=0, t=2$, and $z=1$ in [4, Theorem 2.1 (c) (i)]. □

From the last Theorem, we have the following Corollary which gives sum formulas of Jacobsthal-Narayana numbers (take $W_n = B_n$ with $B_0 = 0, B_1 = 1, B_2 = 1$).

Corollary 20. *For $n \geq 0$, Jacobsthal-Narayana numbers have the following properties.*

(a): $\sum_{k=0}^{n} B_k^2 = \frac{1}{8}(5B_{n+3}^2 + 12B_{n+2}^2 + 12B_{n+1}^2 - 12B_{n+2}B_{n+3} - 4B_{n+1}B_{n+3} - 8B_{n+1}B_{n+2} - 5).$

(b): $\sum_{k=0}^{n} B_{k+1}B_k = \frac{1}{8}(-B_{n+3}^2 - 4B_{n+2}^2 - 4B_{n+1}^2 + 4B_{n+2}B_{n+3} + 4B_{n+1}B_{n+3} + 1).$

(c): $\sum_{k=0}^{n} B_{k+2}B_k = \frac{1}{8}(-3B_{n+3}^2 - 12B_{n+2}^2 - 12B_{n+1}^2 + 12B_{n+2}B_{n+3} + 4B_{n+1}B_{n+3} + 8B_{n+1}B_{n+2} + 3).$

Taking $W_n = C_n$ with $C_0 = 3, C_1 = 1, C_2 = 1$ in the last Theorem, we have the following Corollary which gives sum formulas of Jacobsthal-Narayana-Lucas numbers.

Corollary 21. *For $n \geq 0$, Jacobsthal-Narayana-Lucas numbers have the following properties:*

(a): $\sum_{k=0}^{n} C_k^2 = \frac{1}{8}(5C_{n+3}^2 + 12C_{n+2}^2 + 12C_{n+1}^2 - 12C_{n+2}C_{n+3} - 4C_{n+1}C_{n+3} - 8C_{n+1}C_{n+2} - 77).$

(b): $\sum_{k=0}^{n} C_{k+1}C_k = \frac{1}{8}(-C_{n+3}^2 - 4C_{n+2}^2 - 4C_{n+1}^2 + 4C_{n+2}C_{n+3} + 4C_{n+1}C_{n+3} + 25).$

(c): $\sum_{k=0}^{n} C_{k+2}C_k = \frac{1}{8}(-3C_{n+3}^2 - 12C_{n+2}^2 - 12C_{n+1}^2 + 12C_{n+2}C_{n+3} + 4C_{n+1}C_{n+3} + 8C_{n+1}C_{n+2} + 75)$.

Next, we give the ordinary generating functions $\sum_{n=0}^{\infty} W_n^2 z^n$, $\sum_{n=0}^{\infty} W_{n+1}W_n z^n$, $\sum_{n=0}^{\infty} W_{n+2}W_n z^n$ of the sequences $\{W_n^2\}$, $\{W_{n+1}W_n\}$, $\{W_{n+2}W_n\}$.

THEOREM 22. *Assume that* $|z| < \min\{|\alpha|^{-2}, |\beta|^{-2}, |\gamma|^{-2}, |\alpha\beta|^{-1}, |\alpha\gamma|^{-1}, |\beta\gamma|^{-1}\} = |\alpha|^{-2} \simeq 0.34781$. *Then the ordinary generating functions of the sequences* $\{W_n^2\}$, $\{W_{n+1}W_n\}$, $\{W_{n+2}W_n\}$ *are given as follows:*

(a): $\sum_{n=0}^{\infty} W_n^2 z^n = \frac{1}{-16z^6 - 8z^5 + 4z^4 + 10z^3 + 2z^2 + z - 1}(-W_0^2 + (W_0^2 - W_1^2)z + (2W_0^2 + W_1^2 - W_2^2)z^2 + (6W_0^2 - 4W_2W_0 + 2W_1^2)z^3 + 2(2W_0^2 - 4W_0W_1 + 3W_1^2 - 2W_1W_2 + W_2^2)z^4 + 4(W_1 - W_2)^2 z^5)$.

(b): $\sum_{n=0}^{\infty} W_{n+1}W_n z^n = \frac{1}{-16z^6 - 8z^5 + 4z^4 + 10z^3 + 2z^2 + z - 1}(-W_0W_1 + W_1(W_0 - W_2)z + (2W_0W_1 - W_2^2 - 2W_0W_2 + W_1W_2)z^2 + (6W_0W_1 - 4W_0^2 - 2W_0W_2)z^3 + 4W_1(W_2 - W_1)z^4 + 8W_0(W_2 - W_1)z^5)$.

(c): $\sum_{n=0}^{\infty} W_{n+2}W_n z^n = \frac{1}{-16z^6 - 8z^5 + 4z^4 + 10z^3 + 2z^2 + z - 1}(-W_0W_2 - (2W_0W_1 - W_0W_2 + W_1W_2)z - (W_2^2 + W_1W_2 - 2W_0W_1)z^2 + (-4W_0^2 + 4W_0W_2 - 2W_2^2 + 2W_1W_2)z^3 + 2(-4W_0^2 + 4W_0W_1 - 2W_1^2 + 2W_2W_1)z^4 + 8W_1(W_2 - W_1)z^5)$.

Proof. Take $r = 1, s = 0, t = 2$ in [4, Theorem 3.1]. □

Now, we consider special cases of the last Theorem.

COROLLARY 23. *Assume that* $|z| < \min\{|\alpha|^{-2}, |\beta|^{-2}, |\gamma|^{-2}, |\alpha\beta|^{-1}, |\alpha\gamma|^{-1}, |\beta\gamma|^{-1}\} = |\alpha|^{-2} \simeq 0.34781$. *The ordinary generating functions of the sequences* $\{B_n^2\}$, $\{B_{n+1}B_n\}$, $\{B_{n+2}B_n\}$ *and* $\{C_n^2\}$, $\{C_{n+1}C_n\}$, $\{C_{n+2}C_n\}$ *are given as follows:*

(a):

$$\begin{aligned} \sum_{n=0}^{\infty} B_n^2 z^n &= \frac{4z^4 + 2z^3 - z}{-16z^6 - 8z^5 + 4z^4 + 10z^3 + 2z^2 + z - 1}, \\ \sum_{n=0}^{\infty} C_n^2 z^n &= \frac{16z^4 + 44z^3 + 18z^2 + 8z - 9}{-16z^6 - 8z^5 + 4z^4 + 10z^3 + 2z^2 + z - 1}. \end{aligned}$$

(b):

$$\begin{aligned} \sum_{n=0}^{\infty} B_{n+1}B_n z^n &= \frac{-z}{-16z^6 - 8z^5 + 4z^4 + 10z^3 + 2z^2 + z - 1}, \\ \sum_{n=0}^{\infty} C_{n+1}C_n z^n &= \frac{-24z^3 + 2z - 3}{-16z^6 - 8z^5 + 4z^4 + 10z^3 + 2z^2 + z - 1}. \end{aligned}$$

(c):

$$\begin{aligned} \sum_{n=0}^{\infty} B_{n+2}B_n z^n &= \frac{-2z^2 - z}{-16z^6 - 8z^5 + 4z^4 + 10z^3 + 2z^2 + z - 1}, \\ \sum_{n=0}^{\infty} C_{n+2}C_n z^n &= \frac{-48z^4 - 24z^3 + 4z^2 - 4z - 3}{-16z^6 - 8z^5 + 4z^4 + 10z^3 + 2z^2 + z - 1}. \end{aligned}$$

From the last corollary, we obtain the following results for special cases of generalized Jacobsthal-Narayana numbers.

Corollary 24. *Some infinite sums of* $\{B_n^2\}$, $\{B_{n+1}B_n\}$, $\{B_{n+2}B_n\}$ *and* $\{C_n^2\}$, $\{C_{n+1}C_n\}$, $\{C_{n+2}C_n\}$ *are given as follows:*

(a): $z = \frac{1}{4}$.

$$\sum_{n=0}^{\infty} \frac{B_n^2}{4^n} = \frac{52}{119},$$
$$\sum_{n=0}^{\infty} \frac{C_n^2}{4^n} = \frac{1312}{119}.$$

(b): $z = \frac{1}{4}$.

$$\sum_{n=0}^{\infty} \frac{B_{n+1}B_n}{4^n} = \frac{64}{119},$$
$$\sum_{n=0}^{\infty} \frac{C_{n+1}C_n}{4^n} = \frac{736}{119}.$$

(c): $z = \frac{1}{4}$.

$$\sum_{n=0}^{\infty} \frac{B_{n+2}B_n}{4^n} = \frac{96}{119},$$
$$\sum_{n=0}^{\infty} \frac{C_{n+2}C_n}{4^n} = \frac{1104}{119}.$$

2.6. Generalized Jacobsthal-Narayana Numbers by Matrix Methods. In this section, we present matrix representations of the sequences W_n, B_n and C_n. We also introduce Simson matrix and investigate its properties.

2.6.1. *Matrix Representations of the Sequences* W_n, B_n *and* C_n. We define the square matrix A of order 3 as:

$$A = \begin{pmatrix} 1 & 0 & 2 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

and such that $\det A = 2$. Some properties of matrix A^n can be given as

$$A^n = A^{n-1} + 2A^{n-3},$$
$$A^{n+m} = A^n A^m = A^m A^n,$$

for all integers m and n. Note that we have the following formulas:

$$\begin{pmatrix} W_{n+2} \\ W_{n+1} \\ W_n \end{pmatrix} = \begin{pmatrix} 1 & 0 & 2 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} W_{n+1} \\ W_n \\ W_{n-1} \end{pmatrix},$$

and

$$\begin{pmatrix} W_{n+2} \\ W_{n+1} \\ W_n \end{pmatrix} = \begin{pmatrix} 1 & 0 & 2 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^n \begin{pmatrix} W_2 \\ W_1 \\ W_0 \end{pmatrix},$$

and

$$\begin{pmatrix} B_{n+2} \\ B_{n+1} \\ B_n \end{pmatrix} = \begin{pmatrix} 1 & 0 & 2 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} B_{n+1} \\ B_n \\ B_{n-1} \end{pmatrix}.$$

We also define

$$E_n = \begin{pmatrix} B_{n+1} & 2B_{n-1} & 2B_n \\ B_n & 2B_{n-2} & 2B_{n-1} \\ B_{n-1} & 2B_{n-3} & 2B_{n-2} \end{pmatrix}$$

and

$$D_n = \begin{pmatrix} W_{n+1} & 2W_{n-1} & 2W_n \\ W_n & 2W_{n-2} & 2W_{n-1} \\ W_{n-1} & 2W_{n-3} & 2W_{n-2} \end{pmatrix}.$$

THEOREM 25. *For all integers m, n, we have the following properties:*

(a): $E_n = A^n$, *i.e.,*

$$\begin{pmatrix} 1 & 0 & 2 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^n = \begin{pmatrix} B_{n+1} & 2B_{n-1} & 2B_n \\ B_n & 2B_{n-2} & 2B_{n-1} \\ B_{n-1} & 2B_{n-3} & 2B_{n-2} \end{pmatrix}.$$

(b): $D_1 A^n = A^n D_1$.

(c): $D_{n+m} = D_n E_m = E_m D_n$, *i.e.*,

$$\begin{pmatrix} W_{n+m+1} & 2W_{n+m-1} & 2W_{n+m} \\ W_{n+m} & 2W_{n+m-2} & 2W_{n+m-1} \\ W_{n+m-1} & 2W_{n+m-3} & 2W_{n+m-2} \end{pmatrix}$$

$$= \begin{pmatrix} W_{n+1} & 2W_{n-1} & 2W_n \\ W_n & 2W_{n-2} & 2W_{n-1} \\ W_{n-1} & 2W_{n-3} & 2W_{n-2} \end{pmatrix} \begin{pmatrix} B_{m+1} & 2B_{m-1} & 2B_m \\ B_m & 2B_{m-2} & 2B_{m-1} \\ B_{m-1} & 2B_{m-3} & 2B_{m-2} \end{pmatrix}$$

$$= \begin{pmatrix} B_{m+1} & 2B_{m-1} & 2B_m \\ B_m & 2B_{m-2} & 2B_{m-1} \\ B_{m-1} & 2B_{m-3} & 2B_{m-2} \end{pmatrix} \begin{pmatrix} W_{n+1} & 2W_{n-1} & 2W_n \\ W_n & 2W_{n-2} & 2W_{n-1} \\ W_{n-1} & 2W_{n-3} & 2W_{n-2} \end{pmatrix}.$$

(d):

$$A^n = B_{n-1}A^2 + 2B_{n-3}A + 2B_{n-2}I$$

i.e.,

$$A^n = \frac{1}{2}((B_{n+2} - B_{n+1})A^2 + (-B_{n+2} + B_{n+1} + 2B_n)A + (2B_{n+1} - 2B_n)I)$$

that is,

$$A^n = \frac{1}{2}(B_{n+2}(A^2 - A) + B_{n+1}(-A^2 + A + 2I) + B_n(2A - 2I))$$

where

$$I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Proof. Set $G_n = B_n$, and $r = 1$, $s = 0$, $t = 2$ in [3, Theorem 51.]. $\square$

Next, we present matrix formulas for the generalized Jacobsthal-Narayana and Jacobsthal-Narayana-Lucas numbers numbers.

