PURSUIT PROBLEM OF A LINEAR DIFFERENTIAL GAME FOR MANY NUMBERS 0F PLAYERS WITH GEOMETRIC AND GRONWALL TYPE CONSTRAINTS ON PLAYER'S CONTROL

Abstract. In this paper, we study a Pursuit differential game with many number of pursuers and evaders in $\mathbb{R}^n$. The player's motion obey linear differential equations and the control functions of pursuers and evaders **are** subject to Geometric and Gronwall constraints respectively. If the pursuer's state and the evader's state coincide, the game is deemed completed. We construct a strategy that is distinct from a parallel approach strategy that guaranteed pursuit completion and demonstrated sufficient conditions for which pursuit is completed for many number of pursuers and evaders.

KEYWORDS: evaders, geometric, gronwall, equations

1. introduction

A differential game is considered as a control problem from the perspective of either the pursuing party or the evading party. From this perspective, the game can be reduced to either the pursuit problem (the approach) or the evasion problem (the escape). The fundamental approaches to differential games were developed by Pontryagin [1] and Krasnovskiy [2].

Isaacs introduced the idea of differential games [3]. His research was published in monographs, which included many excellent examples of differential games.Viewing these as Variation Calculus problems, the Author attempted to solve them using the Hamilton-Jacob approach (today called Isaac's method).However, the topic proved to be more intricate for traditional approaches.Isaac's concept was just heuristic in nature.However, the concept sparked curiosity about fresh issues.It was then that differential games were taken into consideration by mathematicians, machinists, professionals, and amateurs alike. This has been studied since 1960, and researchers Pontryagin [1], Krasovskii [2], Bercovitz [4], Elliot and Kalton [5], Isaacs [3], Fleming [6], and Friedman [7],Subbotin ([12] and [13]), Hajek [8], Ho Bryson [9],and a number of others have all produced important findings.

Control functions are largely subject to geometric, integral[16],[17]-[26],[27], and [28], or mixed constraints [29] and [30] in the theory of differential games. Integral inequalities are crucial to the study of differential equations because they can be used to examine the existence, uniqueness, and stability of differential equation solutions. Nevertheless, in [31] and [32], controls functions have most recently been subject to a Gronwall-type integral inequality known as the Gronwall-type constraint. This constraint mechanically expresses the idea that players' maximal speeds will gradually increase during the game.Samatov [32] introduced the idea initially.

Key words and phrases. Differential game, Geometric, Gronwall's inqualities, pursuit, strategy.
2020 Mathematics Subject classification: 47H09, 47H10, 47H05, 47J25.

Ahmed in [33] study a pursuit problem with many finite numbers of players on a closed covex subset K of $\mathbb{R}^n$ with a generalized integral constraints on control functions of the players.

Rilwan et al. in [34] studied a fixed duration pursuit-evasion differential game problem of one pursuer and one evader with GrÃűnwall-type constraints imposed on all players control functions. The players dynamics are governed by simple differential equation, the Authors obtained sufficient conditions for completion of pursuit and evasion to be possible. Players attainability domain and optimal strategies were constructed.

The game value of a pursuit-evasion differential game was studied by Rilwan et al. in [35]. They estimated the value for a fixed duration differential game problem of countably many pursuers and one evader with the GrÃűnwall-type constraints, a generalization of the well known geometric constraints imposed on all the players control functions. The players dynamics are governed by the simple differential equations where they constructed the attainability domain for the players. The constructed strategies were then employed in establishing the estimated value of the game.

The focus of this paper is on a linear pursuit differential game for many numbers of players where control function of the pursuers comply with Geometric constraints while that of the evaders subject to integral inequalities of the Gronwall type. The pursuit problem has been solved by constructing a pursuer strategy.Furthermore, we have discovered the necessary conditions for the pursuit to be Completed for many number of pursuers and evaders as well as the comprehensive proofs.

2. Description of the Game Problem

We consider differential game problem in $\mathbb{R}^n$ where the pursuers P_i and evaders E_j move in accordance with the following differential equations:

$$\begin{cases} P_i : \dot{x}_i(t) + \alpha_i x_i(t) = u_i(t), \quad x_i(0) = x_{i0}, \quad i = 1, 2, 3, ...n, \\ E_j : \dot{y}_j(t) + \alpha_j y_j(t) = \ v_j(t), \quad y_j(0) = y_{j0}, \quad j = 1, 2, 3, ...m, \end{cases} \tag{2.1}$$

where $u_i(\cdot), v_j(\cdot)$ are the control function of the pursuers and evaders respectively, α_i, and α_j,are scalar measurable functions and $n > m$. Let the duration of the game denoted by θ.

