Journal of Advances in Mathematics and Computer Science

xx(x): ....., 20yy; Article no.JAMCS.xxxxx

ISSN: 2456-9968

(Past name: British Journal of Mathematics & Computer Science, Past ISSN: 2231-0851)

On Supra* Generalized Closed and Open Sets in Supra Bitopological Spaces

Original Research Article

Received: XX December 20XX
Accepted: XX December 20XX
Online Ready: XX December 20XX

Abstract

Topological structures hold enormous significance in different mathematical and computational areas, particularly in the analysis of relations and connectivity within networks. Bitopological spaces, which consist of a set equipped with two topologies, provided a richer structure for exploring interactions between different topological properties. Meanwhile, supra topology extends classical topology by allowing a more flexible definition of open sets, leading to a broader class of closure and interior operations. This study introduces a novel class of sets in the context of supra bitopological spaces, focusing on supra star generalized closed ($S_{\tau_{ij}}$-$^*g$-closed) sets. This study is driven by a desire to expand classical topological ideas within the framework of supra bitopological spaces. The research examines essential characteristics of these sets and their complementary structures, investigating their behavior in relation to closure and interior operations. Furthermore, it explores the supra star generalized closure and supra star generalized interior, thereby extending topological structure knowledge. These newly established sets and previously investigated topological structures demonstrate differences, which provide significant new perspectives on their unique properties.

Keywords: Supra bitopological space, supra closed set, supra open set, supra star generalized closure, supra star generalized interior

2010 Mathematics Subject Classification:

1 Introduction

Topological spaces are essential in several fields of mathematics, especially in topology, analysis, and theoretical computer science. Over time, several specialized topological spaces have been developed that encapsulate distinct structural features. The traditional notion of a topological space provides a basis for several developments within topology. A topological space comprises a set X

and a topology τ that delineates open sets according to certain axioms. These spaces have been thoroughly examined owing to their relevance in analysis and geometry [9]. The notion of bitopological spaces was first presented by [8], whereby a set X is endowed with two separate topologies, τ_1 and τ_2. Bitopological spaces provide a more intricate interplay among various topological structures and have been extensively examined in the contexts of separation axioms, convergence, and continuity [7]. New developments have improved the understanding of generalized closed sets in bitopological structures. For instance, [10] introduced fuzzy pairwise regular bitopological spaces, which extend classical bitopological frameworks to accommodate uncertainty and imprecise data. In addition, [13] investigated the role of Pythagorean fuzzy sets in bitopological spaces, providing insights into their applicability in decision-making problems. Further advancements include studies on Hausdorff supra fuzzy bitopological spaces [11] and supra fuzzy R_0 and R_1 bitopological spaces in a quasi-coincidence sense [12], which provide deeper insights into the structural complexity of bitopological frameworks.

The supra topological space, denoted as (X, S_τ), is a significant further development of topological concepts, initially introduced by [9]. Supra topological spaces are more general than regular topological spaces because they allow more open sets. This implies that researchers can study more closure and interior operations. Researchers built on this idea and came up with the idea of supra bitopological spaces, which are shown as $(X, S_{\tau_1}, S_{\tau_2})$. These spaces have two different supra topologies set up on the same underlying set X. [4][5] made significant contributions to this field by analyzing interior and closure operations in supra bitopological spaces. New set structures, such as semi-open and generalized closed sets, were studied in more depth. This expanded the current research on supra-bitopological spaces. Recent years have seen a significant expansion in the study of generalized closed sets. Some extensions have been set up, such as g-$\alpha\rho$-closed sets [2] and g^*p-closed sets [6]. These give us new ways to think about closure operations in various topological fields. The research emphasizes the practical significance of supra bitopological spaces in modeling uncertain and imperfect data.

Recent developments in soft set theory have expanded the investigation of closed sets. The emergence of soft generalized closed sets [3] and nano $\#$-regular generalized closed sets [15] illustrates the increasing focus on expanding closure characteristics throughout several mathematical structures. A notable relevant study in this field is the research studied by [16], which established supra $*g$-closed sets in supra topological spaces. Their research established a basis for understanding the behavior of these sets and their relationships with known generalized closed sets. The notion of supra $*g$-closed sets offered new ideas on closure characteristics in supra topological spaces and laid foundations for further expansions in supra bitopological space. Motivated by these developments, this study introduces and investigates the concept of supra star generalized closed sets, denoted as $S_{\tau_{ij}}$-$^*g$-closed sets, in the context of supra bitopological spaces. The study also investigates associated operations, including supra star generalized closure and supra star generalized interior.

2 Preliminary Notes

This section introduces key definitions and concepts essential for the study. The main results utilize the basic terms and notations for supra bitopological spaces, closures, interior operators, and generalized closed sets, which are defined in this section. To avoid redundancy of using notation within this context, $i, j \in \{1, 2\}$ and $i \neq j$ are being used. Throughout this paper, X denotes the supra bitopological spaces $(X, S_{\tau_1}, S_{\tau_2})$ for where no separation axioms are assumed.

Definition 2.0.1. [9] (X, S_τ) is said to be a *supra topological space* if it is satisfying these conditions:
(i) $X, \varnothing \in S_\tau$.
(ii) The union of any number of sets in S_τ belongs to S_τ.
Each element $A \in S_\tau$ is called S_τ*-open sets* in (X, S_τ) and the complement of S_τ-open is called S_τ*-closed*.

Definition 2.0.2. [9] The *supra closure* of a set A is denoted by S_τ-$cl(A)$ and is defined as S_τ-$cl(A) = \bigcap\{B : B$ is S_τ-closed and $A \subseteq B\}$.

Definition 2.0.3. [9] The *supra interior* of a set A is denoted by S_τ-$int(A)$ and is defined as S_τ-$int(A) = \bigcup\{B : B$ is S_τ-open and $B \subseteq A\}$.

Definition 2.0.4. [14] A subset A of a supra topological space (X, S_τ) is called a S_τ-*semi-open* if $A \subseteq S_\tau$-$cl(S_\tau$-$int(A))$.

