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Complete monotonicity of functions defined by λ generalized psi function and its derivatives

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Abstract

In this work, we firstly give integral representations of λ-psi (or λ-digamma) and λ-zeta functions and then obtain λ-generalization of Binet's first formula for the logarithms of λ-gamma function $\ln\Gamma_\lambda(x)$ as

$$\ln\Gamma_\lambda(x) = \left(x-\frac{1}{2}\right)\ln x - x\ln\lambda + \frac{1}{2}\ln(2\pi) + \int_0^\infty \left[\frac{1}{2}-\frac{1}{t}+\frac{1}{e^t-1}\right]\frac{e^{-tx}}{t}dt$$

for all positive real values of x and λ. As immediate consequences, we get some completely monotonicity properties on functions related to λ-psi function and its derivatives defined by $f_1(x) = \psi_\lambda(x)+\ln\lambda-\ln x+\frac{1}{2x}+\frac{1}{12x^2}$, $f_2(x) = \ln x - \frac{1}{2x} - \ln\lambda - \psi_\lambda(x)$, $f_3(x) = \psi'_\lambda(x) - \frac{1}{x} - \frac{1}{2x^2} - \frac{1}{6x^3} + \frac{1}{30x^5}$, $f_4(x) = \frac{1}{x}+\frac{1}{2x^2}+\frac{1}{6x^3} - \psi'_\lambda(x)$ for all $x, \lambda > 0$. At last, we obtain some mean inequalities on λ-psi function.

Keywords: λ-polygamma function, λ-psi function; inequality; Binet's first formula for $\ln\Gamma_\lambda(x)$

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1 Introduction

The gamma function, which is introduced by Euler, is defined by

$$\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}dt$$

for $x > 0$. The logarithmic derivative of gamma function is called digamma (or psi) function and its integral representation is given by

$$\psi(x) = \int_0^\infty \left(\frac{e^{-t}}{t} - \frac{e^{-tx}}{1-e^t}\right)dt \tag{1.1}$$

for $x > 0$.
Special functions are used in many mathematical subjects such as fractional calculus, inequalities etc. (see Ata and Kıymaz (2022, 2025, 2024a); Ata et al. (2024b); Ata (2023, 2024)). Generalizations of these special functions are also interested by many researchers (see Alzer and Jameson (2017); Batır (2011); Chen and Batır (2012); Gautschi (1974); Alzer and Jameson (2017); Qi et al. (2005); Qi (2010); Qi and Guo (2010); Guo and Qi (2013); Qi (2013); Guo and Luo (2015); Guo et al. (2015); Qi and Guo (2017); Batır (2018); Ata and Kıymaz (2020); Ata (2022, 2018); Ata and Kıymaz (2024b); Ata et al. (2024a)). For instance; in order to define tempered fractional integrals, authors give λ-generalized incomplete gamma function as follows:

Definition 1.1. (Fu and Du, 2021; Mohammed and Baleanu, 2020) For the real number $x > 0$ and $a, \lambda \geq 0$, λ-incomplete gamma function can be defined by

$$\Gamma_\lambda(x, a) = \int_a^\infty t^{x-1} e^{-\lambda t} dt.$$

In 2022, Nantomah and Ege define the λ-analogue of the gamma function

Definition 1.2. (Nantomah and Ege, 2022) λ-gamma function can be defined by

$$\Gamma_\lambda(x) = \int_0^\infty t^{x-1} e^{-\lambda t} dt \tag{1.2}$$

$$= \lim_{k\to\infty} \frac{\lambda^{-x} k! k^x}{x(x+1)(x+2)\ldots(x+k)} \tag{1.3}$$

for $x > 0$ and $\lambda > 0$.

In the same paper, authors also give some properties on the λ-gamma function.