Corollary 26. *For all integers n, we have the following formulas for generalized Jacobsthal-Narayana and Jacobsthal-Narayana-Lucas numbers.*

(a): *Generalized Jacobsthal-Narayana numbers.*

$$\begin{pmatrix} 1 & 0 & 2 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^n = \frac{1}{\Lambda_W(0)} \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

where

$$a_{11} = (W_1^2 + 2W_0^2 - W_1W_2)W_{n+3} + (W_2^2 - W_1W_2 - 2W_0W_1)W_{n+2} + (2W_1^2 - 2W_0W_2)W_{n+1}$$
$$a_{21} = (W_1^2 + 2W_0^2 - W_1W_2)W_{n+2} + (W_2^2 - W_1W_2 - 2W_0W_1)W_{n+1} + (2W_1^2 - 2W_0W_2)W_n$$
$$a_{31} = (W_1^2 + 2W_0^2 - W_1W_2)W_{n+1} + (W_2^2 - W_1W_2 - 2W_0W_1)W_n + (2W_1^2 - 2W_0W_2)W_{n-1}$$

$$a_{12} = 2((W_1^2 + 2W_0^2 - W_1W_2)W_{n+1} + (W_2^2 - W_1W_2 - 2W_0W_1)W_n + 2(W_1^2 - W_0W_2)W_{n-1})$$
$$a_{22} = 2((W_1^2 + 2W_0^2 - W_1W_2)W_n + (W_2^2 - W_1W_2 - 2W_0W_1)W_{n-1} + 2(W_1^2 - W_0W_2)W_{n-2})$$
$$a_{32} = 2((W_1^2 + 2W_0^2 - W_1W_2)W_{n-1} + (W_2^2 - W_1W_2 - 2W_0W_1)W_{n-2} + 2(W_1^2 - W_0W_2)W_{n-3})$$
$$a_{13} = 2((W_1^2 + 2W_0^2 - W_1W_2)W_{n+2} + (W_2^2 - W_1W_2 - 2W_0W_1)W_{n+1} + 2(W_1^2 - W_0W_2)W_n)$$
$$a_{23} = 2((W_1^2 + 2W_0^2 - W_1W_2)W_{n+1} + (W_2^2 - W_1W_2 - 2W_0W_1)W_n + 2(W_1^2 - W_0W_2)W_{n-1})$$
$$a_{33} = 2((W_1^2 + 2W_0^2 - W_1W_2)W_n + (W_2^2 - W_1W_2 - 2W_0W_1)W_{n-1} + 2(W_1^2 - W_0W_2)W_{n-2})$$

and

$$\Lambda_W(0) = W_2^3 + 2W_1^3 + 4W_0^3 - 2W_1W_2^2 + W_2W_1^2 + 2W_0W_1^2 + 2W_0^2W_2 - 6W_2W_1W_0.$$

(b): *Jacobsthal-Narayana-Lucas numbers.*

$$\begin{pmatrix} 1 & 0 & 2 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^n = \frac{1}{116}\begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix}$$

where

$$b_{11} = 18C_{n+3} - 6C_{n+2} - 4C_{n+1}$$
$$b_{21} = 18C_{n+2} - 6C_{n+1} - 4C_n$$
$$b_{31} = 18C_{n+1} - 4C_{n-1} - 6C_n$$
$$b_{12} = 36C_{n+1} - 12C_n - 8C_{n-1}$$
$$b_{22} = 36C_n - 12C_{n-1} - 8C_{n-2}$$
$$b_{32} = 36C_{n-1} - 12C_{n-2} - 8C_{n-3}$$
$$b_{13} = 36C_{n+2} - 12C_{n+1} - 8C_n$$
$$b_{23} = 36C_{n+1} - 12C_n - 8C_{n-1}$$
$$b_{33} = 36C_n - 12C_{n-1} - 8C_{n-2}$$

Proof. Set $r = 1, s = 0, t = 2$ and then take $W_n = C_n$ respectively, in [3, Corollary 52.]. □

Now, we present an identity for W_{n+m}.

Theorem 27. *(Honsberger's Identity) For all integers m and n, we have*

$$W_{n+m} = W_nB_{m+1} + 2W_{n-1}B_{m-1} + 2W_{n-2}B_m$$

Proof. Set $G_n = B_n$ and $r = 1, s = 0, t = 2$ in [3, Theorem 53.]. □

As special cases of the last Theorem, we have the following corollary.

Corollary 28. *For all integers m, n, we have the following properties:*

$$B_{n+m} = B_nB_{m+1} + 2B_{n-1}B_{m-1} + 2B_{n-2}B_m$$
$$C_{n+m} = C_nB_{m+1} + 2C_{n-1}B_{m-1} + 2C_{n-2}B_m$$

Next, we present identies for W_{mn+j} and its special cases.

COROLLARY 29. *For all integers m, n, j, we have the following properties:*

$$\begin{aligned} W_{mn+j} &= B_{mn-1}W_{j+2} + 2B_{mn-3}W_{j+1} + 2B_{mn-2}W_j \\ B_{mn+j} &= B_{mn-1}B_{j+2} + 2B_{mn-3}B_{j+1} + 2B_{mn-2}B_j \\ C_{mn+j} &= B_{mn-1}C_{j+2} + 2B_{mn-3}C_{j+1} + 2B_{mn-2}C_j \end{aligned}$$

is true

Proof. Set $r = 1, s = 0, t = 2$ and then take $W_n = B_n$, $W_n = C_n$, respectively, in [3, Corollary 55.]. □

2.6.2. *Simson Matrix and its Properties.* For $n \in \mathbb{Z}$, we define

$$f_W(n) = \begin{pmatrix} W_{n+2} & W_{n+1} & W_n \\ W_{n+1} & W_n & W_{n-1} \\ W_n & W_{n-1} & W_{n-2} \end{pmatrix}.$$

We call this matrix as Simson matrix of the sequence W_n. Similarly, as special cases of W_n, Simson matrices of the sequences B_n and C_n are

$$f_B(n) = \begin{pmatrix} B_{n+2} & B_{n+1} & B_n \\ B_{n+1} & B_n & B_{n-1} \\ B_n & B_{n-1} & B_{n-2} \end{pmatrix} \text{ and } f_C(n) = \begin{pmatrix} C_{n+2} & C_{n+1} & C_n \\ C_{n+1} & C_n & C_{n-1} \\ C_n & C_{n-1} & C_{n-2} \end{pmatrix}$$

respectively.

LEMMA 30. *For all integers n, m and j, the followings hold.*

(a): $f_W(n) = f_W(n-1) + 2f_W(n-3)$.

(b): $f_W(n) = Af_W(n-1)$ *and* $f_W(n) = A^n f_W(0)$, *i.e.,*

$$\begin{pmatrix} W_{n+2} & W_{n+1} & W_n \\ W_{n+1} & W_n & W_{n-1} \\ W_n & W_{n-1} & W_{n-2} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 2 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} W_{n+1} & W_n & W_{n-1} \\ W_n & W_{n-1} & W_{n-2} \\ W_{n-1} & W_{n-2} & W_{n-3} \end{pmatrix}$$

and

$$\begin{pmatrix} W_{n+2} & W_{n+1} & W_n \\ W_{n+1} & W_n & W_{n-1} \\ W_n & W_{n-1} & W_{n-2} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 2 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^n \begin{pmatrix} W_2 & W_1 & W_0 \\ W_1 & W_0 & W_{-1} \\ W_0 & W_{-1} & W_{-2} \end{pmatrix}.$$

(c): $f_W(n+m) = A^n f_W(m)$ *and* $f_W(n+m) = A^m f_W(n)$ *i.e.,*

$$\begin{pmatrix} W_{n+m+2} & W_{n+m+1} & W_{n+m} \\ W_{n+m+1} & W_{n+m} & W_{n+m-1} \\ W_{n+m} & W_{n+m-1} & W_{n+m-2} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 2 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^n \begin{pmatrix} W_{m+2} & W_{m+1} & W_m \\ W_{m+1} & W_m & W_{m-1} \\ W_m & W_{m-1} & W_{m-2} \end{pmatrix},$$

and

$$\begin{pmatrix} W_{m+n+2} & W_{m+n+1} & W_{m+n} \\ W_{m+n+1} & W_{m+n} & W_{m+n-1} \\ W_{m+n} & W_{m+n-1} & W_{m+n-2} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 2 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^m \begin{pmatrix} W_{n+2} & W_{n+1} & W_n \\ W_{n+1} & W_n & W_{n-1} \\ W_n & W_{n-1} & W_{n-2} \end{pmatrix},$$

and $f_W(n) = A^m f_W(n-m)$, *i.e.,*

$$\begin{pmatrix} W_{n+2} & W_{n+1} & W_n \\ W_{n+1} & W_n & W_{n-1} \\ W_n & W_{n-1} & W_{n-2} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 2 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^m \begin{pmatrix} W_{n-m+2} & W_{n-m+1} & W_{n-m} \\ W_{n-m+1} & W_{n-m} & W_{n-m-1} \\ W_{n-m} & W_{n-m-1} & W_{n-m-2} \end{pmatrix}.$$

Proof. Set $r = 1$, $s = 0$, $t = 2$ in [3, Lemma 56.]. □

Taking the determinant of both sides of the identities given in the last Lemma, we obtain the following Theorem.

THEOREM 31. *For all integers n and m, the following identities hold.*

(a): *Catalan's Identity:*

$$\det(f_W(n+m)) = 2^n \det(f_W(m)) \quad \text{and} \quad \det(f_W(n)) = 2^m \det(f_W(n-m)),$$

i.e.,

$$\begin{vmatrix} W_{n+m+2} & W_{n+m+1} & W_{n+m} \\ W_{n+m+1} & W_{n+m} & W_{n+m-1} \\ W_{n+m} & W_{n+m-1} & W_{n+m-2} \end{vmatrix} = 2^n \begin{vmatrix} W_{m+2} & W_{m+1} & W_m \\ W_{m+1} & W_m & W_{m-1} \\ W_m & W_{m-1} & W_{m-2} \end{vmatrix},$$

and

$$\begin{vmatrix} W_{n+2} & W_{n+1} & W_n \\ W_{n+1} & W_n & W_{n-1} \\ W_n & W_{n-1} & W_{n-2} \end{vmatrix} = 2^m \begin{vmatrix} W_{n-m+2} & W_{n-m+1} & W_{n-m} \\ W_{n-m+1} & W_{n-m} & W_{n-m-1} \\ W_{n-m} & W_{n-m-1} & W_{n-m-2} \end{vmatrix}.$$

(b): *(see Theorem 9) Simson's (or Cassini's) Identity:*

$$\det(f_W(n)) = 2^n \det(f_W(0)),$$

i.e.,

$$\begin{vmatrix} W_{n+2} & W_{n+1} & W_n \\ W_{n+1} & W_n & W_{n-1} \\ W_n & W_{n-1} & W_{n-2} \end{vmatrix} = 2^n \begin{vmatrix} W_2 & W_1 & W_0 \\ W_1 & W_0 & W_{-1} \\ W_0 & W_{-1} & W_{-2} \end{vmatrix}.$$

Proof. Set $r = 1$, $s = 0$, $t = 2$ in [3, Theorem 57.]. □

From the last Theorem, we have the following Corollary which gives determinantal formulas of Jacobsthal-Narayana numbers (take $W_n = B_n$ with $B_0 = 0, B_1 = 1, B_2 = 1$).

COROLLARY 32. *For all integers n and m, the following identities hold.*

(a): *Catalan's Identity:*

$$\det(f_B(n+m)) = 2^n \det(f_B(m)) \quad \textit{and} \quad \det(f_B(n)) = 2^m \det(f_B(n-m)),$$

i.e.,

$$\begin{vmatrix} B_{n+m+2} & B_{n+m+1} & B_{n+m} \\ B_{n+m+1} & B_{n+m} & B_{n+m-1} \\ B_{n+m} & B_{n+m-1} & B_{n+m-2} \end{vmatrix} = 2^n \begin{vmatrix} B_{m+2} & B_{m+1} & B_m \\ B_{m+1} & B_m & B_{m-1} \\ B_m & B_{m-1} & B_{m-2} \end{vmatrix},$$

and

$$\begin{vmatrix} B_{n+2} & B_{n+1} & B_n \\ B_{n+1} & B_n & B_{n-1} \\ B_n & B_{n-1} & B_{n-2} \end{vmatrix} = 2^m \begin{vmatrix} B_{n-m+2} & B_{n-m+1} & B_{n-m} \\ B_{n-m+1} & B_{n-m} & B_{n-m-1} \\ B_{n-m} & B_{n-m-1} & B_{n-m-2} \end{vmatrix}.$$

(b): *Simson's (or Cassini's) Identity:*

$$\det(f_B(n)) = 2^n \det(f_B(0)),$$

i.e.,

$$\begin{vmatrix} B_{n+2} & B_{n+1} & B_n \\ B_{n+1} & B_n & B_{n-1} \\ B_n & B_{n-1} & B_{n-2} \end{vmatrix} = -2^{n-1}.$$

Taking $W_n = C_n$ with $C_0 = 3, C_1 = 1, C_2 = 1$ in the last Theorem, we have the following Corollary which gives determinantal formulas of Jacobsthal-Narayana-Lucas numbers.