Definition 2.1. *A measurable function $u_i(t) = (u_{i1}(t), u_{i2}(t), \ldots, u_{in}(t))$ that satisfy the inequality*

$$||u_i(t)|| \leq \rho_i \quad t \geq 0 \quad i = 1, 2, 3, ..., n, \tag{2.2}$$

where ρ_i is positive numbers, is called permissible control of the pursuer P_i with respect to Geometric constraint.

Definition 2.2. *A measurable function $v_j(t) = (v_{j1}(t), v_{j2}(t), \ldots, v_{jn}(t))$ that satisfy the inequality*

$$||v_j(t)||^2 \leq \sigma_j^2 + 2k \int_0^t ||v_j(s)||^2 dt \quad t \geq 0 \quad j = 1, 2, 3, ...m, \tag{2.3}$$

where σ_j and k are positive numbers, is called permissible control of the evader E_j with respect to Gronwall-type constraint.

Definition 2.3. *A function $U_i(\cdot)$ is called strategy of the pursuers, if for any permissible control of the evaders the system*

$$\begin{cases} P_i : \dot{x}_i(t) + \alpha_i x_i(t) = U_i(t), & x_i(0) = x_{i0}, i = 1, 2, 3, ...n, \\ E_j : \dot{y}_j(t) + \alpha_j y_j(t) = \ v_j(t), & y_j(0) = y_{j0}, j = 1, 2, 3, ..., m \end{cases} \tag{2.4}$$

has a unique solution.

Given the players permissible control $u_i(\cdot)$ and $v_j(\cdot)$, the corresponding paths $x_i(t), y_j(t)$, at any time $t > 0$ of the players for any initial positions x_{i0}, y_{j0}, respectively are given by

$$x_i(t) = x_{i0}e^{-\alpha_i t} + \int_0^t e^{\alpha_i(s-t)} u_i(s)ds, i = 1, 2, 3, ..., m, \tag{2.5}$$

$$y_j(t) = y_{j0}e^{-\alpha_j t} + \int_0^t e^{\alpha_j(s-t)} v_j(s)ds, j = 1, 2, 3, ...n. \tag{2.6}$$

Lemma 2.1. *[32] Let $w(t), t \geq 0$ be a measurable function,φ and k be non-negative real numbers then the inequality*

$$|w(t)| \leq \varphi e^{kt}, \tag{2.7}$$

is true whenever

$$|w(t)|^2 \leq \varphi^2 + 2k \int_0^t |w(s)|^2 ds. \tag{2.8}$$

Definition 2.4. *Pursuit is said to be completed in the game (2.1)- (2.3) if there exists permissible strategy of the pursuers such that for any permissible control of the evaders the relation $x_i(\tau) = y_j(\tau)$ holds for some $\tau \in [0, \theta]$*

3. Main result

In this section, we present the main results of the research work in form of Theorems and their proofs.

let the i^{th} pursuer chases j^{th} evader and uses the strategy

$$u_i(t) = \frac{(e^{\lambda_{ij}\theta} y_{j0} - x_{i0})}{\theta} e^{-\alpha_i t} + e^{\lambda_{ij}(\theta - t)} v_j(t). \tag{3.1}$$

Prior to present our Theorems or results, it is necessary to introduce the following definitions:

$\sigma := \max_j \{\sigma_j\}$. $\rho := \min\{\rho_i\}$. $\Phi := \max_{i,j}\{\frac{\|e^{\lambda_{i,j}} y_{j0} - x_{i0}\|}{\theta}\}$. $\lambda := \max_{i,j}\{\lambda_{i,j}\}$. $\alpha := \max_i\{\alpha_i\}$. $\mu = max\{-\alpha, k\}$.