Lemma 1.1. *(Nantomah and Ege, 2022)*

$$\Gamma_\lambda(x) = \lambda^{-x}\Gamma(x), \tag{1.4}$$

$$\Gamma_\lambda(1) = \frac{1}{\lambda}, \tag{1.5}$$

$$\Gamma_\lambda(x+1) = \frac{x}{\lambda}\Gamma_\lambda(x), \quad x > 0, \tag{1.6}$$

$$\Gamma_\lambda(k+1) = \frac{k!}{\lambda^{k+1}}, \quad k \in \mathbb{N}_0, \tag{1.7}$$

$$\Gamma_\lambda(x)\Gamma_\lambda(1-x) = \frac{\pi}{\lambda \sin(\pi x)}, \quad x \in (0,1), \tag{1.8}$$

$$\Gamma_\lambda(1+x)\Gamma_\lambda(1-x) = \frac{\pi x}{\lambda^2 \sin(\pi x)}, \quad x \in (0,1), \tag{1.9}$$

$$\Gamma_\lambda(x)\Gamma_\lambda\left(x+\frac{1}{2}\right) = 2^{1-2x}\sqrt{\pi}\frac{\Gamma_\lambda(2x)}{\lambda}, \quad x > 0, \tag{1.10}$$

$$\frac{\Gamma_\lambda(x+k)}{\Gamma_\lambda(x)} = \frac{(x)_k}{\lambda^k}, \quad x > 0, \ k \in \mathbb{N}_0, \tag{1.11}$$

$$\Gamma_\lambda\left(k+\frac{1}{2}\right) = \frac{(2k-1)!!}{2^k\lambda^k}\sqrt{\frac{\pi}{\lambda}}, \quad k \in \mathbb{N}_0. \tag{1.12}$$

where $(x)_k = x(x+1)(x+2)\ldots(x+k-1)$ *is Pochhammer symbol and* $m!!$ *is double factorial of* m.

In an usual sense, authors define λ-analogues of beta and psi functions as follows:

Definition 1.3. (Nantomah and Ege, 2022) λ-beta function can be given by

$$\beta_\lambda(x,y) = \frac{\Gamma_\lambda(x)\Gamma_\lambda(y)}{\Gamma_\lambda(x+y)},$$

for $x > 0$ and $y > 0$. The function collide with classical beta function $\beta(x, y)$ since

$$\beta_\lambda(x,y) = \frac{\Gamma_\lambda(x)\Gamma_\lambda(y)}{\Gamma_\lambda(x+y)} = \frac{\lambda^{-x}\Gamma(x)\lambda^{-y}\Gamma(y)}{\lambda^{-(x+y)}\Gamma(x+y)} = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} = \beta(x,y).$$

Definition 1.4. (Nantomah and Ege, 2022) λ-digamma (or λ-psi) function can be given by

$$\psi_\lambda(x) = \frac{d}{dx}\ln\Gamma_\lambda(x) \tag{1.13}$$

for $x > 0$. Some of the integral representations are

$$\begin{aligned}
\psi_\lambda(x) &= \frac{\Gamma'_\lambda(x)}{\Gamma_\lambda(x)} = -\ln\lambda + \psi(x) \\
&= -(\ln\lambda + \gamma) + \int_0^1 \frac{1-t^{x-1}}{1-t}dt \\
&= -(\ln\lambda + \gamma) + \int_0^\infty \frac{e^{-t}-e^{-xt}}{1-e^{-t}}dt \\
&= -(\ln\lambda + \gamma) + \sum_{k=0}^{\infty} \frac{x-1}{(k+1)(k+x)}
\end{aligned}$$

where $\gamma = \lim_{n\to\infty}\left(\sum_{r=1}^{n}\frac{1}{r} - \ln n\right) = 0.5772...$ is Euler-Mascheroni constant and $\psi(x) = \frac{d}{dx}\ln\Gamma(x)$ classical digamma (psi) function.

By using the equation (1.6), authors obtain the recurrence formula for λ-digamma function as

$$\psi_\lambda(x+1) = \frac{1}{x} + \psi_\lambda(x). \tag{1.14}$$

In the paper, authors obtain some well-known theorems, formulas, limit properties and inequalities on these functions, such as in the next theorem, they give arithmetic and geometric mean inequalities on λ-gamma function for x and $1/x$.