Corollary 33. *For all integers n and m, the following identities hold.*

(a): *Catalan's Identity:*

$$\det(f_C(n+m)) = 2^n \det(f_C(m)) \quad \textit{and} \quad \det(f_C(n)) = 2^m \det(f_C(n-m))$$

i.e.,

$$\begin{vmatrix} C_{n+m+2} & C_{n+m+1} & C_{n+m} \\ C_{n+m+1} & C_{n+m} & C_{n+m-1} \\ C_{n+m} & C_{n+m-1} & C_{n+m-2} \end{vmatrix} = 2^n \begin{vmatrix} C_{m+2} & C_{m+1} & C_m \\ C_{m+1} & C_m & C_{m-1} \\ C_m & C_{m-1} & C_{m-2} \end{vmatrix},$$

and

$$\begin{vmatrix} C_{n+2} & C_{n+1} & C_n \\ C_{n+1} & C_n & C_{n-1} \\ C_n & C_{n-1} & C_{n-2} \end{vmatrix} = 2^m \begin{vmatrix} C_{n-m+2} & C_{n-m+1} & C_{n-m} \\ C_{n-m+1} & C_{n-m} & C_{n-m-1} \\ C_{n-m} & C_{n-m-1} & C_{n-m-2} \end{vmatrix}.$$

(b): *Simson's (or Cassini's) Identity:*

$$\det(f_C(n)) = 2^n \det(f_C(0)),$$

i.e.,

$$\begin{vmatrix} C_{n+2} & C_{n+1} & C_n \\ C_{n+1} & C_n & C_{n-1} \\ C_n & C_{n-1} & C_{n-2} \end{vmatrix} = -29 \times 2^n.$$

3. Generalized co-Jacobsthal-Narayana Numbers

If $r = 1$, $s = 0$, $t = 2$, then we get $r_1 = -s = 0$, $s_1 = -rt = -2$, $t_1 = t^2 = 4$. From now on, throughout the paper the chapter, we also use the notation $r = 0$, $s = -2$, $t = 4$ for $r_1 = 0$, $s_1 = -2$, $t_1 = 4$ and we consider the case $r = 0$, $s = -2$, $t = 4$ to use results in the paper [3].

In this section, we define and investigate a new sequence and its two special cases, namely the generalized co-Jacobsthal-Narayana, co-Jacobsthal-Narayana and co-Jacobsthal-Narayana-Lucas numbers. The generalized co-Jacobsthal-Narayana numbers

$$\{Y_n(Y_0, Y_1, Y_2; 0, -2, 4)\}_{n\geq 0}$$

(or shortly $\{Y_n\}_{n\geq 0}$) is defined as follows:

$$Y_n = -2Y_{n-2} + 4Y_{n-3}, \qquad Y_0 = d, Y_1 = e, Y_2 = f, \quad n \geq 3 \tag{3.1}$$

where Y_0, Y_1, Y_2 are arbitrary complex (or real) numbers with real coefficients.

The sequence $\{Y_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$Y_{-n} = \frac{1}{2}Y_{-(n-1)} + \frac{1}{4}Y_{-(n-3)}$$

for $n = 1, 2, 3, ...$ when $t \neq 0$. Therefore, recurrence (3.1) holds for all integer n.

The first few generalized co-Jacobsthal-Narayana numbers with positive subscript and negative subscript are given in the following Table 3.

Table 3. A few generalized co-Jacobsthal-Narayana numbers

n	Y_n	Y_{-n}
0	Y_0	Y_0
1	Y_1	$\frac{1}{2}Y_0 + \frac{1}{4}Y_2$
2	Y_2	$\frac{1}{4}Y_0 + \frac{1}{4}Y_1 + \frac{1}{8}Y_2$
3	$4Y_0 - 2Y_1$	$\frac{3}{8}Y_0 + \frac{1}{8}Y_1 + \frac{1}{16}Y_2$
4	$4Y_1 - 2Y_2$	$\frac{5}{16}Y_0 + \frac{1}{16}Y_1 + \frac{3}{32}Y_2$
5	$4Y_1 - 8Y_0 + 4Y_2$	$\frac{7}{32}Y_0 + \frac{3}{32}Y_1 + \frac{5}{64}Y_2$
6	$16Y_0 - 16Y_1 + 4Y_2$	$\frac{13}{64}Y_0 + \frac{5}{64}Y_1 + \frac{7}{128}Y_2$
7	$16Y_0 + 8Y_1 - 16Y_2$	$\frac{23}{128}Y_0 + \frac{7}{128}Y_1 + \frac{13}{256}Y_2$
8	$48Y_1 - 64Y_0 + 8Y_2$	$\frac{37}{256}Y_0 + \frac{13}{256}Y_1 + \frac{23}{512}Y_2$
9	$32Y_0 - 80Y_1 + 48Y_2$	$\frac{63}{512}Y_0 + \frac{23}{512}Y_1 + \frac{37}{1024}Y_2$
10	$192Y_0 - 64Y_1 - 80Y_2$	$\frac{109}{1024}Y_0 + \frac{37}{1024}Y_1 + \frac{63}{2048}Y_2$
11	$352Y_1 - 320Y_0 - 64Y_2$	$\frac{183}{2048}Y_0 + \frac{63}{2048}Y_1 + \frac{109}{4096}Y_2$
12	$352Y_2 - 192Y_1 - 256Y_0$	$\frac{309}{4096}Y_0 + \frac{109}{4096}Y_1 + \frac{183}{8192}Y_2$
13	$1408Y_0 - 960Y_1 - 192Y_2$	$\frac{527}{8192}Y_0 + \frac{183}{8192}Y_1 + \frac{309}{16\,384}Y_2$

REMARK 34. *In this paper we will extensively use the paper [3]. Note that in the notation of [3], here we have $r = 1$, $s = 0$, $t = 2$ and $r_1 = 0$, $s_1 = -2$, $t_1 = 4$. For simplicity, we can use the result of [3] by taking and replacing $r = 0$, $s = -2$, $t = 4$.*

As $\{Y_n\}$ is a third-order recurrence sequence (difference equation), it's characteristic equation (cubic equation) is

$$y^3 + 2y - 4 = 0.$$

The roots $\theta_1, \theta_2, \theta_3$ of characteristic equation of $\{Y_n\}$ are given as

$$\begin{aligned}
\theta_1 &= \left(2 + \sqrt{\frac{116}{27}}\right)^{1/3} - \left(-2 + \sqrt{\frac{116}{27}}\right)^{1/3}, \\
\theta_2 &= \omega\left(2 + \sqrt{\frac{116}{27}}\right)^{1/3} - \omega^2\left(-2 + \sqrt{\frac{116}{27}}\right)^{1/3}, \\
\theta_3 &= \omega^2\left(2 + \sqrt{\frac{116}{27}}\right)^{1/3} - \omega\left(-2 + \sqrt{\frac{116}{27}}\right)^{1/3},
\end{aligned}$$

where

$$\omega = \frac{-1 + i\sqrt{3}}{2} = \exp(2\pi i/3).$$

There are the following relations between the roots of characteristic equation:

$$\begin{cases}
\theta_1 + \theta_2 + \theta_3 = 0, \\
\theta_1\theta_2 + \theta_1\theta_3 + \theta_2\theta_3 = 2, \\
\theta_1\theta_2\theta_3 = 4.
\end{cases}$$

Note that there are an important relation between $\theta_1, \theta_2, \theta_3$ and α, β, γ:

$$\theta_1 = \beta\gamma,$$

$$\theta_2 = \alpha\beta,$$

$$\theta_3 = \alpha\gamma.$$

The sequence $\{Y_n\}$ can be expressed with Binet's formula. Using the roots of characteristic equation and the recurrence relation of Y_n, Binet's formula of Y_n can be given as follows:

THEOREM 35. *For all integers n, Binet's formula of generalized co-Jacobsthal-Narayana numbers is given as follows.*

$$\begin{aligned} Y_n &= \frac{p_1\theta_1^n}{(\theta_1-\theta_2)(\theta_1-\theta_3)} + \frac{p_2\theta_2^n}{(\theta_2-\theta_1)(\theta_2-\theta_3)} + \frac{p_3\theta_3^n}{(\theta_3-\theta_1)(\theta_3-\theta_2)} \\ &= A_1\theta_1^n + A_2\theta_2^n + A_3\theta_3^n, \end{aligned}$$

where

$$p_1 = Y_2 - (\theta_2+\theta_3)Y_1 + \theta_2\theta_3 Y_0,\ p_2 = Y_2 - (\theta_1+\theta_3)Y_1 + \theta_1\theta_3 Y_0,\ p_3 = Y_2 - (\theta_1+\theta_2)Y_1 + \theta_1\theta_2 Y_0,$$

and

$$\begin{aligned} A_1 &= \frac{p_1}{(\theta_1-\theta_2)(\theta_1-\theta_3)} = \frac{Y_2-(\theta_2+\theta_3)Y_1+\theta_2\theta_3Y_0}{(\theta_1-\theta_2)(\theta_1-\theta_3)} \\ &= \frac{(\theta_1Y_2+\theta_1\theta_1Y_1+tY_0)}{-4\theta_1+12}, \\ A_2 &= \frac{p_2}{(\theta_2-\theta_1)(\theta_2-\theta_3)} = \frac{Y_2-(\theta_1+\theta_3)Y_1+\theta_1\theta_3Y_0}{(\theta_2-\theta_1)(\theta_2-\theta_3)} \\ &= \frac{(\theta_2Y_2+\theta_2\theta_2Y_1+tY_0)}{-4\theta_2+12}, \\ A_3 &= \frac{p_3}{(\theta_3-\theta_1)(\theta_3-\theta_2)} = \frac{Y_2-(\theta_1+\theta_2)Y_1+\theta_1\theta_2Y_0}{(\theta_3-\theta_1)(\theta_3-\theta_2)} \\ &= \frac{(\theta_3Y_2+\theta_3\theta_3Y_1+tY_0)}{-4\theta_3+12}. \end{aligned}$$

Proof. For the proof, take $r = 0$, $s = -2$, $t = 4$ in [3, Theorem 3 (a)] or $r = 0$, $s = -2$, $t = 4$ in [3, Theorem 19 (a)]. □

Next, we give the ordinary generating function $\sum_{n=0}^{\infty} Y_n z^n$ of the sequence Y_n.

LEMMA 36. *Suppose that $f_{Y_n}(z) = \sum_{n=0}^{\infty} Y_n z^n$ is the ordinary generating function of the generalized co-Jacobsthal-Narayana numbers $\{Y_n\}_{n\geq 0}$. Then, $\sum_{n=0}^{\infty} Y_n z^n$ is given by*

$$\sum_{n=0}^{\infty} Y_n z^n = \frac{Y_0 + Y_1 z + (Y_2 + 2Y_0)z^2}{1 + 2z^2 - 4z^3}.$$

Proof. Set $r = 0$, $s = -2$, $t = 4$ in [3, Lemma 9.] or $r = 0$, $s = -2$, $t = 4$ in [3, Lemma 24.]. $\square$

In this paper, we define and investigate, in detail, two special cases of the generalized co-,Jacobsthal-Narayana numbers $\{Y_n\}$ which we call them co-,Jacobsthal-Narayana and co-,Jacobsthal-Narayana-Lucas numbers. co-,Jacobsthal-Narayana numbers $\{U_n\}_{n\geq 0}$ and co-Jacobsthal-Narayana-Lucas numbers $\{S_n\}_{n\geq 0}$ are defined, respectively, by the third-order recurrence relations

$$U_{n+3} = -2U_{n+2} + 4U_n, \qquad U_0 = 0, U_1 = 1, U_2 = 0, \tag{3.2}$$

$$S_{n+3} = -2S_{n+2} + 4S_n, \qquad S_0 = 3, S_1 = 0, S_2 = -4. \tag{3.3}$$

i.e.,

$$U_n = -2U_{n-2} + 4U_{n-3}, \qquad U_0 = 0, U_1 = 1, U_2 = 0,$$

$$S_n = -2S_{n-2} + 4S_{n-3}, \qquad S_0 = 3, S_1 = 0, S_2 = -4.$$

The sequences $\{U_n\}_{n\geq 0}$ and $\{S_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$U_{-n} = \frac{1}{2}U_{-(n-1)} + \frac{1}{4}U_{-(n-3)},$$

$$S_{-n} = \frac{1}{2}S_{-(n-1)} + \frac{1}{4}S_{-(n-3)},$$

for $n = 1, 2, 3, ...$ respectively. Therefore, recurrences (3.2) and (3.3) hold for all integers n.

Next, we present the first few values of the co-Jacobsthal-Narayana and co-Jacobsthal-Narayana-Lucas numbers with positive and negative subscripts.

Table 4. The first few values of the special third-order numbers with positive and negative subscripts.

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13
U_n	0	1	0	-2	4	4	-16	8	48	-80	-64	352	-192	-960
U_{-n}	0	0	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{16}$	$\frac{3}{32}$	$\frac{5}{64}$	$\frac{7}{128}$	$\frac{13}{256}$	$\frac{23}{512}$	$\frac{37}{1024}$	$\frac{63}{2048}$	$\frac{109}{4096}$	$\frac{183}{8192}$
S_n	3	0	-4	12	8	-40	32	112	-224	-96	896	-704	-2176	4992
S_{-n}	3	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{7}{8}$	$\frac{9}{16}$	$\frac{11}{32}$	$\frac{25}{64}$	$\frac{43}{128}$	$\frac{65}{256}$	$\frac{115}{512}$	$\frac{201}{1024}$	$\frac{331}{2048}$	$\frac{561}{4096}$	$\frac{963}{8192}$

For all integers n, Binet's formula of co-Jacobsthal-Narayana and co-Jacobsthal-Narayana-Lucas numbers (using initial conditions (3.2) and (3.3) in Theorem theo:smeako7) can be expressed as follows:

Theorem 37. *For all integers* n*, Binet's formulas of co-Jacobsthal-Narayana and co-Jacobsthal-Narayana-Lucas numbers are*

$$U_n = \frac{\theta_1^{n+1}}{(\theta_1-\theta_2)(\theta_1-\theta_3)} + \frac{\theta_2^{n+1}}{(\theta_2-\theta_1)(\theta_2-\theta_3)} + \frac{\theta_3^{n+1}}{(\theta_3-\theta_1)(\theta_3-\theta_2)}$$

$$= \frac{\theta_1^{n+2}}{-4\theta_1+12} + \frac{\theta_2^{n+2}}{-4\theta_2+12} + \frac{\theta_3^{n+2}}{-4\theta_3+12},$$

and

$$S_n = \theta_1^n + \theta_2^n + \theta_3^n,$$

respectively.