Theorem 3.1. *Suppose in the game(2.1)- (2.3) that $\lambda = 0$ and $\rho > \Phi$. If the i^{th} pursuer uses the strategy (3.1)then pursuit can be completed whenever*

$$\sigma \leq \begin{cases} e^{-k\theta}(\rho - \Phi), & \alpha \geq 0. \\ \rho e^{-\mu\theta} - \Phi, & \alpha < 0. \end{cases}$$

Proof. If $\lambda = 0$ then $\alpha_i = \alpha_j$,observe that the strategy(3.1) becomes

$$u_i(t) = \frac{(y_{j0} - x_{i0})}{\theta} e^{-\alpha_i t} + v_j(t). \tag{3.2}$$

consequently, we have that $x_i(\theta) = y_j(\theta)$. Indeed,

$$\begin{aligned}
x_i(\theta) &= x_{i0}e^{-\alpha_i\theta} + \int_0^\theta e^{\alpha_i(s-\theta)} u_i(s)ds \\
&= x_{i0}e^{-\alpha_i\theta} + \int_0^\theta e^{\alpha_i(s-\theta)} \left(\frac{(y_{j0} - x_{i0})}{\theta} e^{-\alpha_i s} + v_j(s) \right) ds \\
&= x_{i0}e^{-\alpha_i\theta} + \int_0^\theta \frac{(y_{j0} - x_{i0})}{\theta} e^{-\alpha_j\theta} ds + \int_0^\theta e^{\alpha_i(s-\theta)} v_j(s)ds \\
&= x_0e^{-\alpha_i\theta} + \frac{(y_{j0} - x_{i0})}{\theta} e^{-\alpha_i\theta} \int_0^\theta ds + \int_0^\theta e^{\alpha_i(s-\theta)} v_j(s)ds \\
&= y_{j0}e^{-\alpha_i\theta} + \int_0^\theta e^{\alpha_i(s-\theta)} v_j(s)ds = y_j(\theta).
\end{aligned}$$

The permissible of the pursuer strategy (3.2) can be shown using lamma2.1 as follows:

$$\begin{aligned}
\|u_i(t)\| &= \left\| \frac{(y_{j0} - x_{i0})}{\theta} e^{-\alpha_i t} + v_j(t) \right\| \\
&\leq \frac{e^{-\alpha_i t}}{\theta} \|y_{j0} - x_{i0}\| + \|v_j(t)\| \\
&\leq \frac{e^{-\alpha_i t}}{\theta} \|y_{j0} - x_{i0}\| + \sigma_j e^{kt} \\
&\leq \frac{e^{-\alpha t}}{\theta} \|y_{j0} - x_{i0}\| + \sigma_j e^{kt}
\end{aligned} \tag{3.3}$$

when $\alpha \geq 0$,then from (3.3) we have

$$\begin{aligned}
\|u_i(t)\|^2 &\leq \frac{1}{\theta} \|y_{j0} - x_{i0}\| + \sigma_j e^{kt} \\
&\leq \frac{\|y_{j0} - x_{i0}\|}{\theta} + \sigma_j e^{k\theta} \\
&= \frac{\|y_{j0} - x_{i0}\|}{\theta} + e^{k\theta}\sigma_j \\
&\leq \frac{\|y_{j0} - x_{i0}\|}{\theta} + e^{k\theta}\sigma \\
&\leq \Phi + e^{k\theta} (\rho - \Phi) e^{-k\theta} \\
&= \rho \leq \rho_i.
\end{aligned}$$

when $\alpha < 0$,then from (3.3) we have

$$\begin{aligned}
\|u_i(t)\|^2 &= e^{\mu t}\frac{\|y_{j0} - x_{i0}\|}{\theta} + \sigma_j e^{\mu t} \\
&\leq e^{\mu\theta}\frac{\|y_{j0} - x_{i0}\|}{\theta} + \sigma_j e^{\mu\theta} \\
&\leq e^{\mu\theta}\Phi + e^{\mu\theta}\sigma \\
&= (\Phi + \sigma)\, e^{\mu\theta} \\
&\leq \left(\Phi + \rho e^{-\mu\theta} - \Phi\right) e^{\mu\theta} \\
&= \rho \leq \rho_i.
\end{aligned}$$