Definition 2.0.5. [1] A subset A of a supra topological space (X, S_τ) is called S_τ-ω-*closed* if S_τ-$cl(A) \subseteq U$ whenever $A \subseteq U$ and U is S_τ-semi-open in (X, S_τ). The complement of S_τ-ω-closed is S_τ-ω-*open*.

Definition 2.0.6. [16] A subset A of supra topological space (X, S_τ) is called a *supra star g-closed set* (briefly S_τ-$^*g$-*closed*) if S_τ-$cl(A) \subseteq U$ whenever $A \subseteq U$ and U is S_τ-ω-open in (X, S_τ).

Definition 2.0.7. [4] If S_{τ_1} and S_{τ_2} are two supra topologies on a non-empty set X, then the triplet $(X, S_{\tau_1}, S_{\tau_2})$ is said to be a *supra bitopological space*. Each element of S_{τ_i} is called a *supra* τ_i-*open sets* (briefly S_{τ_i}-*open sets*) in X. Then the complement of τ_i-open sets are called *supra* τ_i-*closed sets* (briefly S_{τ_i}-*closed sets*), for $i \in \{1, 2\}$.

Definition 2.0.8. [4] The S_{τ_i}-*closure* of a the set A is denoted by S_{τ_i}-$cl(A)$ and is defined as S_{τ_i}-$cl(A) = \bigcap\{B : B$ is a S_{τ_i}-$closed$ and $A \subseteq B$ for $i \in \{1, 2\}\}$.

Definition 2.0.9. [4] The S_{τ_i}-*interior* of a the set A is denoted by S_{τ_i}-$int(A)$ and is defined as S_{τ_i}-$int(A) = \bigcup\{B : B$ is a S_{τ_i}-$open$ and $B \subseteq A$ for $i \in \{1, 2\}\}$.

Definition 2.0.10. [5] Let $(X, S_{\tau_1}, S_{\tau_2})$ be a supra bitopological space, $A \subseteq X$, A is said to be $S_{\tau_{ij}}$-*semi-open* (briefly $S_{\tau_{ij}}$-*s-open*), if $A \subseteq S_{\tau_j}$-$cl(S_{\tau_i}$-$int(A))$, where $i, j \in \{1, 2\}$ and $i \neq j$.

3 Main Results

This section presents key findings on supra star generalized closed sets in supra bitopological spaces. The results extend classical closure and interior concepts and provide insights into their properties and relationships with other topological constructs.

3.1 Supra*-g Closed Sets and Supra*-g Open Sets

This subsection introduces supra star generalized closed sets ($S_{\tau_{ij}}$-$^*g$-closed) and their complementary supra star generalized open sets ($S_{\tau_{ij}}$-$^*g$-open). Their properties and interactions with existing closed and open sets are analyzed to highlight their significance in supra bitopological spaces. Throughout this paper X denotes the supra bitopological spaces $(X, S_{\tau_1}, S_{\tau_2})$ where no separation axioms are assumed.

Definition 3.1.1. A subset A of a supra bitopological space X is said to be *supra* τ_{ij}-*star generalized closed* (briefly $S_{\tau_{ij}}$-$^*g$-*closed*) if S_{τ_i}-$cl(A) \subseteq U$ whenever $A \subseteq U$ and U is S_{τ_j}-ω-open, where $i, j \in \{1, 2\}$ and $i \neq j$. The complement of $S_{\tau_{ij}}$-$^*g$-closed set is $S_{\tau_{ij}}$-$^*g$-*open set*. The family of all $S_{\tau_{ij}}$-$^*g$-closed (resp. $S_{\tau_{ij}}$-$^*g$-open) $sets$ of X is denoted by $S_{\tau_{ij}}$-$^*g$-$C(X)$ (resp. $S_{\tau_{ij}}$-$^*g$-$O(X)$).

Definition 3.1.2. A subset A of a supra bitopological space X is supra τ_{ij}-star generalized open (briefly $S_{\tau_{ij}}$-$^*g$-open) if its complement is $S_{\tau_{ij}}$-$^*g$-closed set.

Example 3.1.3. *Let* $X = \{1, 2, 3, 4\}$*, consider the following two supra topological spaces on* X*,*

$S_{\tau_1} = \{X, \varnothing, \{1\}, \{4\}, \{1,4\}, \{2,3\}, \{1,2,3\}, \{1,3,4\}, \{2,3,4\}\}$; *and*
$S_{\tau_2} = \{X, \varnothing, \{1\}, \{3\}, \{1,3\}, \{2,4\}, \{1,2,3\}, \{1,2,4\}, \{2,3,4\}\}$.
Then,
S_{τ_1}*-closed* $= \{X, \varnothing, \{1\}, \{2\}, \{4\}, \{1,4\}, \{2,3\}, \{1,2,3\}, \{2,3,4\}\}$; *and*
S_{τ_2}*-closed* $= \{X, \varnothing, \{1\}, \{3\}, \{4\}, \{1,3\}, \{2,4\}, \{1,2,4\}, \{2,3,4\}\}$.
Now, S_{τ_2}*-*ω*-open* $= \{X, \varnothing, \{1\}, \{3\}, \{1,3\}, \{2,4\}, \{1,2,3\}, \{1,2,4\}, \{2,3,4\}\}$.
Suppose $A = \{1,4\}$*, where* $\{1,4\}$ *is a subset of* $\{1,2,4\}$*, and* X *in* U*, which is* $U = S_{\tau_2}$*-*ω*-open. This implies that* $S_{\tau_1}\text{-}cl(A) = S_{\tau_1}\text{-}cl(\{1,4\}) = \{1,4\}$*. So,* $\{1,4\} \subseteq \{1,2,4\}$ *and* X*. Hence,* $\{1,4\}$ *is a* $S_{\tau_{12}}\text{-}^*g$*-closed set. The family of all* $S_{\tau_{12}}\text{-}^*g$*-closed sets and its complements are,*
$S_{\tau_{12}}\text{-}^*g\text{-}C(X) = \{X, \varnothing, \{1\}, \{2\}, \{4\}, \{1,4\}, \{2,3\}, \{3,4\}, \{1,2,3\}, \{1,3,4\}, \{2,3,4\}\}$; *and*
$S_{\tau_{12}}\text{-}^*g\text{-}O(X) = \{X, \varnothing, \{1\}, \{2\}, \{4\}, \{1,2\}, \{1,4\}, \{2,3\}, \{1,2,3\}, \{1,3,4\}, \{2,3,4\}\}$.