Lemma 36 gives the following results as particular examples (generating functions of co-Jacobsthal-Narayana and co-Jacobsthal-Narayana-Lucas numbers).

COROLLARY 38. *Generating functions of co-Jacobsthal-Narayana and co-Jacobsthal-Narayana-Lucas numbers are*

$$\sum_{n=0}^{\infty} U_n z^n = \frac{z}{1+2z^2-4z^3},$$
$$\sum_{n=0}^{\infty} S_n z^n = \frac{3+2z^2}{1+2z^2-4z^3},$$

respectively.

3.1. Connections between B_n, C_n and U_n, S_n. S_n can be given as follows.

LEMMA 39. *For all integers n, we have the following formula for S_n:*

$$\begin{aligned} S_n &= \theta_1^n + \theta_2^n + \theta_3^n \\ &= \alpha^n\beta^n + \alpha^n\gamma^n + \beta^n\gamma^n. \end{aligned}$$

Proof. Use [3, Lemma 30.]. □

We can present the relations between U_n, S_n and B_n, C_n as follows.

LEMMA 40. *For all integers n, we have the following formulas:*

(a): $S_n = \frac{1}{2}(C_n^2 - C_{2n})$.

(b): $U_n = 2^n B_{-n-1}$ *and* $U_{-n} = 2^{-n} B_{n-1}$.

(c): $S_n = 2^n C_{-n}$ *and* $S_{-n} = 2^{-n} C_n$.

Proof. Use [3, Lemma 32.]. □

3.2. Some Identities of Generalized co-Jacobsthal-Narayana Numbers. In this section, we obtain some identities of generalized co-Jacobsthal-Narayana, co-Jacobsthal-Narayana and co-Jacobsthal-Narayana-Lucas numbers. First, we can give a few basic relations between $\{U_n\}$ and $\{S_n\}$.

LEMMA 41. *The following equalities are true:*

(a): $16S_n = 14U_{n+4} + 4U_{n+3} + 36U_{n+2}$.

(b): $4S_n = U_{n+3} + 2U_{n+2} + 14U_{n+1}$.

(c): $2S_n = U_{n+2} + 6U_{n+1} + 2U_n$.

(d): $S_n = 3U_{n+1} + 2U_{n-1}$.

(e): $S_n = -4U_{n-1} + 12U_{n-2}$.

(f): $116U_n = S_{n+4} + 3S_{n+3} + 11S_{n+2}$.

(g): $116U_n = 3S_{n+3} + 9S_{n+2} + 4S_{n+1}$.

(h): $116U_n = 9S_{n+2} - 2S_{n+1} + 12S_n$.

(i): $116U_n = -2S_{n+1} - 6S_n + 36S_{n-1}$.

(j): $116U_n = -6S_n + 40S_{n-1} - 8S_{n-2}$.

Proof. Set $G_n = U_n$, $H_n = S_n$ and $r = 0$, $s = -2$, $t = 4$ in [3, Lemma 36.]. □

Note that all the identities in the above lemma can be proved by induction as well.

Next, we give a few basic relations between $\{U_n\}$ and $\{Y_n\}$.

LEMMA 42. *The following equalities are true:*

(a): $(Y_2^3+4Y_1^3+16Y_0^3+2Y_1^2Y_2+2Y_0Y_2^2+4Y_0Y_1^2-16Y_0^2Y_1-12Y_0Y_1Y_2)U_n = (4Y_0^2-Y_1Y_2-2Y_0Y_1)Y_{n+2}+(Y_2^2 + 2Y_0Y_2 - 4Y_0Y_1)Y_{n+1} + 4(Y_1^2 - Y_0Y_2)Y_n$.

(b): $(Y_2^3+4Y_1^3+16Y_0^3+2Y_1^2Y_2+2Y_0Y_2^2+4Y_0Y_1^2-16Y_0^2Y_1-12Y_0Y_1Y_2)U_n = (Y_2^2+2Y_0Y_2-4Y_0Y_1)Y_{n+1}+(4Y_1^2 - 8Y_0^2 + 2Y_1Y_2 - 4Y_0Y_2 + 4Y_0Y_1)Y_n + (16Y_0^2 - 4Y_1Y_2 - 8Y_0Y_1)Y_{n-1}$.

(c): $(Y_2^3+4Y_1^3+16Y_0^3+2Y_1^2Y_2+2Y_0Y_2^2+4Y_0Y_1^2-16Y_0^2Y_1-12Y_0Y_1Y_2)U_n = (4Y_1^2 - 8Y_0^2 + 2Y_1Y_2 - 4Y_0Y_2 + 4Y_0Y_1)Y_n + (-2Y_2^2 + 16Y_0^2 - 4Y_1Y_2 - 4Y_0Y_2)Y_{n-1} + (4Y_2^2 + 8Y_0Y_2 - 16Y_0Y_1)Y_{n-2}$.

(d): $4Y_n = (Y_2 + 2Y_0)U_{n+2} + 4Y_0U_{n+1} + (2Y_2 + 4Y_1 + 4Y_0)U_n$.

(e): $Y_n = Y_0U_{n+1} + Y_1U_n + (Y_2 + 2Y_0)U_{n-1}$.

(f): $Y_n = Y_1U_n + Y_2U_{n-1} + 4Y_0U_{n-2}$.

Proof. Set $W_n = Y_n$, $G_n = U_n$ and $r = 0$, $s = -2$, $t = 4$ in [3, Lemma 37.]. □

Now, we present a few basic relations between $\{S_n\}$ and $\{Y_n\}$.

LEMMA 43. *The following equalities are true:*

(a): $(Y_2^3+4Y_1^3+16Y_0^3+2Y_1^2Y_2+2Y_0Y_2^2+4Y_0Y_1^2-16Y_0^2Y_1-12Y_0Y_1Y_2)S_n = (3Y_2^2 + 2Y_1^2 + 4Y_0Y_2 - 12Y_0Y_1)Y_{n+2} + (12Y_1^2 + 4Y_1Y_2 - 12Y_0Y_2 - 16Y_0^2 + 8Y_0Y_1)Y_{n+1} + (2Y_2^2 + 4Y_1^2 + 48Y_0^2 - 12Y_1Y_2 - 32Y_0Y_1)Y_n$.

(b): $(Y_2^3+4Y_1^3+16Y_0^3+2Y_1^2Y_2+2Y_0Y_2^2+4Y_0Y_1^2-16Y_0^2Y_1-12Y_0Y_1Y_2)S_n = (12Y_1^2+4Y_1Y_2-12Y_0Y_2-16Y_0^2+8Y_0Y_1)Y_{n+1}+(48Y_0^2-4Y_2^2-12Y_1Y_2-8Y_0Y_2-8Y_0Y_1)Y_n+(12Y_2^2+8Y_1^2+16Y_0Y_2-48Y_0Y_1)Y_{n-1}$.

(c): $(Y_2^3+4Y_1^3+16Y_0^3+2Y_1^2Y_2+2Y_0Y_2^2+4Y_0Y_1^2-16Y_0^2Y_1-12Y_0Y_1Y_2)S_n = (-4Y_2^2+48Y_0^2-12Y_1Y_2-8Y_0Y_2 - 8Y_0Y_1)Y_n + (12Y_2^2 - 16Y_1^2 + 32Y_0^2 - 8Y_1Y_2 + 40Y_0Y_2 - 64Y_0Y_1)Y_{n-1} + (48Y_1^2 - 64Y_0^2 + 16Y_1Y_2 - 48Y_0Y_2 + 32Y_0Y_1)Y_{n-2}$.

(d): $116Y_n = (3Y_2 + 9Y_1 + 4Y_0)S_{n+2} + (9Y_2 - 2Y_1 + 12Y_0)S_{n+1} + (4Y_2 + 12Y_1 + 44Y_0)S_n$.

(e): $116Y_n = (9Y_2 - 2Y_1 + 12Y_0)S_{n+1} + (-2Y_2 - 6Y_1 + 36Y_0)S_n + (12Y_2 + 36Y_1 + 16Y_0)S_{n-1}$.

(f): $58Y_n = (-Y_2 - 3Y_1 + 18Y_0)S_n + (-3Y_2 + 20Y_1 - 4Y_0)S_{n-1} + (18Y_2 - 4Y_1 + 24Y_0)S_{n-2}$.

Proof. Set $W_n = Y_n$, $H_n = S_n$, and $r = 0$, $s = -2$, $t = 4$ in [3, Lemma 38.]. □

We can present identities between B_n, C_n and U_n, S_n by using Lemmas given above.

LEMMA 44. *For all integers n, we have the following formulas:*

(a): $S_{-n} = 2^{-n}(3B_{n+1} - 2B_n)$.

(b): $S_n = \frac{1}{2}((3B_{n+1} - 2B_n)^2 - (3B_{2n+1} - 2B_{2n}))$.

(c): $58U_{-n} = 2^{-n}(-C_{n+2} + 10C_{n+1} - 3C_n)$.

(d): $C_{-n} = 2^{-n}(3U_{n+1} + 2U_{n-1})$.

(e): $116B_{-n-1} = 2^{-n}(9S_{n+2} - 2S_{n+1} + 12S_n)$.

(f): $29B_{-n} = 2^{-n-1}(3S_{n+2} + 9S_{n+1} + 4S_n)$.

Prof. Use Lemmas 5, 40, 41. □

Now, we present some identities of generalized co-Jacobsthal-Narayana numbers and its special cases.

LEMMA 45. *Suppose that $\{X_n\}_{n\geq 0} = \{X_n(X_0, X_1, X_2)\}_{n\geq 0}$ is also defined by the third-order recurrence relations*

$$X_n = -2X_{n-2} + 4X_{n-3} \tag{3.4}$$

i.e.,

$$X_{n+3} = -2X_{n+1} + 4X_n$$

with the initial values X_0, X_1, X_2 not all being zero and

$$X_{-n} = \frac{1}{2}X_{-(n-1)} + \frac{1}{4}X_{-(n-3)}$$

so that (3.4) is true for all integer n.

Then the following equalities are true:

(a): $(X_0X_3^2 + X_1^2X_4 + X_2^3 - X_0X_2X_4 - 2X_1X_2X_3)Y_n = ((X_1^2 - X_0X_2)Y_2 + (X_0X_3 - X_1X_2)Y_1 + (X_2^2 - X_1X_3)Y_0)X_{n+2} + ((X_0X_3 - X_1X_2)Y_2 + (X_2^2 - X_0X_4)Y_1 + (X_1X_4 - X_2X_3)Y_0)X_{n+1} + ((X_2^2 - X_1X_3)Y_2 + (X_1X_4 - X_2X_3)Y_1 + (X_3^2 - X_2X_4)Y_0)X_n$.

(b): $(Y_0Y_3^2 + Y_1^2Y_4 + Y_2^3 - Y_0Y_2Y_4 - 2Y_1Y_2Y_3)U_n = (4Y_0^2 - Y_1Y_2 - 2Y_0Y_1)Y_{n+2} + (Y_2^2 + 2Y_0Y_2 - 4Y_0Y_1)Y_{n+1} + 4(Y_1^2 - Y_0Y_2)Y_n$.

(c): $4Y_n = (Y_2 + 2Y_0)U_{n+2} + (4Y_0)U_{n+1} + (2Y_2 + 4Y_1 + 4Y_0)U_n$.

(d): $(Y_0Y_3^2 + Y_1^2Y_4 + Y_2^3 - Y_0Y_2Y_4 - 2Y_1Y_2Y_3)S_n = (3Y_2^2 + 2Y_1^2 + 4Y_0Y_2 - 12Y_0Y_1)Y_{n+2} + (12Y_1^2 - 16Y_0^2 + 4Y_1Y_2 - 12Y_0Y_2 + 8Y_0Y_1)Y_{n+1} + (2Y_2^2 + 4Y_1^2 + 48Y_0^2 - 12Y_1Y_2 - 32Y_0Y_1)Y_n$.

(e): $116Y_n = (3Y_2 + 9Y_1 + 4Y_0)S_{n+2} + (9Y_2 - 2Y_1 + 12Y_0)S_{n+1} + (4Y_2 + 12Y_1 + 44Y_0)S_n$.

Proof.

(a): Writing

$$Y_n = q_1 \times X_{n+2} + q_2 \times X_{n+1} + q_3 \times X_n$$

and solving the system of equations

$$\begin{aligned} Y_0 &= q_1 \times X_2 + q_2 \times X_1 + q_3 \times X_0 \\ Y_1 &= q_1 \times X_3 + q_2 \times X_2 + q_3 \times X_1 \\ Y_2 &= q_1 \times X_4 + q_2 \times X_3 + q_3 \times X_2 \end{aligned}$$

we find the required identity.