□

Theorem 3.2. *Suppose in the game(2.1)- (2.3) that $\lambda < 0$ and $\rho > \Phi$. If the i^{th} pursuer uses the strategy (3.1)then pursuit can be completed whenever*
$\sigma \leq \begin{cases} e^{-k\theta}(\rho - \Phi), & \alpha \geq 0. \\ \rho e^{-\mu\theta} - \Phi, & \alpha < 0. \end{cases}$

Proof. If the i^{th} pursuer apply strategy (3.1) we have that $x_i(\theta) = y_j(\theta)$. Indeed,

$$\begin{aligned}
x_i(\theta) &= x_{i0}e^{-\alpha_i\theta} + \int_0^\theta e^{\alpha_i(s-\theta)}u_i(s)ds \\
&= x_{i0}e^{-\alpha_i\theta} + \int_0^\theta e^{\alpha_i(s-\theta)}\left(\frac{(e^{\lambda_{ij}\theta}y_{j0} - x_{i0})}{\theta}e^{-\alpha_i s} + e^{\lambda_{ij}(\theta-s)}v_j(s)\right)ds \\
&= x_{i0}e^{-\alpha_i\theta} + \int_0^\theta \frac{y_{j0}e^{\alpha_{i(s-\theta)}}e^{\lambda_r\theta}e^{-\alpha_i s}}{\theta}ds - \int_0^\theta \frac{x_{i0}e^{\alpha_i(s-\theta)}e^{-\alpha_i s}}{\theta}ds \\
&+ \int_0^\theta e^{\alpha_i(s-\theta)}e^{\lambda_{ij}(\theta-s)}v_j(s)ds \\
&= x_{i0}e^{-\alpha_i\theta} + \int_0^\theta \frac{y_{j0}e^{-\alpha_j\theta}}{\theta}ds - \int_0^\theta \frac{x_{i0}e^{-\alpha_i\theta}}{\theta} + \int_0^\theta e^{\alpha_j(s-\theta)}v_j(s)ds \\
&= x_{i0}e^{-\alpha_i\theta} + y_{j0}e^{-\alpha_j\theta} - x_{i0}e^{-\alpha_i\theta} + \int_0^\theta e^{\alpha_j(s-\theta)}v_j(s)ds \\
&= y_{j0}e^{-\alpha_j\theta} + \int_0^\theta e^{\alpha_j(s-\theta)}v_j(s)ds = y_j(\theta).
\end{aligned}$$

The permissible of the pursuer strategy (3.1) is followed using lamma2.1 as,

$$\begin{aligned}
\|u_i(t)\| &= \left\|\frac{(e^{\lambda_{ij}\theta}y_{j0} - x_{i0})}{\theta}e^{-\alpha_i t} + e^{\lambda_{ij}(\theta-t)}v_j(t)\right\| \\
&\leq \frac{e^{-\alpha_i t}\|e^{\lambda_{ij}\theta}y_{j0} - x_{i0}\|}{\theta} + \|e^{\lambda_{ij}(\theta-t)}v_j(t)\| \\
&\leq \frac{e^{-\alpha_i t}\|e^{\lambda_{ij}\theta}y_{j0} - x_{i0}\|}{\theta} + e^{\lambda_{ij}(\theta-t)}\|v_j(t)\| \\
&\leq \frac{e^{-\alpha t}\|e^{\lambda_{ij}\theta}y_{j0} - x_{i0}\|}{\theta} + \|v_j(t)\| \\
&\leq \frac{e^{-\alpha t}\|e^{\lambda_{i,j}\theta}y_{j0} - x_{i0}\|}{\theta} + \sigma_j e^{kt}. \qquad (3.4)
\end{aligned}$$

when $\alpha \geq 0$, then from (3.4), we have

$$\begin{aligned}
\|u_i(t)\| &\leq \frac{1}{\theta}\|e^{\lambda_{i,j}\theta}y_{j0} - x_{i0}\| + \sigma_j e^{kt} \\
&\leq \frac{\|e^{\lambda_{i,j}\theta}y_{j0} - x_{i0}\|}{\theta} + \sigma_j e^{k\theta} \\
&= \frac{\|e^{\lambda_{i,j}\theta}y_{j0} - x_{i0}\|}{\theta} + e^{k\theta}\sigma_j \\
&\leq \frac{\|e^{\lambda_{i,j}\theta}y_{j0} - x_{i0}\|}{\theta} + e^{k\theta}\sigma \\
&\leq \Phi + e^{k\theta}\left(\rho - \Phi\right)e^{-k\theta} \\
&= \rho \leq \rho_i.
\end{aligned}$$