Theorem 3.1.4. *A subset* A *of a supra bitopological space* X *is said to be supra* τ_{ij}*-star generalized open (briefly* $S_{\tau_{ij}}\text{-}^*g$*-open) if and only if* $F \subseteq S_{\tau_i}\text{-}int(A)$ *whenever* $F \subseteq A$ *and* F *is a* S_{τ_j}*-*ω*-closed.*

Proof. Suppose that A is $S_{\tau_{ij}}\text{-}^*g$-open. Let $F \subseteq A$ and F be S_{τ_j}-ω-closed. Then F^c is S_{τ_j}-ω-open and $A^c \subseteq F^c$. Since A is $S_{\tau_{ij}}\text{-}^*g$-open, A^c is $S_{\tau_{ij}}\text{-}^*g$-closed, by the Definition 3.1.1, $S_{\tau_i}\text{-}cl(A^c) \subseteq F^c$ and note that $S_{\tau_i}\text{-}cl(A^c) = (S_{\tau_i}\text{-}int(A))^c$. Hence, $F \subseteq S_{\tau_i}\text{-}int(A)$.

Conversely, suppose that $F \subseteq S_{\tau_i}\text{-}int(A)$ where F is a S_{τ_j}-ω-closed and $F \subseteq A$. Then $A^c \subseteq F^c$, where F^c is S_{τ_j}-ω-open. Since $F \subseteq S_{\tau_i}\text{-}int(A)$, it follows $(S_{\tau_i}\text{-}int(A))^c \subseteq F^c$ that is $S_{\tau_i}\text{-}cl(A^c) \subseteq F^c$ since $S_{\tau_i}\text{-}cl(A^c) = (S_{\tau_i}\text{-}int(A))^c$. Thus A^c is $S_{\tau_{ij}}\text{-}^*g$-closed and A is $S_{\tau_{ij}}\text{-}^*g$-open. □

Remark 3.1.5. *The union of two* $S_{\tau_{ij}}\text{-}^*g$*-closed (resp.* $S_{\tau_{ij}}\text{-}^*g$*-open) sets need not be* $S_{\tau_{ij}}\text{-}^*g$*-closed (resp.* $S_{\tau_{ij}}\text{-}^*g$*-open) as seen from the following example.*

Example 3.1.6. *Consider* $S_{\tau_{12}}\text{-}^*g\text{-}C(X) = \{X, \varnothing, \{1\}, \{2\}, \{4\}, \{1,4\}, \{2,3\}, \{3,4\}, \{1,2,3\}, \{1,3,4\}, \{2,3,4\}\}$*. Suppose* $A = \{2\}$ *and* $B = \{1,4\}$*. Now,* $A \cup B = \{2\} \cup \{1,4\} = \{1,2,4\}$*, implies* $\{1,2,4\} \notin S_{\tau_{ij}}\text{-}^*g$*-closed set.*

Consider $S_{\tau_{12}}\text{-}^*g\text{-}O(X) = \{X, \varnothing, \{1\}, \{2\}, \{4\}, \{1,2\}, \{1,4\}, \{2,3\}, \{1,2,3\}, \{1,3,4\}, \{2,3,4\}\}$*. Suppose* $A = \{4\}$ *and* $B = \{1,2\}$*. Now,* $A \cup B = \{4\} \cup \{1,2\} = \{1,2,4\}$*, implies that* $\{1,2,4\} \notin S_{\tau_{ij}}\text{-}^*g$*-open set.*

Remark 3.1.7. *The intersection of two* $S_{\tau_{ij}}\text{-}^*g$*-closed (resp.* $S_{\tau_{ij}}\text{-}^*g$*-open) sets need not be* $S_{\tau_{ij}}\text{-}^*g$*-closed (resp.* $S_{\tau_{ij}}\text{-}^*g$*-open) as seen from the following example.*

Example 3.1.8. *Consider* $S_{\tau_{12}}\text{-}^*g\text{-}C(X) = \{X, \varnothing, \{1\}, \{2\}, \{4\}, \{1,4\}, \{2,3\}, \{3,4\}, \{1,2,3\}, \{1,3,4\}, \{2,3,4\}\}$*. Suppose* $A = \{2,3\}$ *and* $B = \{3,4\}$*. Now,* $A \cap B = \{2,3\} \cap \{3,4\} = \{3\}$*. Hence,* $\{3\} \notin S_{\tau_{12}}\text{-}^*g$*-closed set.*

Consider $S_{\tau_{12}}\text{-}^*g\text{-}O(X) = \{X, \varnothing, \{1\}, \{2\}, \{4\}, \{1,2\}, \{1,4\}, \{2,3\}, \{1,2,3\}, \{1,3,4\}, \{2,3,4\}\}$*. Suppose* $A = \{1,2,3\}$ *and* $B = \{1,3,4\}$*. Now,* $A \cap B = \{1,2,3\} \cap \{1,3,4\} = \{3\}$*. Thus,* $\{3\} \notin S_{\tau_{12}}\text{-}^*g$*-open set.*

Remark 3.1.9. $S_{\tau_{12}}\text{-}^*g\text{-}C(X)$ *(resp.* $S_{\tau_{12}}\text{-}^*g\text{-}O(X)$*) is generally not equal to* $S_{\tau_{21}}\text{-}^*g\text{-}C(X)$ *(resp.* $S_{\tau_{12}}\text{-}^*g\text{-}O(X)$*) as can be seen from the following example.*