(b): Replace Y_n and X_n with U_n and Y_n, respectively in (a).

(c): Replace X_n with U_n in (a).

(d): Replace Y_n and X_n with S_n and Y_n, respectively in (a).

(e): Replace X_n with S_n in (a). □

3.3. Simson's Formulas of co-Jacobsthal-Narayana Numbers. The following theorem gives Simson's formula of the generalized co-Jacobsthal-Narayana numbers $\{Y_n\}$.

THEOREM 46 (Simson's Formula of Generalized co-Jacobsthal-Narayana Numbers). *For all integers n, we have*

$$\begin{vmatrix} Y_{n+2} & Y_{n+1} & Y_n \\ Y_{n+1} & Y_n & Y_{n-1} \\ Y_n & Y_{n-1} & Y_{n-2} \end{vmatrix} = 4^n \begin{vmatrix} Y_2 & Y_1 & Y_0 \\ Y_1 & Y_0 & Y_{-1} \\ Y_0 & Y_{-1} & Y_{-2} \end{vmatrix}$$

$$= 4^n \begin{vmatrix} Y_2 & Y_1 & Y_0 \\ Y_1 & Y_0 & \frac{1}{4}(Y_2+2Y_0) \\ Y_0 & \frac{1}{4}(Y_2+2Y_0) & \frac{1}{8}(Y_2+2Y_1+2Y_0) \end{vmatrix}$$

Proof. Set $W_n = Y_n$ and $r = 0$, $s = -2$, $t = 4$ in [3, Theorem 33.]. □

The previous theorem gives the following results as particular examples.

COROLLARY 47. *For all integers n, Simson's formula of co-Jacobsthal-Narayana and co-Jacobsthal-Narayana-Lucas numbers are given as*

$$\begin{vmatrix} U_{n+2} & U_{n+1} & U_n \\ U_{n+1} & U_n & U_{n-1} \\ U_n & U_{n-1} & U_{n-2} \end{vmatrix} = -4^{n-1},$$

$$\begin{vmatrix} S_{n+2} & S_{n+1} & S_n \\ S_{n+1} & S_n & S_{n-1} \\ S_n & S_{n-1} & S_{n-2} \end{vmatrix} = -29 \times 2^{2n},$$

respectively.

Proof. Set $Y_n = U_n$ and $Y_n = S_n$ in Theorem 46, respectively. □

3.4. Recurrence Properties of Generalized co-Jacobsthal-Narayana Numbers. The generalized co-Jacobsthal-Narayana numbers Y_n at negative indices can be expressed by the sequence itself at positive indices.

THEOREM 48. *For $n \in \mathbb{Z}$, we have*

$$Y_{-n} = 2^{-2n}(Y_{2n} - S_n Y_n + \frac{1}{2}(S_n^2 - S_{2n})Y_0).$$

Proof. Set $Y_n = Y_n$, $C_n = S_n$ and $r = 0$, $s = -2$, $t = 4$ in [3, Theorem 39.]. □

As special cases of the above Theorem 48, we have the following Corollary.

COROLLARY 49. *For $n \in \mathbb{Z}$, we have*

(a):
$$U_{-n} = -\frac{1}{2^{2n+1}}(2U_n^2 - 2U_{2n} + U_n U_{n+2} + 6U_n U_{n+1}).$$

(b):
$$S_{-n} = \frac{1}{2^{2n+1}}(S_n^2 - S_{2n}).$$

Proof. Take $r = 0$, $s = -2$, $t = 4$, and $G_n = U_n$ and $H_n = S_n$, respectively, in [3, Corollary 42.] or set $Y_n = U_n$ and $Y_n = S_n$, respectively, in Theorem 48. □

The last Corollary can be written in the following form by using Lemma 40.

COROLLARY 50. *For $n \in \mathbb{Z}$, we have*

(a):
$$B_{n-1} = -\frac{1}{2^{n+1}}(2U_n^2 - 2U_{2n} + U_n U_{n+2} + 6U_n U_{n+1}).$$

(b):
$$C_n = \frac{1}{2^{n+1}}(S_n^2 - S_{2n}).$$

Proof. Use Lemma 40 and Corollary 49. □

3.5. Sum Formulas $\sum_{k=0}^{n} Y_k$, $\sum_{k=0}^{n} Y_{2k}$, $\sum_{k=0}^{n} Y_{2k+1}$, $\sum_{k=0}^{n} Y_{-k}$, $\sum_{k=0}^{n} Y_{-2k}$, $\sum_{k=0}^{n} Y_{-2k+1}$ and Generating Functions $\sum_{n=0}^{\infty} Y_n z^n$, $\sum_{n=0}^{\infty} Y_{2n} z^n$, $\sum_{n=0}^{\infty} Y_{2n+1} z^n$, $\sum_{n=0}^{\infty} Y_{-n} z^n$, $\sum_{n=0}^{\infty} Y_{-2n} z^n$, $\sum_{n=0}^{\infty} Y_{-2n+1} z^n$ of Generalized co-Jacobsthal-Narayana Numbers. Next, we present sum formulas of generalized co-Jacobsthal-Narayana numbers

THEOREM 51. *For $n \geq 0$, we have the following sum formulas for generalized co-Jacobsthal-Narayana numbers:*

(a): $\sum_{k=0}^{n} Y_k = Y_{n+2} + Y_{n+1} + 4Y_n - Y_1 - Y_2 - 3Y_0.$

(b): $\sum_{k=0}^{n} Y_{2k} = \frac{1}{7}(3Y_{2n+2} + 4Y_{2n+1} + 16Y_{2n} - 4Y_1 - 3Y_2 - 9Y_0).$

(c): $\sum_{k=0}^{n} Y_{2k+1} = \frac{1}{7}(4Y_{2n+2} + 10Y_{2n+1} + 12Y_{2n} - 4Y_2 - 3Y_1 - 12Y_0).$

(d): $\sum_{k=0}^{n} Y_{-k} = -Y_{-n+2} - Y_{-n+1} - 3Y_{-n} + Y_2 + Y_1 + 4Y_0.$

(e): $\sum_{k=0}^{n} Y_{-2k} = \frac{1}{7}(-3Y_{-2n} - 4Y_{-2n-1} - 16Y_{-2n-2} + 3Y_2 + 4Y_1 + 16Y_0).$

(f): $\sum_{k=0}^{n} Y_{-2k+1} = \frac{2}{7}(-2Y_{-2n} - 5Y_{-2n-1} - 6Y_{-2n-2} + 2Y_2 + 5Y_1 + 6Y_0).$

Proof.

(a): Set $W_n = Y_n$, $r = 0, s = -2, t = 4$ and $z = 1$ in [3, Theorem 62 (a) (i)].

(b): Set $W_n = Y_n$, $r = 0, s = -2, t = 4$ and $z = 1$ in [3, Theorem 62 (b) (i)].

(c): Set $W_n = Y_n$, $r = 0, s = -2, t = 4$ and $z = 1$ in [3, Theorem 62 (c) (i)].

(d): Set $W_n = Y_n$, $r = 0, s = -2, t = 4$ and $z = 1$ in [3, Theorem 62 (d) (i)].

(e): Set $W_n = Y_n$, $r = 0, s = -2, t = 4$ and $z = 1$ in [3, Theorem 62 (e) (i)].

(f): Set $W_n = Y_n$, $r = 0, s = -2, t = 4$ and $z = 1$ in [3, Theorem 62 (f) (i)]. □

From the last Theorem, we have the following Corollary which gives sum formulas of co-Jacobsthal-Narayana numbers (take $Y_n = U_n$ with $U_0 = 0, U_1 = 1, U_2 = 0$).

Corollary 52. *For $n \geq 0$, co-Jacobsthal-Narayana numbers have the following properties.*

(a): $\sum_{k=0}^{n} U_k = U_{n+2} + U_{n+1} + 4U_n - 1.$

(b): $\sum_{k=0}^{n} U_{2k} = \frac{1}{7}(3U_{2n+2} + 4U_{2n+1} + 16U_{2n} - 4).$

(c): $\sum_{k=0}^{n} U_{2k+1} = \frac{1}{7}(4U_{2n+2} + 10U_{2n+1} + 12U_{2n} - 3).$

(d): $\sum_{k=0}^{n} U_{-k} = -U_{-n+2} - U_{-n+1} - 3U_{-n} + 1.$

(e): $\sum_{k=0}^{n} U_{-2k} = \frac{1}{7}(-3U_{-2n} - 4U_{-2n-1} - 16U_{-2n-2} + 4).$

(f): $\sum_{k=0}^{n} U_{-2k+1} = \frac{2}{7}(-2U_{-2n} - 5U_{-2n-1} - 6U_{-2n-2} + 5).$

Taking $Y_n = S_n$ with $S_0 = 3, S_1 = 0, S_2 = -4$ in the last Theorem, we have the following Corollary which gives sum formulas of co-Jacobsthal-Narayana-Lucas numbers.

Corollary 53. *For $n \geq 0$, co-Jacobsthal-Narayana-Lucas numbers have the following properties:*

(a): $\sum_{k=0}^{n} S_k = S_{n+2} + S_{n+1} + 4S_n - 5.$

(b): $\sum_{k=0}^{n} S_{2k} = \frac{1}{7}(3S_{2n+2} + 4S_{2n+1} + 16S_{2n} - 15).$

(c): $\sum_{k=0}^{n} S_{2k+1} = \frac{1}{7}(4S_{2n+2} + 10S_{2n+1} + 12S_{2n} - 20).$

(d): $\sum_{k=0}^{n} S_{-k} = -S_{-n+2} - S_{-n+1} - 3S_{-n} + 8.$

(e): $\sum_{k=0}^{n} S_{-2k} = \frac{1}{7}(-3S_{-2n} - 4S_{-2n-1} - 16S_{-2n-2} + 36).$

(f): $\sum_{k=0}^{n} S_{-2k+1} = \frac{2}{7}(-2S_{-2n} - 5S_{-2n-1} - 6S_{-2n-2} + 10).$

Next, we give the ordinary generating function of special cases of the generalized co-Jacobsthal-Narayana numbers $\{Y_{mn+j}\}$.

Corollary 54. *The ordinary generating functions of the sequences Y_n, Y_{2n}, Y_{2n+1}, Y_{-n}, Y_{-2n}, Y_{-2n+1} are given as follows:*

(a): $(|z| < \min\{|\theta_1|^{-1}, |\theta_2|^{-1}, |\theta_3|^{-1}\} = |\theta_2|^{-1} = |\theta_3|^{-1} \simeq 0.543026\,).$

$$\sum_{n=0}^{\infty} Y_n z^n = \frac{(2Y_0 + Y_2)z^2 + Y_1 z + Y_0}{-4z^3 + 2z^2 + 1}.$$

(b): $(|z| < \min\{|\theta_1|^{-2}, |\theta_2|^{-2}, |\theta_3|^{-2}\} = |\theta_2|^{-2} = |\theta_3|^{-2} \simeq 0.294877)$.

$$\sum_{n=0}^{\infty} Y_{2n}z^n = \frac{(4Y_0+4Y_1+2Y_2)z^2+(4Y_0+Y_2)z+Y_0}{-16z^3+4z^2+4z+1}.$$

(c): $(|z| < \min\{|\theta_1|^{-2}, |\theta_2|^{-2}, |\theta_3|^{-2}\} = |\theta_2|^{-2} = |\theta_3|^{-2} \simeq 0.294877)$.

$$\sum_{n=0}^{\infty} Y_{2n+1}z^n = \frac{Y_1+(4Y_0+2Y_1)z+4(2Y_0+Y_2)z^2}{-16z^3+4z^2+4z+1}.$$

(d): $(|z| < \min\{|\theta_1|, |\theta_2|, |\theta_3|\} = |\theta_1| \simeq 1.179509)$.

$$\sum_{n=0}^{\infty} Y_{-n}z^n = \frac{-Y_1z^2-Y_2z-4Y_0}{z^3+2z-4}.$$

(e): $(|z| < \min\{|\theta_1|^{2}, |\theta_2|^{2}, |\theta_3|^{2}\} = |\theta_1|^{2} \simeq 1.391241)$.

$$\sum_{n=0}^{\infty} Y_{-2n}z^n = \frac{-Y_2z^2-(4Y_1+2Y_2)z-16Y_0}{z^3+4z^2+4z-16}.$$

(f): $(|z| < \min\{|\theta_1|^{2}, |\theta_2|^{2}, |\theta_3|^{2}\} = |\theta_1|^{2} \simeq 1.391241)$.

$$\sum_{n=0}^{\infty} Y_{-2n+1}z^n = \frac{2(Y_1-2Y_0)z^2+4(Y_1-2Y_0-Y_2)z-16Y_1}{z^3+4z^2+4z-16}.$$

Proof. Set $W_n = Y_n$ and $r = 0, s = -2, t = 4$ in [3, Corollary 67.]. □

Now, we consider special cases of the last corollary.