when $\alpha < 0$,then from (3.4),we have

$$\begin{aligned}
\|u_i(t)\| &= \frac{e^{\mu t}\|e^{\lambda_{i,j}\theta}y_{j0} - x_{i0}\|}{\theta} + \sigma_j e^{\mu t} \\
&\leq e^{\mu\theta}\frac{\|e^{\lambda_{i,j}\theta}y_{j0} - x_{i0}\|}{\theta} + \sigma_j e^{\mu\theta} \\
&\leq e^{\mu\theta}\Phi + e^{\mu\theta}\sigma \\
&= \left(\Phi + \sigma\right)e^{\mu\theta} \\
&\leq \left(\Phi + \rho e^{-\mu\theta} - \Phi\right)e^{\mu\theta} \\
&= \rho \leq \rho_i.
\end{aligned}$$

□

Theorem 3.3. *Suppose in the game(2.1)- (2.3) that $\lambda > 0$ and $\rho > \Phi$. If the i^{th} pursuer uses the strategy (3.1)then pursuit can be completed whenever*
$\sigma \leq \begin{cases} e^{-\theta(k+\lambda)}\left(\rho - \Phi\right), \alpha \geq 0. \\ \left(e^{-\mu\theta}\rho - \Phi\right)e^{-\lambda\theta}, \alpha < 0. \end{cases}$

Proof. First, we showed that $x_i(\theta) = y_j(\theta)$ using same argument as in theorem (3.2). Secondly, we prove that strategy (3.1) is permissible, that is,

$$\begin{aligned}
\|u_i(t)\| &= \left\|\frac{(e^{\lambda_{ij}\theta}y_{j0} - x_{i0})}{\theta}e^{-\alpha_i t} + e^{\lambda_{ij}(\theta-t)}v_j(t)\right\| \\
&\leq \frac{e^{-\alpha_i t}\|e^{\lambda_{ij}\theta}y_{j0} - x_{i0}\|}{\theta} + \|e^{\lambda_{ij}(\theta-t)}v_j(t)\| \\
&\leq \frac{e^{-\alpha_i t}\|e^{\lambda_{ij}\theta}y_{j0} - x_{i0}\|}{\theta} + e^{\lambda_{ij}(\theta-t)}\|v_j(t)\| \\
&\leq \frac{e^{-\alpha t}\|e^{\lambda_{ij}\theta}y_{j0} - x_{i0}\|^2}{\theta^2} + e^{\lambda\theta}\|v_j(t)\| \qquad (3.5)
\end{aligned}$$

when $\alpha \geq 0$, then from (3.5), we have

$$\begin{aligned}
\|u_i(t)\| &\leq \frac{\|e^{\lambda_{ij}\theta}y_{j0} - x_{i0}\|}{\theta} + e^{\lambda\theta}\|v_j(t)\| \\
&\leq \frac{\|e^{\lambda_{ij}\theta}y_{j0} - x_{i0}\|}{\theta} + \sigma_j e^{\theta(\lambda+k)} \\
&\leq \frac{\|e^{\lambda_{i,j}\theta}y_{j0} - x_{i0}\|}{\theta} + \sigma e^{\theta(\lambda+k)} \\
&\leq \frac{\|e^{\lambda_{ij}\theta}y_{j0} - x_{i0}\|}{\theta} + e^{-\theta(\lambda+k)}\left(\rho - \Phi\right)e^{\theta(\lambda+k)} \\
&\leq \Phi + \rho - \Phi \\
&= \rho \leq \rho_i.
\end{aligned}$$

when $\alpha < 0$ then from (3.5),we have

$$\begin{aligned}
\|u_i(t)\| &\leq \frac{e^{\mu\theta}\|e^{\lambda_{ij}\theta}y_{j0} - x_{i0}\|}{\theta} + \sigma e^{(\lambda+\mu)\theta} \\
&\leq e^{\mu\theta}\Phi + \left(e^{-\mu\theta}\rho - \Phi\right)e^{-\lambda\theta}e^{(\lambda+\mu)\theta} \\
&= e^{\mu\theta}\Phi + \left(e^{-\mu\theta}\rho - \Phi\right)e^{\mu\theta} \\
&= \rho \leq \rho_i.
\end{aligned}$$

□

4. Conclusion

In this research, we studied pursuit differential game problem of many players for a fixed duration with linear dynamic equation in Hilbert space $\mathbb{R}^n$. The dynamic equation of the players is described by first order linear differential equation. The control functions of the pursuers and evaders are subjected to geometric constraints and Gronwall inequality constraints respectively.