Example 3.1.10. *Consider,*
$S_{\tau_{12}}\text{-}^*g\text{-}C(X) = \{X, \varnothing, \{1\}, \{2\}, \{4\}, \{1,4\}, \{2,3\}, \{3,4\}, \{1,2,3\}, \{1,3,4\}, \{2,3,4\}\}$; *and*
$S_{\tau_{21}}\text{-}^*g\text{-}C(X) = \{X, \varnothing, \{1\}, \{3\}, \{4\}, \{1,3\}, \{2,4\}, \{1,2,4\}, \{2,3,4\}\}$.
Thus, $S_{\tau_{12}}\text{-}^*g\text{-}C(X) \neq S_{\tau_{21}}\text{-}^*g\text{-}C(X)$*. Also,*
$S_{\tau_{12}}\text{-}^*g\text{-}O(X) = \{X, \varnothing, \{1\}, \{2\}, \{4\}, \{1,2\}, \{1,4\}, \{2,3\}, \{1,2,3\}, \{1,3,4\}, \{2,3,4\}\}$; *and*
$S_{\tau_{21}}\text{-}^*g\text{-}O(X) = \{X, \varnothing, \{1\}, \{3\}, \{4\}, \{1,3\}, \{2,4\}, \{1,2,4\}, \{2,3,4\}\}$.
Hence, $S_{\tau_{12}}\text{-}^*g\text{-}O(X) \neq S_{\tau_{21}}\text{-}^*g\text{-}O(X)$.

Theorem 3.1.11. *Let A be a supra bitopological space X. If A is $S_{\tau_{ij}}$-$^*g$-closed, then S_{τ_i}-$cl(A) \setminus A$ contains no non empty S_{τ_j}-ω-closed set.*

Proof. Let A be $S_{\tau_{ij}}$-$^*g$-closed. Suppose F be S_{τ_j}-ω-closed set and S_{τ_i}-$cl(A) \setminus A$ contains F where $F \neq \varnothing$. That is $F \subseteq S_{\tau_i}$-$cl(A) \setminus A$, where $F \neq \varnothing$. Note that S_{τ_i}-$cl(A) \setminus A = S_{\tau_i}$-$cl(A) \cap A^c$. This implies $F \subseteq S_{\tau_i}$-$cl(A) \cap A^c$. Thus, $F \subseteq S_{\tau_i}$-$cl(A)$ and $F \subseteq A^c$. Since A is $S_{\tau_{ij}}$-$^*g$-closed, by Definition 3.1.1, S_{τ_i}-$cl(A) \subseteq F^c$, where F^c be S_{τ_j}-ω-open and $A \subseteq F^c$, then $F \subseteq (S_{\tau_i}$-$cl(A))^c$. Now, $F \subseteq S_{\tau_i}$-$cl(A) \cap (S_{\tau_i}$-$cl(A))^c = \varnothing$. It follows that $F = \varnothing$, which is a contradiction. Therefore, S_{τ_i}-$cl(A) \setminus A$ contains no non empty S_{τ_j}-ω-closed set. $\square$

Remark 3.1.12. *The converse of the above Theorem 3.1.11 is not true as seen from the following example.*

Example 3.1.13. *Consider $S_{\tau_{12}}$-$^*g$-$C(X) = \{X, \varnothing, \{1\}, \{2\}, \{4\}, \{1,4\}, \{2,3\}, \{3,4\}, \{1,2,3\}, \{1,3,4\}, \{2,3,4\}\}$. And, S_{τ_2}-ω-open $= \{X, \varnothing, \{1\}, \{3\}, \{1,3\}, \{2,4\}, \{1,2,3\}, \{1,2,4\}, \{2,3,4\}\}$. Suppose $A = \{1,2\}$, then S_{τ_1}-$cl(A) \setminus A = S_{\tau_i}$-$cl(\{1,2\}) \setminus \{1,2\} = \{1,2,3\} \setminus \{1,2\} = \{3\}$ which does not contain any non empty S_{τ_2}-ω-closed set. Hence, $A \notin S_{\tau_{12}}$-$^*g$-closed.*

Theorem 3.1.14. *If subset A is S_{τ_j}-ω-closed set, then S_{τ_i}-$cl(A) \setminus A$ is S_{τ_j}-ω-open.*

Proof. Let A be S_{τ_j}-ω-closed set. Let $F \subseteq S_{\tau_i}$-$cl(A) \setminus A$ where F is S_{τ_j}-ω-closed set. Since A is S_{τ_j}-ω-closed, we have S_{τ_i}-$cl(A) \setminus A$ does not contain non empty S_{τ_j}-ω-closed, by Theorem 3.1.11. Consequently, $F = \varnothing$. Therefore, $F = \varnothing \subseteq S_{\tau_i}$-$cl(A) \setminus A$, and so $F = \varnothing \subseteq S_{\tau_i}$-$int(S_{\tau_i}$-$cl(A) \setminus A)$. Hence, S_{τ_i}-$cl(A) \setminus A$ is S_{τ_j}-ω-open. $\square$

Theorem 3.1.15. *If A is $S_{\tau_{ij}}$-$^*g$-closed set of X such that $A \subseteq B \subseteq S_{\tau_i}$-$cl(A)$, then B is a $S_{\tau_{ij}}$-$^*g$-closed set of X.*

Proof. Suppose that A is $S_{\tau_{ij}}$-$^*g$-closed set and $A \subseteq B \subseteq S_{\tau_i}$-$cl(A)$. Let $B \subseteq U$ and U is S_{τ_j}-ω-open. Given $A \subseteq B$, the $A \subseteq U$. Since A is $S_{\tau_{ij}}$-$^*g$-closed, we have S_{τ_i}-$cl(A) \subseteq U$. Since $B \subseteq S_{\tau_i}$-$cl(A)$, then S_{τ_i}-$cl(B) \subseteq S_{\tau_i}$-$cl(S_{\tau_i}$-$cl(A)) = S_{\tau_i}$-$cl(A) \subseteq U$, implies S_{τ_i}-$cl(B) \subseteq U$. Therefore, B is $S_{\tau_{ij}}$-$^*g$-closed set. $\square$