Corollary 55. *The ordinary generating functions of special cases of the generalized co-Jacobsthal-Narayana numbers are given as follows:*

(a): $(|z| < \min\{|\theta_1|^{-1}, |\theta_2|^{-1}, |\theta_3|^{-1}\} = |\theta_2|^{-1} = |\theta_3|^{-1} \simeq 0.543026)$.

$$\sum_{n=0}^{\infty} U_n z^n = \frac{z}{-4z^3+2z^2+1},$$
$$\sum_{n=0}^{\infty} S_n z^n = \frac{2z^2+3}{-4z^3+2z^2+1}.$$

(b): $(|z| < \min\{|\theta_1|^{-2}, |\theta_2|^{-2}, |\theta_3|^{-2}\} = |\theta_2|^{-2} = |\theta_3|^{-2} \simeq 0.294877)$.

$$\sum_{n=0}^{\infty} U_{2n} z^n = \frac{4z^2}{-16z^3+4z^2+4z+1},$$
$$\sum_{n=0}^{\infty} S_{2n} z^n = \frac{4z^2+8z+3}{-16z^3+4z^2+4z+1}.$$

(c): $(|z| < \min\{|\theta_1|^{-2}, |\theta_2|^{-2}, |\theta_3|^{-2}\} = |\theta_2|^{-2} = |\theta_3|^{-2} \simeq 0.294877)$.

$$\sum_{n=0}^{\infty} U_{2n+1} z^n = \frac{2z+1}{-16z^3+4z^2+4z+1},$$
$$\sum_{n=0}^{\infty} S_{2n+1} z^n = \frac{8z^2+12z}{-16z^3+4z^2+4z+1}.$$

(d): $(|z| < \min\{|\theta_1|, |\theta_2|, |\theta_3|\} = |\theta_1| \simeq 1.179509\,)$.

$$\begin{aligned}\sum_{n=0}^{\infty} U_{-n} z^n &= \frac{-z^2}{z^3+2z-4},\\ \sum_{n=0}^{\infty} S_{-n} z^n &= \frac{4z-12}{z^3+2z-4}.\end{aligned}$$

(e): $(|z| < \min\{|\theta_1|^2, |\theta_2|^2, |\theta_3|^2\} = |\theta_1|^2 \simeq 1.391241)$.

$$\begin{aligned}\sum_{n=0}^{\infty} U_{-2n} z^n &= \frac{-4z}{z^3+4z^2+4z-16},\\ \sum_{n=0}^{\infty} S_{-2n} z^n &= \frac{4z^2+8z-48}{z^3+4z^2+4z-16}.\end{aligned}$$

(f): $(|z| < \min\{|\theta_1|^2, |\theta_2|^2, |\theta_3|^2\} = |\theta_1|^2 \simeq 1.391241)$.

$$\begin{aligned}\sum_{n=0}^{\infty} U_{-2n+1} z^n &= \frac{2z^2+4z-16}{z^3+4z^2+4z-16},\\ \sum_{n=0}^{\infty} S_{-2n+1} z^n &= \frac{-12z^2-8z}{z^3+4z^2+4z-16}.\end{aligned}$$

From the last corollary, we obtain the following results for special cases of z.

Corollary 56. *We have the following infinite sums .*

(a): $z = \frac{1}{2}$.

$$\begin{aligned}\sum_{n=0}^{\infty} \frac{U_n}{2^n} &= \frac{1}{2},\\ \sum_{n=0}^{\infty} \frac{S_n}{2^n} &= \frac{7}{2}.\end{aligned}$$

(b): $z = \frac{1}{4}$.

$$\begin{aligned}\sum_{n=0}^{\infty} \frac{U_{2n}}{4^n} &= \frac{1}{8},\\ \sum_{n=0}^{\infty} \frac{S_{2n}}{4^n} &= \frac{21}{8}.\end{aligned}$$

(c): $z = \frac{1}{4}$.

$$\begin{aligned}\sum_{n=0}^{\infty} \frac{U_{2n+1}}{4^n} &= \frac{3}{4},\\ \sum_{n=0}^{\infty} \frac{S_{2n+1}}{4^n} &= \frac{7}{4}.\end{aligned}$$

(d): $z = 1$.

$$\sum_{n=0}^{\infty} U_{-n} = 1,$$
$$\sum_{n=0}^{\infty} S_{-n} = 8.$$

(e): $z = 1$.

$$\sum_{n=0}^{\infty} U_{-2n} = \frac{4}{7},$$
$$\sum_{n=0}^{\infty} S_{-2n} = \frac{36}{7}.$$

(f): $z = 1$.

$$\sum_{n=0}^{\infty} U_{-2n+1} = \frac{10}{7},$$
$$\sum_{n=0}^{\infty} S_{-2n+1} = \frac{20}{7}.$$

3.6. Sum Formulas $\sum_{k=0}^{n} z^k Y_k^2$, $\sum_{k=0}^{n} z^k Y_{k+1} Y_k$, $\sum_{k=0}^{n} z^k Y_{k+2} Y_k$ and Generating Functions $\sum_{n=0}^{\infty} Y_n^2 z^n$, $\sum_{n=0}^{\infty} Y_{n+1} Y_n z^n$, $\sum_{n=0}^{\infty} Y_{n+2} Y_n z^n$ of Generalized co-Jacobsthal-Narayana Numbers. Next, we present sum formulas of generalized co-Jacobsthal-Narayana numbers.

Theorem 57. *For $n \geq 0$, we have the following sum formulas for generalized co-Jacobsthal-Narayana numbers:*

(a): $\sum_{k=0}^{n} Y_k^2 = \frac{1}{119}(13Y_{n+3}^2 + 13Y_{n+2}^2 + 89Y_{n+1}^2 + 16Y_{n+2}Y_{n+3} + 64Y_{n+1}Y_{n+3} + 48Y_{n+1}Y_{n+2} - 13Y_2^2 - 13Y_1^2 - 89Y_0^2 - 16Y_1Y_2 - 64Y_0Y_2 - 48Y_0Y_1)$.

(b): $\sum_{k=0}^{n} Y_{k+1}Y_k = \frac{1}{119}(8Y_{n+3}^2 + 8Y_{n+2}^2 + 128Y_{n+1}^2 + 19Y_{n+2}Y_{n+3} + 76Y_{n+1}Y_{n+3} + 57Y_{n+1}Y_{n+2} - 8Y_2^2 - 8Y_1^2 - 128Y_0^2 - 57Y_0Y_1 - 19Y_1Y_2 - 76Y_0Y_2)$.

(c): $\sum_{k=0}^{n} Y_{k+2}Y_k = \frac{1}{119}(6Y_{n+3}^2 + 6Y_{n+2}^2 + 96Y_{n+1}^2 + 44Y_{n+2}Y_{n+3} + 57Y_{n+1}Y_{n+3} + 132Y_{n+1}Y_{n+2} - 6Y_2^2 - 6Y_1^2 - 96Y_0^2 - 44Y_1Y_2 - 57Y_0Y_2 - 132Y_0Y_1)$.

Proof.

(a): Set $W_n = Y_n$, $r = 0$, $s = -2$, $t = 4$ and $z = 1$ in [5, Theorem 2.1 (a) (i)].

(b): Set $W_n = Y_n$, $r = 0$, $s = -2$, $t = 4$ and $z = 1$ in [5, Theorem 2.1 (b) (i)].

(c): Set $W_n = Y_n$, $r = 0$, $s = -2$, $t = 4$ and $z = 1$ in [5, Theorem 2.1 (c) (i)]. □

From the last Theorem, we have the following Corollary which gives sum formulas of co-Jacobsthal-Narayana numbers (take $Y_n = U_n$ with $U_0 = 0, U_1 = 1, U_2 = 0$).

Corollary 58. *For $n \geq 0$, co-Jacobsthal-Narayana numbers have the following properties.*

(a): $\sum_{k=0}^{n} U_k^2 = \frac{1}{119}(13U_{n+3}^2 + 13U_{n+2}^2 + 89U_{n+1}^2 + 16U_{n+2}U_{n+3} + 64U_{n+1}U_{n+3} + 48U_{n+1}U_{n+2} - 13)$

(b): $\sum_{k=0}^{n} U_{k+1}U_k = \frac{1}{119}(8U_{n+3}^2 + 8U_{n+2}^2 + 128U_{n+1}^2 + 19U_{n+2}U_{n+3} + 76U_{n+1}U_{n+3} + 57U_{n+1}U_{n+2} - 8)$

(c): $\sum_{k=0}^{n} U_{k+2}U_k = \frac{1}{119}(6U_{n+3}^2 + 6U_{n+2}^2 + 96U_{n+1}^2 + 44U_{n+2}U_{n+3} + 57U_{n+1}U_{n+3} + 132U_{n+1}U_{n+2} - 6)$

Taking $Y_n = S_n$ with $S_0 = 3, S_1 = 0, S_2 = -4$ in the last Theorem, we have the following Corollary which gives sum formulas of co-Jacobsthal-Narayana-Lucas numbers.

Corollary 59. *For $n \geq 0$, co-Jacobsthal-Narayana-Lucas numbers have the following properties:*

(a): $\sum_{k=0}^{n} S_k^2 = \frac{1}{119}(13S_{n+3}^2 + 13S_{n+2}^2 + 89S_{n+1}^2 + 16S_{n+2}S_{n+3} + 64S_{n+1}S_{n+3} + 48S_{n+1}S_{n+2} - 241)$

(b): $\sum_{k=0}^{n} S_{k+1}S_k = \frac{1}{119}(8S_{n+3}^2 + 8S_{n+2}^2 + 128S_{n+1}^2 + 19S_{n+2}S_{n+3} + 76S_{n+1}S_{n+3} + 57S_{n+1}S_{n+2} - 368)$

(c): $\sum_{k=0}^{n} S_{k+2}S_k = \frac{1}{119}(6S_{n+3}^2 + 6S_{n+2}^2 + 96S_{n+1}^2 + 44S_{n+2}S_{n+3} + 57S_{n+1}S_{n+3} + 132S_{n+1}S_{n+2} - 276)$

Next, we give the ordinary generating functions $\sum_{n=0}^{\infty} Y_n^2 z^n$, $\sum_{n=0}^{\infty} Y_{n+1}Y_n z^n$, $\sum_{n=0}^{\infty} Y_{n+2}Y_n z^n$ of the sequences $\{Y_n^2\}$, $\{Y_{n+1}Y_n\}$, $\{Y_{n+2}Y_n\}$.

Theorem 60. *Assume that* $|z| < \min\{|\theta_1|^{-2}, |\theta_2|^{-2}, |\theta_3|^{-2}, |\theta_1\theta_2|^{-1}, |\theta_1\theta_3|^{-1}, |\theta_2\theta_3|^{-1}\} = |\theta_2|^{-2} = |\theta_3|^{-2} = |\theta_2\theta_3|^{-1} \simeq 0.294877$. *Then the ordinary generating functions of the sequences* $\{Y_n^2\}$, $\{Y_{n+1}Y_n\}$, $\{Y_{n+2}Y_n\}$ *are given as follows:*

(a): $\sum_{n=0}^{\infty} Y_n^2 z^n = \frac{1}{-256z^6 + 64z^5 + 32z^4 + 40z^3 + 4z^2 - 2z - 1}(16(2Y_0 + Y_2)^2 z^5 + 4(4Y_1^2 + 8Y_0Y_1 + 4Y_1Y_2)z^4 + (24Y_0^2 + 16Y_1Y_0 - 2Y_2^2)z^3 - z^2(-4Y_0^2 + 2Y_1^2 + Y_2^2) - (2Y_0^2 + Y_1^2)z - Y_0^2)$.

(b): $\sum_{n=0}^{\infty} Y_{n+1}Y_n z^n = \frac{1}{-256z^6 + 64z^5 + 32z^4 + 40z^3 + 4z^2 - 2z - 1}(64Y_0(2Y_0+Y_2)z^5 + 4(2Y_0+Y_2)(4Y_1 + 2Y_2)z^4 + (8Y_1^2 + 24Y_0Y_1 + 4Y_1Y_2)z^3 + (4Y_0Y_1 - 4Y_0Y_2)z^2 - Y_1(2Y_0 + Y_2)z - Y_0Y_1)$.

(c): $\sum_{n=0}^{\infty} Y_{n+2}Y_n z^n = \frac{1}{-256z^6 + 64z^5 + 32z^4 + 40z^3 + 4z^2 - 2z - 1}(64Y_1(2Y_0+Y_2)z^5 + 4(16Y_0^2 + 8Y_0Y_1 + 8Y_0Y_2 - 4Y_1Y_2)z^4 + (32Y_0^2 + 24Y_0Y_2 - 16Y_1Y_0 + 4Y_2^2)z^3 + (4Y_1^2 - 4Y_1Y_2 - 8Y_0Y_1 + 2Y_2^2 + 4Y_0Y_2)z^2 - Y_0Y_2 - (-2Y_1^2 + 4Y_0Y_1 + 2Y_0Y_2)z)$.

Proof. Set $Y_n = Y_n$ and $r = 0$, $s = -2$, $t = 4$ in [4, Theorem 3.1] or in [5, Theorem 3.1]. □

Now, we consider special cases of the last Theorem.