We found conditions that guaranteed completion of pursuit by the pursuers in a finite time in forms of Theorems and their comprehensive proofs. We construct a new strategy for completion of pursuit if the dynamic motion of the players is described by linear differential equation. Indeed, this is a significant contribution to the literature on pursuit differential game problems.

References

[1] Pontryagin L. S. Izbrannye trudy ,Moscow: MAKS Press, 2004. 551 p. (in Russian
[2] Krasovskii N. N., Subbotin A. I. Game-Theoretical Control Problems. New York: Springer, 2011. 517 p.
[3] Isaacs R. Differential Games. New York: John Wiley and Sons, 1965. 385 p.
[4] . Berkovitz L.D. Differential game of generalized pursuit and evasion. SIAM J. Control Optim., 1986. Vol. 24, No. 3, P. 361-373. DOI: 10.1137/0324021
[5] Elliott R.J., Kalton N. J. The existence of value in differential games of pursuit and evasion. J. Differential Equations, 1972. Vol. 12, No. 3. P. 504-523. DOI: 10.1016/0022-0396(72)90022-8

[6] Fleming W.H. The convergence problem for differential games, II. In: Advances in Game Theory, M. Dresher, L.S. Shapley, A.W. Tucker (eds.). Ann. of Math. Stud., vol. 52. Princeton University Press, 1964. P. 195-210. DOI: 10.1515/9781400882014-013

[7] Friedman A. Differential Games. Pure Appl. Math., vol. 25. New York: Wiley Interscience, 1971. 350 p.

[8] HÂťajek O. Pursuit Games: An Introduction to the Theory and Applications of Differential Games of Pursuit and Evasion. New York: Dover Pub., 2008. 288 p.

[9] Ho Y., Bryson A., Baron S. Differential games and optimal pursuit-evasion strategies. IEEE Trans. Automat. Control, 1965. Vol. 10, No. 4. P. 385-389. DOI: 10.1109/TAC.1965.1098197

[10] Pshenichnyi B. N. Simple pursuit by several objects. Cybern. Syst. Anal., 1976. Vol. 12, No. 5. P. 484-485. DOI: 10.1007/BF01070036

[11] Pshenichnyi B. N., Chikrii A. A., Rappoport I. S. An efficient method of solving differential games with many pursuers. Dokl. Akad. Nauk SSSR, 1981. Vol. 256, No. 3. P. 530-535.

[12] Subbotin A. I., Chentsov A. G. Optimizaciya garantii v zadachah upravleniya [Optimization of Guarantee in Control Problems]. Moscow: Nauka, 1981. 288 p. (in Russian)

[13] Subbotin A. I. Generalization of the main equation of differential game theory. J. Optim. Theory Appl., 1984. Vol. 43, No. 1. P. 103-133. DOI: 10.1007/BF00934749

[14] Ushakov V. N. Extremal strategies in differential games with integral constraints. J. Appl. Math. Mech., 1972. Vol. 36, No. 1. P. 12-19. DOI: 10.1016/0021-8928(72)90076-7

[15] Chikrii A.A. Conflict-Controlled Processes. Dordrecht: Springer, 1997. DOI: 10.1007/978-94-017-1135-7

[16] Badakaya, A., Halliru, A., Adamu, J.: Game value for a pursuit-evasion diferential game problem in a Hilbert space. J. Dyn. Games 9, 1 (2021)

[17] Ferrara, M., Ibragimov, G., Scalzo, V.: Diferential games and discrete dynamics in applied sciences.Discret. Dyn. Nat. Soc. 2017, 6023140 (2017). https://doi.org/10.1155/2017/6023140

[18] Ibragimov, G.: A game problem on a closed convex set. Sib. Adv. Math. 12, 16-31 (2002)

[19] Ibragimov, G.: Optimal pursuit with countably many pursuers and one evader. Difer. Equ. 41, 627-635 (2005)