Theorem 3.1.16. *Let A and B be subset of supra bitopological space X such that S_{τ_i}-$int(A) \subseteq B \subseteq A$. If A is a $S_{\tau_{ij}}$-$^*g$-open set, then B is $S_{\tau_{ij}}$-$^*g$-open set.*

Proof. Let A be $S_{\tau_{ij}}$-$^*g$-open. Let U be a S_{τ_j}-ω-closed such that $U \subseteq B$. Since $U \subseteq B$ and $B \subseteq A$, we have $U \subseteq A$. by assumption, $U \subseteq S_{\tau_i}$-$int(A)$. Since S_{τ_i}-$int(A) \subseteq B$, we have S_{τ_i}-$int(S_{\tau_i}$-$int(A)) \subseteq S_{\tau_i}$-$int(B)$. Therefore, S_{τ_i}-$int(A) \subseteq S_{\tau_i}$-$int(B)$. Consequently, $U \subseteq S_{\tau_i}$-$int(B)$. Hence, B is $S_{\tau_{ij}}$-$^*g$-open. $\square$

3.2 Supra*-g Closure

The supra star generalized closure operator ($S_{\tau_{ij}}$-$^*g$-$cl(A)$) defined the smallest supra star generalized closed set containing a given set. This subsection explores its fundamental properties and relationships with existing closure operations. Throughout this paper X denotes the supra bitopological spaces $(X, S_{\tau_1}, S_{\tau_2})$ where no separation axioms are assumed.

Definition 3.2.1. Let X be a supra bitopological space and $A \subseteq X$. An element $x \in X$ is called $S_{\tau_{ij}}$-$^*g$-$adherent$ to A if $V \cap A \neq \varnothing$ for every $S_{\tau_{ij}}$-$^*g$-open set V containing x. The set of all $S_{\tau_{ij}}$-$^*g$-$adherent\ points\ of\ A$ is called the $S_{\tau_{ij}}$-$^*g$-$closure\ of\ A$ denoted by $S_{\tau_{ij}}$-$^*g$-$cl(A)$.

Example 3.2.2. *Consider the Example 3.1.3 as supra bitopological space on X, such that $S_{\tau_{12}}$-$^*g$-$C(X) = X, \varnothing, \{1\}, \{2\}, \{4\}, \{1,4\}, \{2,3\}, \{3,4\}, \{1,2,3\}, \{1,3,4\}$, and $\{2,3,4\}$. Suppose $A = \{1\}$ where $\{1\}$ is a subset of $\{1\}, \{1,4\}, \{1,2,3\}, \{1,3,4\}$, and X. Thus, $\{1\}, \{1,4\}, \{1,2,3\}, \{1,3,4\}, X \in S_{\tau_{12}}$-$^*g$-$closed\ sets$ containing x. Then $S_{\tau_{12}}$-$^*g$-$cl(A) = S_{\tau_{12}}$-$^*g$-$cl(\{1\}) = \{1\} \cap \{1,4\} \cap \{1,2,3\} \cap \{1,3,4\} \cap X = \{1\}$. Hence, $\{1\}$ is a $S_{\tau_{12}}$-$^*g$-closure of A.*

Theorem 3.2.3. *Let X be a supra bitopological space and $A \subseteq X$, then $S_{\tau_{ij}}$-$^*g$-$cl(A) = \bigcap\{F : F$ is $S_{\tau_{ij}}$-$^*g$-$closed$ and $A \subseteq F\}$.*

Proof. Let A be a subset of a supra bitopological space. Suppose $x \in S_{\tau_{ij}}$-$^*g$-$cl(A)$. Then $V \cap A \neq \varnothing$ for every $S_{\tau_{ij}}$-$^*g$-open set V containing x. Suppose $x \notin \bigcap\{F : F$ is $S_{\tau_{ij}}$-$^*g$-closed and $A \subseteq F\}$. Then $x \notin F$ for some $S_{\tau_{ij}}$-$^*g$-closed set F and so $x \in F^c$ for some $S_{\tau_{ij}}$-$^*g$-open F^c. Thus $F^c \cap A = \varnothing$ for some $S_{\tau_{ij}}$-$^*g$-open F^c containing x, which is a contradiction. Hence, $x \in \bigcap\{F : F$ is $S_{\tau_{ij}}$-$^*g$-closed and $A \subseteq F\}$. Consequently, $S_{\tau_{ij}}$-$^*g$-$cl(A) \subseteq \bigcap\{F : F$ is $S_{\tau_{ij}}$-$^*g$-closed and $A \subseteq F\}$. Next, ley $y \in \bigcap\{F : F$ is $S_{\tau_{ij}}$-$^*g$-closed and $A \subseteq F\}$. Then $y \in F$ for all $S_{\tau_{ij}}$-$^*g$-closed set such that $A \subseteq F$. Suppose $y \notin S_{\tau_{ij}}$-$^*g$-$cl(A)$. Then $V \cap A = \varnothing$ for some $S_{\tau_{ij}}$-$^*g$-closed set V containing y. Hence V^c is $S_{\tau_{ij}}$-$^*g$-closed set such that $A \subseteq V^c$ and $y \notin V^c$, a contradiction. Thus $y \in S_{\tau_{ij}}$-$^*g$-$cl(A)$. □

Lemma 3.2.4. *Let X be a supra bitopological space. The following properties hold:*

(i) *$A \subseteq S_{\tau_{ij}}$-$^*g$-$cl(A)$; and*

(ii) *If $A \subseteq B$, then $S_{\tau_{ij}}$-$^*g$-$cl(A) \subseteq S_{\tau_{ij}}$-$^*g$-$cl(B)$*

Proof. Let $A, B \subseteq X$.

(i) Let $x \in A$ and suppose $x \notin S_{\tau_{ij}}$-$^*g$-$cl(A)$. Then there exists a $S_{\tau_{ij}}$-$^*g$-open set V containing x such that $V \cap A = \varnothing$,this is a contradiction since $x \in A$. Thus, $x \in S_{\tau_{ij}}$-$^*g$-$cl(A)$.