Corollary 61. *Assume that* $|z| < \min\{|\theta_1|^{-2}, |\theta_2|^{-2}, |\theta_3|^{-2}, |\theta_1\theta_2|^{-1}, |\theta_1\theta_3|^{-1}, |\theta_2\theta_3|^{-1}\} = |\theta_2|^{-2} = |\theta_3|^{-2} = |\theta_2\theta_3|^{-1} \simeq 0.294877$. *The ordinary generating functions of the sequences* $\{U_n^2\}$, $\{U_{n+1}U_n\}$, $\{U_{n+2}U_n\}$ *and* $\{S_n^2\}$, $\{S_{n+1}S_n\}$, $\{S_{n+2}S_n\}$ *are given as follows:*

(a):

$$\sum_{n=0}^{\infty} U_n^2 z^n = \frac{16z^4 - 2z^2 - z}{-256z^6 + 64z^5 + 32z^4 + 40z^3 + 4z^2 - 2z - 1},$$

$$\sum_{n=0}^{\infty} S_n^2 z^n = \frac{64z^5 + 184z^3 + 20z^2 - 18z - 9}{-256z^6 + 64z^5 + 32z^4 + 40z^3 + 4z^2 - 2z - 1}.$$

(b):

$$\sum_{n=0}^{\infty} U_{n+1}U_n z^n = \frac{8z^3}{-256z^6+64z^5+32z^4+40z^3+4z^2-2z-1},$$
$$\sum_{n=0}^{\infty} S_{n+1}S_n z^n = \frac{384z^5-64z^4+48z^2}{-256z^6+64z^5+32z^4+40z^3+4z^2-2z-1}.$$

(c):

$$\sum_{n=0}^{\infty} U_{n+2}U_n z^n = \frac{4z^2+2z}{-256z^6+64z^5+32z^4+40z^3+4z^2-2z-1},$$
$$\sum_{n=0}^{\infty} S_{n+2}S_n z^n = \frac{192z^4+64z^3-16z^2+24z+12}{-256z^6+64z^5+32z^4+40z^3+4z^2-2z-1}.$$

From the last corollary, we obtain the following results for special cases of z.

Corollary 62. *Some infinite sums of* $\{U_n^2\}$, $\{U_{n+1}U_n\}$, $\{U_{n+2}U_n\}$ *and* $\{S_n^2\}$, $\{S_{n+1}S_n\}$, $\{S_{n+2}S_n\}$ *are given as follows:*

(a): $z = \frac{1}{4}$.

$$\sum_{n=0}^{\infty} \frac{U_n^2}{4^n} = \frac{5}{8},$$
$$\sum_{n=0}^{\infty} \frac{S_n^2}{4^n} = \frac{149}{8}.$$

(b): $z = \frac{1}{4}$.

$$\sum_{n=0}^{\infty} \frac{U_{n+1}U_n}{4^n} = -\frac{1}{4},$$
$$\sum_{n=0}^{\infty} \frac{S_{n+1}S_n}{4^n} = -\frac{25}{4}.$$

(c): $z = \frac{1}{4}$.

$$\sum_{n=0}^{\infty} \frac{U_{n+2}U_n}{4^n} = -\frac{3}{2},$$
$$\sum_{n=0}^{\infty} \frac{S_{n+2}S_n}{4^n} = -\frac{75}{2}.$$

3.7. Generalized co-Jacobsthal-Narayana Numbers by Matrix Methods. In this section, we present matrix representations of the sequences Y_n, U_n and S_n. We also introduce Simson matrix and investigate its properties.

3.7.1. *Matrix Representations of the Sequences Y_n, U_n and S_n.* We define the square matrix K of order 3 as:

$$K=\begin{pmatrix} 0 & -2 & 4 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

such that $\det K = 4$. Some properties of matrix K^n can be given as

$$\begin{aligned} K^n &= -2K^{n-2}+4K^{n-3}, \\ K^{n+m} &= K^nK^m = K^mK^n, \end{aligned}$$

for all integers m and n. Note that we have the following formulas:

$$\begin{pmatrix} Y_{n+2} \\ Y_{n+1} \\ Y_n \end{pmatrix}=\begin{pmatrix} 0 & -2 & 4 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} Y_{n+1} \\ Y_n \\ Y_{n-1} \end{pmatrix},$$

and

$$\begin{pmatrix} Y_{n+2} \\ Y_{n+1} \\ Y_n \end{pmatrix}=\begin{pmatrix} 0 & -2 & 4 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^n\begin{pmatrix} Y_2 \\ Y_1 \\ Y_0 \end{pmatrix},$$

and

$$\begin{pmatrix} U_{n+2} \\ U_{n+1} \\ U_n \end{pmatrix}=\begin{pmatrix} 0 & -2 & 4 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} U_{n+1} \\ U_n \\ U_{n-1} \end{pmatrix}.$$

We also define

$$N_n=\begin{pmatrix} U_{n+1} & -2U_n+4U_{n-1} & 4U_n \\ U_n & -2U_{n-1}+4U_{n-2} & 4U_{n-1} \\ U_{n-1} & -2U_{n-2}+4U_{n-3} & 4U_{n-2} \end{pmatrix}$$

and

$$M_n=\begin{pmatrix} Y_{n+1} & -2Y_n+4Y_{n-1} & 4Y_n \\ Y_n & -2Y_{n-1}+4Y_{n-2} & 4Y_{n-1} \\ Y_{n-1} & -2sY_{n-2}+4Y_{n-3} & 4Y_{n-2} \end{pmatrix}.$$

THEOREM 63. *For all integers m, n, we have the following properties:*

(a): $N_n = K^n$, *i.e.,*

$$\begin{pmatrix} 0 & -2 & 4 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^n=\begin{pmatrix} U_{n+1} & -2U_n+4U_{n-1} & 4U_n \\ U_n & -2U_{n-1}+4U_{n-2} & 4U_{n-1} \\ U_{n-1} & -2U_{n-2}+4U_{n-3} & 4U_{n-2} \end{pmatrix}.$$

(b): $M_1K^n = K^nM_1$.

(c): $M_{n+m} = M_n N_m = N_m M_n$, *i.e.*,

$$\begin{pmatrix} Y_{n+m+1} & -2Y_{n+m}+4Y_{n+m-1} & 4Y_{n+m} \\ Y_{n+m} & -2Y_{n+m-1}+4Y_{n+m-2} & 4Y_{n+m-1} \\ Y_{n+m-1} & -2Y_{n+m-2}+4Y_{n+m-3} & 4Y_{n+m-2} \end{pmatrix}$$

$$= \begin{pmatrix} Y_{n+1} & -2Y_n+4Y_{n-1} & 4Y_n \\ Y_n & -2Y_{n-1}+4Y_{n-2} & 4Y_{n-1} \\ Y_{n-1} & -2Y_{n-2}+4Y_{n-3} & 4Y_{n-2} \end{pmatrix} \begin{pmatrix} U_{m+1} & -2U_m+4U_{m-1} & 4U_m \\ U_m & -2U_{m-1}+4U_{m-2} & 4U_{m-1} \\ U_{m-1} & -2U_{m-2}+4U_{m-3} & 4U_{m-2} \end{pmatrix}$$

$$= \begin{pmatrix} U_{m+1} & -2U_m+4U_{m-1} & 4U_m \\ U_m & -2U_{m-1}+4U_{m-2} & 4U_{m-1} \\ U_{m-1} & -2U_{m-2}+4U_{m-3} & 4U_{m-2} \end{pmatrix} \begin{pmatrix} Y_{n+1} & -2Y_n+4Y_{n-1} & 4Y_n \\ Y_n & -2Y_{n-1}+4Y_{n-2} & 4Y_{n-1} \\ Y_{n-1} & -2Y_{n-2}+4Y_{n-3} & 4Y_{n-2} \end{pmatrix}.$$

(d):

$$K^n = U_{n-1}K^2 + (-2U_{n-2}+4U_{n-3})K + 4U_{n-2}I$$

i.e.,

$$K^n = \frac{1}{4}((U_{n+2}+2U_n)K^2 + 4U_nK + (2U_{n+2}+4U_{n+1}+4U_n)I)$$

that is,

$$K^n = \frac{1}{4}(U_{n+2}(K^2+2I) + 4U_{n+1}I + U_n(2K^2+4K+4I))$$

where

$$I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Proof. Set $W_n = Y_n$, $r = 0$, $s = -2$, $t = 4$ and $G_n = U_n$ in [3, Theorem 51.]. □

Next, we present matrix formulas for the generalized co-Jacobsthal-Narayana and co-Jacobsthal-Narayana-Lucas numbers.

Corollary 64. *For all integers n, we have the following formulas for generalized co-Jacobsthal-Narayana numbers and co-Jacobsthal-Narayana-Lucas numbers.*

(a): *Generalized co-Jacobsthal-Narayana numbers.*

$$\begin{pmatrix} 0 & -2 & 4 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^n = \frac{1}{\Lambda_Y(0)} \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

where

$$a_{11} = (4Y_0^2 - Y_1Y_2 - 2Y_0Y_1)Y_{n+3} + (Y_2^2 + 2Y_0Y_2 - 4Y_0Y_1)Y_{n+2} + 4(Y_1^2 - Y_0Y_2)Y_{n+1}$$
$$a_{21} = (4Y_0^2 - Y_1Y_2 - 2Y_0Y_1)Y_{n+2} + (Y_2^2 + 2Y_0Y_2 - 4Y_0Y_1)Y_{n+1} + 4(Y_1^2 - Y_0Y_2)Y_n$$
$$a_{31} = (4Y_0^2 - Y_1Y_2 - 2Y_0Y_1)Y_{n+1} + (Y_2^2 + 2Y_0Y_2 - 4Y_0Y_1)Y_n + 4(Y_1^2 - Y_0Y_2)Y_{n-1}$$

$a_{12} = -2((4Y_0^2 - Y_1Y_2 - 2Y_0Y_1)Y_{n+2} + (Y_2^2 + 2Y_0Y_2 - 4Y_0Y_1)Y_{n+1} + 4(Y_1^2 - Y_0Y_2)Y_n) + 4((4Y_0^2 - Y_1Y_2 - 2Y_0Y_1)Y_{n+1} + (Y_2^2 + 2Y_0Y_2 - 4Y_0Y_1)Y_n + 4(Y_1^2 - Y_0Y_2)Y_{n-1})$

$a_{22} = -2((4Y_0^2 - Y_1Y_2 - 2Y_0Y_1)Y_{n+1} + (Y_2^2 + 2Y_0Y_2 - 4Y_0Y_1)Y_n + 4(Y_1^2 - Y_0Y_2)Y_{n-1}) + 4((4Y_0^2 - Y_1Y_2 - 2Y_0Y_1)Y_n + (Y_2^2 + 2Y_0Y_2 - 4Y_0Y_1)Y_{n-1} + 4(Y_1^2 - Y_0Y_2)Y_{n-2})$

$a_{32} = -2((4Y_0^2 - Y_1Y_2 - 2Y_0Y_1)Y_n + (Y_2^2 + 2Y_0Y_2 - 4Y_0Y_1)Y_{n-1} + 4(Y_1^2 - Y_0Y_2)Y_{n-2}) + 4((4Y_0^2 - Y_1Y_2 - 2Y_0Y_1)Y_{n-1} + (Y_2^2 + 2Y_0Y_2 - 4Y_0Y_1)Y_{n-2} + 4(Y_1^2 - Y_0Y_2)Y_{n-3})$

$a_{13} = 4((4Y_0^2 - Y_1Y_2 - 2Y_0Y_1)Y_{n+2} + (Y_2^2 + 2Y_0Y_2 - 4Y_0Y_1)Y_{n+1} + 4(Y_1^2 - Y_0Y_2)Y_n)$

$a_{23} = 4((4Y_0^2 - Y_1Y_2 - 2Y_0Y_1)Y_{n+1} + (Y_2^2 + 2Y_0Y_2 - 4Y_0Y_1)Y_n + 4(Y_1^2 - Y_0Y_2)Y_{n-1})$

$a_{33} = 4((4Y_0^2 - Y_1Y_2 - 2Y_0Y_1)Y_n + (Y_2^2 + 2Y_0Y_2 - 4Y_0Y_1)Y_{n-1} + 4(Y_1^2 - Y_0Y_2)Y_{n-2})$

$\Lambda_Y(0) = Y_2^3 + 4Y_1^3 + 16Y_0^3 + 2Y_0Y_2^2 + 2Y_2Y_1^2 + 4Y_0Y_1^2 - 16Y_0^2Y_1 - 12Y_2Y_1Y_0$

(b): *co-Jacobsthal-Narayana-Lucas numbers.*

$$\begin{pmatrix} 0 & -2 & 4 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^n = \frac{1}{464}\begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix}$$

where

$b_{11} = 36S_{n+3} - 8S_{n+2} + 48S_{n+1}$

$b_{21} = 36S_{n+2} - 8S_{n+1} + 48S_n$

$b_{31} = 36S_{n+1} - 8S_n + 48S_{n-1}$

$b_{12} = -72S_{n+2} + 160S_{n+1} - 128S_n + 192S_{n-1}$

$b_{22} = -72S_{n+1} + 160S_n - 128S_{n-1} + 192S_{n-2}$

$b_{32} = -72S_n + 160S_{n-1} - 128S_{n-2} + 192S_{n-3}$

$b_{13} = 144S_{n+2} - 32S_{n+1} + 192S_n$

$b_{23} = 144S_{n+1} - 32S_n + 192S_{n-1}$

$b_{33} = 144S_n - 32S_{n-1} + 192S_{n-2}$

Proof. Set $W_n = Y_n, r = 0,\ s = -2,\ t = 4$ and then take $Y_n = S$, respectively, in [3, Corollary 52.]. □

Note that, a_{12}, a_{22}, a_{32} and b_{12}, b_{22}, b_{32} can be written in the following form:

$a_{12} = (-2Y_2^2 + 16Y_0^2 - 4Y_1Y_2 - 4Y_0Y_2)Y_{n+1} + (4Y_2^2 - 8Y_1^2 + 16Y_0^2 - 4Y_1Y_2 + 16Y_0Y_2 - 24Y_0Y_1)Y_n + 4(4Y_1^2 - 8Y_0^2 + 2Y_1Y_2 - 4Y_0Y_2 + 4Y_0Y_1)Y_{n-1}$

$a_{22} = (-2Y_2^2 + 16Y_0^2 - 4Y_1Y_2 - 4Y_0Y_2)Y_n + (4Y_2^2 - 8Y_1^2 + 16Y_0^2 - 4Y_1Y_2 + 16Y_0Y_2 - 24Y_0Y_1)Y_{n-1} + 4(4Y_1^2 - 8Y_0^2 + 2Y_1Y_2 - 4Y_0Y_2 + 4Y_0Y_1)Y_{n-2}$

$a_{32} = (-2Y_2^2 + 16Y_0^2 - 4Y_1Y_2 - 4Y_0Y_2)Y_{n-1} + (4Y_2^2 - 8Y_1^2 + 16Y_0^2 - 4Y_1Y_2 + 16Y_0Y_2 - 24Y_0Y_1)Y_{n-2} + 4(4Y_1^2 - 8Y_0^2 + 2Y_1Y_2 - 4Y_0Y_2 + 4Y_0Y_1)Y_{n-3}$

and

$b_{12} = 160S_{n+1} + 16S_n - 96S_{n-1}$

$b_{22} = 160S_n + 16S_{n-1} - 96S_{n-2}$

$b_{32} = 160S_{n-1} + 16S_{n-2} - 96S_{n-3}$.