[20] Ibragimov, G., Hussin, N.: A pursuit-evasion diferential game with many pursuers and one evader. Malays. J. Math. Sci. 4, 183-194 (2010)

[21] Ibragimov, G., Rikhsiev, B.: On some sufcient conditions for optimality of the pursuit time in the differential game with multiple pursuers. Autom. Remote Control 67, 529-537 (2006)

[22] Ibragimov, G.I., Rikhsiev, B.B.: On some sufcient conditions for optimality of the pursuit time in the diferential game with multiple pursuers. Autom. Remote Control 67(4), 529-537 (2006)

[23] Ibragimov, G., Salimi, M.: Pursuit-evasion diferential game with many inertial players. Math. Probl. Eng. 2009, 653723 (2009)

[24] Ibragimov, G., Satimov, N.: A multiplayer pursuit diferential game on a closed convex set with integral constraints. Abstr. Appl. Anal. 2012 (2012)

[25] Ibragimov, G., Satimov, N.: A multiplayer pursuit diferential game on a convex set with integral constraints. Abstr. Appl. Anal. (2012).

https://doi.org/10.1155/2012/460171

[26] Ibragimov, G., Salimi, M., Amini, M.: Evasion from many pursuers in simple motion diferential game with integral constraints. Eur. J. Oper. Res. 218, 505-511 (2012)

[27] Ibragimov, G., Alias, A.I., Waziri, U., JaâĂŹafaru, A.B.: Diferential game of optimal pursuit for an infnite system of of diferential equations. Bull. Malays. Math. Sci. Soc. 19(3), 963-976 (2015)

[28] Ibragimov, G., Ferrara, M., Kuchkarov, A., Pansera, B.: Simple motion evasion diferential game of many pursuers and evaders with integral constraints. Dyn. Games Appl. 8, 352âĂŞ378 (2018)

[29] DarâĂŹin A.N., Kurzhanskii A.B. Control under indeterminacy and double constraints. Differ. Equ., 2003. Vol. 39, No. 11. P. 1554âĂŞ1567. DOI: 10.1023/B:DIEQ.0000019347.24930.a3

[30] Kornev D. V., Lukoyanov N. Yu. On a minimax control problem for a positional functional under geometric and integral constraints on control actions. Proc. Steklov Inst. Math., 2016. Vol. 293, P. 85-100. DOI: 10.1134/S0081543816050096

[31] Rilwan, J., Massimiliano F., Badakaya, A. J. and Bruno A. P.: On pursuit and evasion game problems with Gronwall type constraints, Springer Nature, https://doi.org/10.1007/s11135-023-01616-9 (2023).

[32] Samatov B.T. Ulmasjon S and Adhambek A., A linear differential game with Gronwall-type constraints on controls, ural mathematical journal, 6, 4, (2020)95-107

[33] Ahmed, I. Wiyada, K. W. Ibragimov, G.I. and Rilwan, J. "Pursuit Differential Game Problem with Multiple Players on a Closed Convex Set with More General Integral Constraints", Thai Journal of Mathematics, **18**(2) (2020) 551-561. http://thaijmath.in.cmu.ac.th.

[34] Rilwan, J.,Ferrara M. and Badakaya, A. J. On Pursuit and Evasion Game problems with Gronwall-type Constraints. Quality and Quantity (2023) 57:5551âĂŞ5562 https://doi.org/10.1007/s11135-023-01616-9.

[35] Rilwan, J., Fotia, P. and Ferrara, M. On game Value of a Differential Game problem with Gronwall-type Constraints on Players Control Functions. Decisions Econ Finan 46, 319âĂŞ333 (2023). https://doi.org/10.1007/s10203-023-00389-y.

Disclaimer (Artificial intelligence)
Option 1:
Author(s) hereby declare that NO generative AI technologies such as Large Language Models (ChatGPT, COPILOT, etc.) and text-to-image generators have been used during the writing or editing of this manuscript.
Option 2:
Author(s) hereby declare that generative AI technologies such as Large Language Models, etc. have been used during the writing or editing of manuscripts. This explanation will include the name, version, model, and source of the generative AI technology and as well as all input prompts provided to the generative AI technology
Details of the AI usage are given below:
1.
2.
3.