(ii) Let $x \in S_{\tau_{ij}}$-$^*g$-$cl(A)$. Thus, for all $S_{\tau_{ij}}$-$^*g$-open set V containing x, $V \cap A \neq \varnothing$. Since $A \subseteq B$, $\varnothing \neq V \cap A \subseteq V \cap B$, and so $V \cap B \neq \varnothing$ for every V such that $x \in V$. Therefore x is $S_{\tau_{ij}}$-$^*g$-$cl(B)$. □

Theorem 3.2.5. *Let A and B be subsets of a supra bitopological space X. Then the following properties hold:*

(i) *$S_{\tau_{ij}}$-$^*g$-$cl(A) \cup S_{\tau_{ij}}$-$^*g$-$cl(B) \subseteq S_{\tau_{ij}}$-$^*g$-$cl(A \cup B)$; and*

(ii) *$S_{\tau_{ij}}$-$^*g$-$cl(A \cap B) \subseteq S_{\tau_{ij}}$-$^*g$-$cl(A) \cap S_{\tau_{ij}}$-$^*g$-$cl(B)$*

Proof. Let $A, B \subseteq X$

(i) Note that $A \subseteq A \cup B$ and $B \subseteq A \cup B$. By Lemma 3.2.4 (ii), it follows that $S_{\tau_{ij}}$-$^*g$-$cl(A) \subseteq S_{\tau_{ij}}$-$^*g$-$cl(A \cup B)$ and $S_{\tau_{ij}}$-$^*g$-$cl(B) \subseteq S_{\tau_{ij}}$-$^*g$-$cl(A \cup B)$. Thus, $S_{\tau_{ij}}$-$^*g$-$cl(A) \cup S_{\tau_{ij}}$-$^*g$-$cl(B) \subseteq S_{\tau_{ij}}$-$^*g$-$cl(A \cup B)$.

(ii) Note that $A \cap B \subseteq A$ and $A \cap B \subseteq B$. Then $S_{\tau_{ij}}$-$^*g$-$cl(A \cap B) \subseteq S_{\tau_{ij}}$-$^*g$-$cl(A)$ and $S_{\tau_{ij}}$-$^*g$-$cl(A \cap B) \subseteq S_{\tau_{ij}}$-$^*g$-$cl(B)$, by Lemma 3.2.4 (ii). Thus, $S_{\tau_{ij}}$-$^*g$-$cl(A \cap B) \subseteq S_{\tau_{ij}}$-$^*g$-$cl(A) \cap S_{\tau_{ij}}$-$^*g$-$cl(B)$. □

Theorem 3.2.6. *Let X be a supra bitopological space. If A is $S_{\tau_{ij}}$-$^*g$-closed, then $S_{\tau_{ij}}$-$^*g$-$cl(A) = A$.*

Proof. Let A be $S_{\tau_{ij}}$-$^*g$-closed. Note that by Lemma 3.2.4 (i), $A \subseteq S_{\tau_{ij}}$-$^*g$-$cl(A)$. It suffices to show that $S_{\tau_{ij}}$-$^*g$-$cl(A) \subseteq A$. Suppose $x \in S_{\tau_{ij}}$-$^*g$-$cl(A)$. Then for all $S_{\tau_{ij}}$-$^*g$-open set V containing x, $V \cap A \neq \varnothing$. Suppose on the contrary $x \notin A$. Then $x \in A^c$ where A^c is $S_{\tau_{ij}}$-$^*g$-open set and $A^c \cap A = \varnothing$. This is a contradiction since $x \in S_{\tau_{ij}}$-$^*g$-$cl(A)$. Thus, $x \in A$, consequently $S_{\tau_{ij}}$-$^*g$-$cl(A) \subseteq A$. Therefore, $S_{\tau_{ij}}$-$^*g$-$cl(A) = A$. □

Proposition 3.2.7. *X and $\varnothing$ are both $S_{\tau_{ij}}$-$^*g$-closed sets.*

Proof. Let U be S_{τ_j}-ω-open such that $X \subseteq U$. Note that X is the only S_{τ_j}-ω-open containing X. Now, $S_{\tau_{ij}}$-$^*g$-$cl(X) \subseteq X$, and so X is $S_{\tau_{ij}}$-$^*g$-closed.

Additionally, let U be S_{τ_j}-ω-open such that $\varnothing \subseteq U$. Note that $\varnothing$ is the subset of all S_{τ_j}-ω-open $sets$. Now, $S_{\tau_{ij}}$-$^*g$-$cl(\varnothing) = \varnothing \subseteq U$. Therefore, $\varnothing$ is also a $S_{\tau_{ij}}$-$^*g$-closed. □

Theorem 3.2.6 and Proposition 3.2.7, support the claim of the remrk below.

Remark 3.2.8. *If a subset A of supra bitopological space, then*
(i) $S_{\tau_{ij}}$-$^*g$-$cl(X) = X$*; and*
(ii) $S_{\tau_{ij}}$-$^*g$-$cl(\varnothing) = \varnothing$.

3.3 Supra*-g Interior

The supra star generalized interior ($S_{\tau_{ij}}$-$^*g$-$int(A)$) determines the largest supra star generalized open subset within a given set. This subsection discusses its properties and interactions with the closure operator in supra bitopological spaces. Throughout this paper X denotes the supra bitopological spaces $(X, S_{\tau_1}, S_{\tau_2})$ where no separation axioms are assumed.

Definition 3.3.1. Let X be a supra bitopological space and $A \subseteq X$. An element $x \in A$ is called $S_{\tau_{ij}}$-$^*g$-$interior\ point\ of\ A$ if there exists an $S_{\tau_{ij}}$-$^*g$-open set G such that $x \in G \subseteq A$. The set of all $S_{\tau_{ij}}$-$^*g$-$interior\ points\ of\ A$ is called the $S_{\tau_{ij}}$-$^*g$-$interior\ point\ of\ A$ denoted by $S_{\tau_{ij}}$-$^*g$-$int(A)$.