Now, we present an identity for Y_{n+m}.

THEOREM 65. *(Honsberger's Identity) For all integers m and n, we have*

$$\begin{aligned} Y_{n+m} &= Y_n U_{m+1} + Y_{n-1}(-2U_m + 4U_{m-1}) + 4Y_{n-2}U_m \\ &= Y_n U_{m+1} + (-2Y_{n-1} + 4Y_{n-2})\, U_m + 4Y_{n-1}U_{m-1} \end{aligned}$$

Proof. Set $W_n = Y_n$, $r = 0,\ s = -2,\ t = 4$ and then $G_n = U_n$ in [3, Theorem 53.]. □

As special cases of the last Theorem, we have the following corollary.

COROLLARY 66. *For all integers m, n, we have the following properties:*

$$\begin{aligned} U_{n+m} &= U_n U_{m+1} + U_{n-1}(-2U_m + 4U_{m-1}) + 4U_{n-2}U_m \\ S_{n+m} &= S_n U_{m+1} + S_{n-1}(-2U_m + 4U_{m-1}) + 4S_{n-2}U_m \end{aligned}$$

Next, we present identies for Y_{mn+j} and its special cases.

COROLLARY 67. *For all integers m, n, j, we have the following properties:*

$$\begin{aligned} Y_{mn+j} &= U_{mn-1}Y_{j+2} + (-2U_{mn-2} + 4U_{mn-3})\, Y_{j+1} + 4U_{mn-2}Y_j \\ U_{mn+j} &= U_{mn-1}U_{j+2} + (-2U_{mn-2} + 4U_{mn-3})\, U_{j+1} + 4U_{mn-2}U_j \\ S_{mn+j} &= U_{mn-1}S_{j+2} + (-2U_{mn-2} + 4U_{mn-3})\, S_{j+1} + 4U_{mn-2}S_j \end{aligned}$$

Proof. Set $r = 0,\ s = -2,\ t = 4$ and $W_n = Y_n$, then take $Y_n = U_n$ and $Y_n = S_n$, respectively, in [3, Corollary 55.]. □

3.7.2. *Simson Matrix and its Properties.* For $n \in \mathbb{Z}$, we define

$$f_Y(n) = \begin{pmatrix} Y_{n+2} & Y_{n+1} & Y_n \\ Y_{n+1} & Y_n & Y_{n-1} \\ Y_n & Y_{n-1} & Y_{n-2} \end{pmatrix}.$$

We call this matrix as Simson matrix of the sequence Y_n. Similarly, as special cases of Y_n, Simson matrices of the sequences U_n and S_n are

$$f_U(n) = \begin{pmatrix} U_{n+2} & U_{n+1} & U_n \\ U_{n+1} & U_n & U_{n-1} \\ U_n & U_{n-1} & U_{n-2} \end{pmatrix} \text{ and } f_S(n) = \begin{pmatrix} S_{n+2} & S_{n+1} & S_n \\ S_{n+1} & S_n & S_{n-1} \\ S_n & S_{n-1} & S_{n-2} \end{pmatrix}$$

respectively.

LEMMA 68. *For all integers n, m and j, the followings hold.*

(a): $f_Y(n) = -2f_Y(n-2) + 4f_Y(n-3)$.

(b): $f_Y(n) = Kf_Y(n-1)$ *and* $f_Y(n) = K^n f_Y(0)$, *i.e.,*

$$\begin{pmatrix} Y_{n+2} & Y_{n+1} & Y_n \\ Y_{n+1} & Y_n & Y_{n-1} \\ Y_n & Y_{n-1} & Y_{n-2} \end{pmatrix} = \begin{pmatrix} 0 & -2 & 4 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} Y_{n+1} & Y_n & Y_{n-1} \\ Y_n & Y_{n-1} & Y_{n-2} \\ Y_{n-1} & Y_{n-2} & Y_{n-3} \end{pmatrix}$$

and

$$\begin{pmatrix} Y_{n+2} & Y_{n+1} & Y_n \\ Y_{n+1} & Y_n & Y_{n-1} \\ Y_n & Y_{n-1} & Y_{n-2} \end{pmatrix} = \begin{pmatrix} 0 & -2 & 4 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^n \begin{pmatrix} Y_2 & Y_1 & Y_0 \\ Y_1 & Y_0 & Y_{-1} \\ Y_0 & Y_{-1} & Y_{-2} \end{pmatrix}.$$

(c): $f_Y(n+m) = K^n f_Y(m)$ *and* $f_Y(n+m) = K^m f_Y(n)$ *i.e.,*

$$\begin{pmatrix} Y_{n+m+2} & Y_{n+m+1} & Y_{n+m} \\ Y_{n+m+1} & Y_{n+m} & Y_{n+m-1} \\ Y_{n+m} & Y_{n+m-1} & Y_{n+m-2} \end{pmatrix} = \begin{pmatrix} 0 & -2 & 4 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^n \begin{pmatrix} Y_{m+2} & Y_{m+1} & Y_m \\ Y_{m+1} & Y_m & Y_{m-1} \\ Y_m & Y_{m-1} & Y_{m-2} \end{pmatrix},$$

and

$$\begin{pmatrix} Y_{m+n+2} & Y_{m+n+1} & Y_{m+n} \\ Y_{m+n+1} & Y_{m+n} & Y_{m+n-1} \\ Y_{m+n} & Y_{m+n-1} & Y_{m+n-2} \end{pmatrix} = \begin{pmatrix} 0 & -2 & 4 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^m \begin{pmatrix} Y_{n+2} & Y_{n+1} & Y_n \\ Y_{n+1} & Y_n & Y_{n-1} \\ Y_n & Y_{n-1} & Y_{n-2} \end{pmatrix},$$

and $f_Y(n) = K^m f_Y(n-m)$, *i.e.,*

$$\begin{pmatrix} Y_{n+2} & Y_{n+1} & Y_n \\ Y_{n+1} & Y_n & Y_{n-1} \\ Y_n & Y_{n-1} & Y_{n-2} \end{pmatrix} = \begin{pmatrix} 0 & -2 & 4 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^m \begin{pmatrix} Y_{n-m+2} & Y_{n-m+1} & Y_{n-m} \\ Y_{n-m+1} & Y_{n-m} & Y_{n-m-1} \\ Y_{n-m} & Y_{n-m-1} & Y_{n-m-2} \end{pmatrix}.$$

Proof. Set $W_n = Y_n$, and $r = 0$, $s = -2$, $t = 4$ in [3, Lemma 56.]. □

Taking the determinant of both sides of the identities given in the last Lemma, we obtain the following Theorem.

Theorem 69. *For all integers n and m, the following identities hold.*

(a): *Catalan's Identity:*

$$\det(f_Y(n+m)) = 4^n \det(f_Y(m)) \quad \text{and} \quad \det(f_Y(n)) = 4^m \det(f_Y(n-m)),$$

i.e.,

$$\begin{vmatrix} Y_{n+m+2} & Y_{n+m+1} & Y_{n+m} \\ Y_{n+m+1} & Y_{n+m} & Y_{n+m-1} \\ Y_{n+m} & Y_{n+m-1} & Y_{n+m-2} \end{vmatrix} = 4^n \begin{vmatrix} Y_{m+2} & Y_{m+1} & Y_m \\ Y_{m+1} & Y_m & Y_{m-1} \\ Y_m & Y_{m-1} & Y_{m-2} \end{vmatrix},$$

and

$$\begin{vmatrix} Y_{n+2} & Y_{n+1} & Y_n \\ Y_{n+1} & Y_n & Y_{n-1} \\ Y_n & Y_{n-1} & Y_{n-2} \end{vmatrix} = 4^m \begin{vmatrix} Y_{n-m+2} & Y_{n-m+1} & Y_{n-m} \\ Y_{n-m+1} & Y_{n-m} & Y_{n-m-1} \\ Y_{n-m} & Y_{n-m-1} & Y_{n-m-2} \end{vmatrix}.$$

(b): *(see Theorem 46) Simson's (or Cassini's) Identity:*

$$\det(f_Y(n)) = 4^n \det(f_Y(0)),$$

i.e.,

$$\begin{vmatrix} Y_{n+2} & Y_{n+1} & Y_n \\ Y_{n+1} & Y_n & Y_{n-1} \\ Y_n & Y_{n-1} & Y_{n-2} \end{vmatrix} = 4^n \begin{vmatrix} Y_2 & Y_1 & Y_0 \\ Y_1 & Y_0 & Y_{-1} \\ Y_0 & Y_{-1} & Y_{-2} \end{vmatrix}.$$

Proof. Set $W_n = Y_n$, and $r = 0,\ s = -2,\ t = 4$ in [3, Theorem 57.]. □

From the last Theorem, we have the following Corollary which gives determinantal formulas of co-Jacobsthal-Narayana numbers (take $Y_n = U_n$ with $U_0 =, U_1 =, U_2 =$).

Corollary 70. *For all integers n and m, the following identities hold.*

(a): *Catalan's Identity:*

$$\det(f_U(n+m)) = 4^n \det(f_U(m)) \quad \textit{and} \quad \det(f_U(n)) = 4^m \det(f_U(n-m)),$$

i.e.,

$$\begin{vmatrix} U_{n+m+2} & U_{n+m+1} & U_{n+m} \\ U_{n+m+1} & U_{n+m} & U_{n+m-1} \\ U_{n+m} & U_{n+m-1} & U_{n+m-2} \end{vmatrix} = 4^n \begin{vmatrix} U_{m+2} & U_{m+1} & U_m \\ U_{m+1} & U_m & U_{m-1} \\ U_m & U_{m-1} & U_{m-2} \end{vmatrix},$$

and

$$\begin{vmatrix} U_{n+2} & U_{n+1} & U_n \\ U_{n+1} & U_n & U_{n-1} \\ U_n & U_{n-1} & U_{n-2} \end{vmatrix} = 4^m \begin{vmatrix} U_{n-m+2} & U_{n-m+1} & U_{n-m} \\ U_{n-m+1} & U_{n-m} & U_{n-m-1} \\ U_{n-m} & U_{n-m-1} & U_{n-m-2} \end{vmatrix}.$$

(b): *Simson's (or Cassini's) Identity:*

$$\det(f_U(n)) = 4^n \det(f_U(0)),$$

i.e.,

$$\begin{vmatrix} U_{n+2} & U_{n+1} & U_n \\ U_{n+1} & U_n & U_{n-1} \\ U_n & U_{n-1} & U_{n-2} \end{vmatrix} = -4^{n-1}.$$

Taking $Y_n = S_n$ with $S_0 =, S_1 =, S_2 =$ in the last Theorem, we have the following Corollary which gives determinantal formulas of co-Jacobsthal-Narayana-Lucas numbers.

Corollary 71. *For all integers n and m, the following identities hold.*

(a): *Catalan's Identity:*

$$\det(f_S(n+m)) = 4^n \det(f_S(m)) \quad \textit{and} \quad \det(f_S(n)) = 4^m \det(f_S(n-m))$$

i.e.,

$$\begin{vmatrix} S_{n+m+2} & S_{n+m+1} & S_{n+m} \\ S_{n+m+1} & S_{n+m} & S_{n+m-1} \\ S_{n+m} & S_{n+m-1} & S_{n+m-2} \end{vmatrix} = 4^n \begin{vmatrix} S_{m+2} & S_{m+1} & S_m \\ S_{m+1} & S_m & S_{m-1} \\ S_m & S_{m-1} & S_{m-2} \end{vmatrix},$$

and

$$\begin{vmatrix} S_{n+2} & S_{n+1} & S_n \\ S_{n+1} & S_n & S_{n-1} \\ S_n & S_{n-1} & S_{n-2} \end{vmatrix} = 4^m \begin{vmatrix} S_{n-m+2} & S_{n-m+1} & S_{n-m} \\ S_{n-m+1} & S_{n-m} & S_{n-m-1} \\ S_{n-m} & S_{n-m-1} & S_{n-m-2} \end{vmatrix}.$$

(b): *Simson's (or Cassini's) Identity:*

$$\det(f_S(n)) = 4^n \det(f_S(0)),$$

i.e.,

$$\begin{vmatrix} S_{n+2} & S_{n+1} & S_n \\ S_{n+1} & S_n & S_{n-1} \\ S_n & S_{n-1} & S_{n-2} \end{vmatrix} = -29 \times 2^{2n}.$$

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