Example 3.3.2. *Consider the Example 3.1.3 as supra bitopological space on X, thus $S_{\tau_{12}}$-$^*g$-$O(X) = X, \varnothing, \{1\}, \{2\}, \{4\}, \{1,2\}, \{1,4\}, \{2,3\}, \{1,2,3\}, \{1,3,4\}$, and $\{2,3,4\}$. Suppose $A = \{1,2,3\}$ where $\{1,2,3\}, \{2,3\}, \{1,2\}, \{2\}, \{1\}$, and $\varnothing$ are the subset of $\{1,2,3\}$. Thus, $\{1\}, \{2\}, \{1,2\}, \{2,3\}, \{1,2,3\}$, $\varnothing \in S_{\tau_{12}}$-$^*g$-$open\ set$. Then $S_{\tau_{12}}$-$^*g$-$int(A) = S_{\tau_{12}}$-$^*g$-$int(\{1,2,3\}) = \{1,2,3\} \cup \{2,3\} \cup \{1,2\} \cup \{2\} \cup \{1\} \cup \varnothing = \{1,2,3\}$. Hence, $\{1,2,3\}$ is a $S_{\tau_{12}}$-$^*g$-interior point of A.*

Theorem 3.3.3. *Let X be a supra bitopological space and $A \subseteq X$. Then $S_{\tau_{ij}}$-$^*g$-$int(A) = \bigcup\{G : G$ is $S_{\tau_{ij}}$-$^*g$-open and $G \subseteq A\}$.*

Proof. Let $A \subseteq X$. Suppose $x \in S_{\tau_{ij}}$-$^*g$-$int(A)$. Then x is a $S_{\tau_{ij}}$-$^*g$-$interior\ of\ A$. Then there exists a $S_{\tau_{ij}}$-$^*g$-open set G such that $x \in G \subseteq A$. Hence, $x \in \bigcup\{G : G$ is $S_{\tau_{ij}}$-$^*g$-open and $G \subseteq A\}$. Thus, $S_{\tau_{ij}}$-$^*g$-$int(A) = \bigcup\{G : G$ is $S_{\tau_{ij}}$-$^*g$-open and $G \subseteq A\}$. Suppose, $y \in \bigcup\{G : G$ is $S_{\tau_{ij}}$-$^*g$-open and $G \subseteq A\}$. Then there exists G which is $S_{\tau_{ij}}$-$^*g$-open such that $y \in G \subseteq A$. Hence, by Definition 3.3.1, $y \in S_{\tau_{ij}}$-$^*g$-$int(A)$. □

Lemma 3.3.4. *Let A and B be subsets of a supra bitopological space X. The following properties hold:*
(i) $S_{\tau_{ij}}$-$^*g$-$int(A) \subseteq A$*; and*
(ii) *If $A \subseteq B$, then $S_{\tau_{ij}}$-$^*g$-$int(A) \subseteq S_{\tau_{ij}}$-$^*g$-$int(B)$.*

Proof. Let $A, B \subseteq X$. Then,
(i) Let $x \in S_{\tau_{ij}}$-$^*g$-$int(A)$. Then by Theorem 3.3.3, $x \in \bigcup\{G : G$ is $S_{\tau_{ij}}$-$^*g$-open and $G \subseteq A\}$. It means that $x \in G$ for some $S_{\tau_{ij}}$-$^*g$-open set G such that $G \subseteq A$ and so $x \in A$. Hence, $S_{\tau_{ij}}$-$^*g$-$int(A) \subseteq A$.
(ii) Let $x \in S_{\tau_{ij}}$-$^*g$-$int(A)$. Thus, for some $S_{\tau_{ij}}$-$^*g$-open set G containing x, $G \subseteq A$. Since $G \subseteq A \subseteq B$, then $G \subseteq B$. So $x \in G \subseteq B$, where G is $S_{\tau_{ij}}$-$^*g$-open, it follows that $x \in S_{\tau_{ij}}$-$^*g$-$int(B)$. Therefore, $S_{\tau_{ij}}$-$^*g$-$int(A) \subseteq S_{\tau_{ij}}$-$^*g$-$int(B)$. □

Theorem 3.3.5. *Let A and B be subsets of a supra bitopological space X. Then the following properties hold:*
(i) $S_{\tau_{ij}}$-$^*g$-$int(A) \cup S_{\tau_{ij}}$-$^*g$-$int(B) \subseteq S_{\tau_{ij}}$-$^*g$-$int(A \cup B)$*; and*

(ii) $S_{\tau_{ij}}$-$^*g$-$int(A \cap B) \subseteq S_{\tau_{ij}}$-$^*g$-$int(A) \cap S_{\tau_{ij}}$-$^*g$-$int(B)$.

Proof. Let $A, B \subseteq X$.

(i) Note that $A \subseteq A \cup B$ and $B \subseteq A \cup B$. By Lemma 3.3.4 (ii), it follows that $S_{\tau_{ij}}$-$^*g$-$int(A) \subseteq S_{\tau_{ij}}$-$^*g$-$int(A \cup B)$ and $S_{\tau_{ij}}$-$^*g$-$int(B) \subseteq S_{\tau_{ij}}$-$^*g$-$int(A \cup B)$. Thus, $S_{\tau_{ij}}$-$^*g$-$int(A) \cup S_{\tau_{ij}}$-$^*g$-$int(B) \subseteq S_{\tau_{ij}}$-$^*g$-$int(A \cup B)$.

(ii) Note that $A \cap B \subseteq A$ and $A \cap B \subseteq B$. By Lemma 3.3.4 (ii), it follows that $S_{\tau_{ij}}$-$^*g$-$int(A \cap B) \subseteq S_{\tau_{ij}}$-$^*g$-$int(A)$ and $S_{\tau_{ij}}$-$^*g$-$int(A \cap B) \subseteq S_{\tau_{ij}}$-$^*g$-$int(B)$. Thus, $S_{\tau_{ij}}$-$^*g$-$int(A \cap B) \subseteq S_{\tau_{ij}}$-$^*g$-$int(A) \cap S_{\tau_{ij}}$-$^*g$-$int(B)$. ☐

Theorem 3.3.6. *Let X be a supra bitopological space and $A \subseteq X$. If A is $S_{\tau_{ij}}$-$^*g$-open, then $A = S_{\tau_{ij}}$-$^*g$-$int(A)$.*

Proof. Let A be a $S_{\tau_{ij}}$-$^*g$-open set. Then,$A \in \bigcup\{G : G$ is $S_{\tau_{ij}}$-$^*g$-open and $G \subseteq A\}$. Since every member in the collection is $S_{\tau_{ij}}$-$^*g$-open and A is in collection, it follows that the union of this collection is A. Thus, by Lemma 3.3.4, $A = S_{\tau_{ij}}$-$^*g$-$int(A)$ ☐

Proposition 3.2.7 supports the claim of remrk 3.3.7 (i) as complement of $S_{\tau_{ij}}$-$^*g$-closed set. By Theorem 3.3.6, the remrk 3.3.7 (ii) and (iii) hold.

Remark 3.3.7. *If a subset A be a supra bitopological space, then,*

(i) *X and $\varnothing$ are both $S_{\tau_{ij}}$-$^*g$-open set s;*

(ii) *$S_{\tau_{ij}}$-$^*g$-$int(X) = X$; and*

(iii) *$S_{\tau_{ij}}$-$^*g$-$int(\varnothing) = \varnothing$.*

4 CONCLUSIONS

This paper introduced the concept of supra star generalized closed sets ($S_{\tau_{ij}}$-$^*g$-closed) in supra bitopological spaces, investigating its characteristics and distinctions with existing topological structures. The study demonstrates that the union and intersection of these sets do not inherently preserve closure characteristics, contrasting them from previous ideas of closed sets. Furthermore, the supra star generalized closure and interior operations were examined, extending the understanding of closure and interior features in topological spaces. The further avenues for investigation are as follows:

1. Extending these concepts to fuzzy supra bitopological spaces.
2. Investigating separation axioms and continuity properties concerning $S_{\tau_{ij}}$-$^*g$-closed sets.
3. Exploring real-world applications in mathematical modeling and computational topology.
4. Studying the role of supra star generalized closed sets in decision-making models and their applications in various fields such as network topology and data analysis.

Motivation and Novelty of the Study

This study is motivated by the need to enhance and expand current closure operations in supra bitopological areas. Although previous research has examined many types of closed sets, including supra g-closed and supra semi-closed sets [16], a challenge remains in determining the connection of closure characteristics under two supra topologies.

This study introduces supra star generalized closed ($S_{\tau_{ij}}$-$^*g$-closed) sets, establishing a more extended structure for investigating closure properties. Unlike previous definitions, $S_{\tau_{ij}}$-$^*g$-closed sets refine closure operations in supra bitopological spaces, providing a novel perspective on set interactions. The distinctiveness of this study is in its ability to integrate and enhance existing concepts, hence extending the understanding of closure and interior operations in bitopological spaces. This contribution supplements the theoretical structure of supra bitopological spaces and promotes future advancements in topological research.

Acknowledgment

The authors would like to thank everyone who has been with us in this research study. The support you have given is well appreciated.

Competing Interests

The authors declare that they have no competing interests.

References

[1] Chandramathi, N., Bhuvaneswari, K., and Bharathi, S. (2012). On ω^{μ}-closed sets and continuous functions in topological spaces. *International Journal of Mathematical Archive*, 3(2):712–721.

[2] Devi, R. and Sivaraj, M. (2019). On generalized $\alpha\rho$-closed sets in topological spaces. *AIP Conference Proceedings*, 2177(1).

[3] Gomathi, N. and Indira, T. (2021). On soft generalized closed sets in soft topological spaces. *Advances in Mathematics: Scientific Journal*, 10(8):5575–5595.

[4] Gowri, R. and Rajayal, A. (2017a). On supra bitopological spaces. *IOSR Journal of Mathematics*, 13(5):55–58.

[5] Gowri, R. and Rajayal, A. (2017b). Supra α-open sets and supra α-separation axioms in supra bitopological spaces. *International Journal of Mathematics Trends and Technology*, 52.

[6] Jaya Kumar, P. (2016). New notion of generalized closed sets in topological spaces. *International Journal of Mathematics Trends and Technology*, 36(2):123–130.

[7] Jayasudha, B. and Bhuvaneswari, K. (2019). A study on bitopological spaces and their applications. *Journal of Mathematics and Computing*, 10(4):251–267.

[8] Kelly, J. (1963). Bitopological spaces. *Proceedings of the London Mathematical Society*, 3(13):71–89.

[9] Mashhour, A., Allam, A., Mahmoud, F., and Khedr, F. (1983). On supra topological spaces. *Indian Journal of Pure and Applied Mathematics*, 14(4):502–510.

[10] Miah, S., Amin, R., and Hossain, S. (2020). Fuzzy pairwise regular bitopological spaces in quasi-coincidence sense. *Journal of Bangladesh Academy of Sciences*, 44(2):139–143.

[11] Miah, S. and Sharin, S. (2022). Study on hausdorff supra fuzzy bitopological space - approach of quasi-coincidence. *International Journal of Mathematics and Computer Science*, 3:87–93.

[12] Miah, S., Sharin, S., Shahjalal, M., and Sultana, N. (2024). An in-depth exploration on supra fuzzy R_0 and R_1 bitopological space in quasi-coincidence sense. *International Journal of Mathematics and Computer Science*, 2:90–99.

[13] Pal, M., Mukherjee, A., and Samanta, S. K. (2021). Bitopological spaces in the context of pythagorean fuzzy set. *Afrika Matematika*, 32:1487–1502.

[14] Ravi, O., Ramkumar, G., and Kamaraj, M. (2011). On supra g-closed sets. *International Journal of Advances in Pure and Applied Mathematics*, 1(2):52–66.

[15] Saleh, S., Birkea, F., Al-shami, T., Arar, M., and Omran, M. (2025).

[16] Selvaraj, R. and Chandrasekar, S. (2018). supra-$^*g$-closed sets in supra topological spaces. *International Journal of Mathematics Trends and Technology*, 56(1):15–20.