Theorem 1.2. *For $x > 0$,the inequalities*

$$\Gamma_\lambda(x)\Gamma_\lambda\left(\frac{1}{x}\right) \geq \lambda^{-\left(x+\frac{1}{x}\right)}, \tag{1.15}$$

$$\Gamma_\lambda(x) + \Gamma_\lambda\left(\frac{1}{x}\right) \geq 2\lambda^{-\frac{1}{2}\left(x+\frac{1}{x}\right)} \tag{1.16}$$

are satisfied. With equality when $x = 1$.

The interested readers can find more information about properties, inequalities and generalizations of special functions in (Gautschi, 1974; Qi, 2010; Whittaker and Watson, 1996; Qi and Guo, 2010; Batır, 2011; Chen and Batır, 2012; Guo and Qi, 2013; Qi, 2013; Guo and Luo, 2015; Guo et al., 2015; Qi and Guo, 2017; Batır, 2018; Qi et al., 2005; Alzer and Jameson, 2017; Kim et al., 2018; Dıaz and Pariguan, 2007) and references therein.

Motivated by previous works, we introduce the definition of λ-Hurwitz zeta function and Binet's first formula for logarithms of λ-gamma function $\ln\Gamma_\lambda(x)$.Then we give other integral representation on λ-psi function and some complete monotonicity properties on the function related to λ-psi function and its derivatives. As applications, we lastly obtain arithmetic, geometric and harmonic mean inequalities on λ-psi function between x and $1/x$ for all positive values of x and λ.

2 Useful Lemmas and λ-Analogue of Hurwitz Zeta Function

In this section, we give some properties that help us to prove our main results

Lemma 2.1. *(Qi et al., 2005) For $x > 0$ and any non-negative integer n, the integral*

$$\frac{1}{x^{n+1}} = \frac{1}{n!}\int_0^\infty t^n e^{-xt}dt$$

holds true.

Lemma 2.2. *(Spiegel and Ribero, 1970, pg.98, eq. 15.71)*

$$\ln\frac{a}{b} = \int_0^\infty \frac{e^{-at}-e^{-bt}}{t}dt.$$

In the next result, we obtain new integral representation of λ-psi function:

Lemma 2.3. *The λ-psi function can be also given by*

$$\psi_\lambda(x) = \int_0^\infty \left[e^{-u} - \frac{\lambda^x}{(u+\lambda)^x}\right]\frac{du}{u} \tag{2.1}$$

$$= \int_0^\infty \left[e^{-u} - \frac{1}{\left(\frac{u}{\lambda}+1\right)^x}\right]\frac{du}{u} \tag{2.2}$$

$$= \int_0^\infty \left[e^{-\lambda t} - \frac{1}{(t+1)^x}\right]\frac{dt}{t} \tag{2.3}$$

for all positive real values of x and λ.

Proof. By differentiating the λ-gamma function with respect to x and using Lemma 2.2, we get

$$\begin{aligned}
\Gamma'_\lambda(x) &= \int_0^\infty t^{x-1}e^{-\lambda t}\ln t dt \\
&= \int_0^\infty t^{x-1}e^{-\lambda t}\int_0^\infty \frac{e^{-u}-e^{-ut}}{u}du\, dt \\
&= \int_0^\infty\int_0^\infty \frac{t^{x-1}e^{-\lambda t}e^{-u} - t^{x-1}e^{-\lambda t}e^{-ut}}{u}du\, dt \\
&= \int_0^\infty \left(e^{-u}\int_0^\infty t^{x-1}e^{-\lambda t}dt - \int_0^\infty t^{x-1}e^{-t(u+\lambda)}dt\right)\frac{du}{u}.
\end{aligned}$$

Now, let us denote the second integral in the parentheses by I. Substituting $t(u+\lambda) = \lambda v$ yields that

$$\begin{aligned}
I &= \int_0^\infty t^{x-1}e^{-t(u+\lambda)}dt = \int_0^\infty \left(\frac{\lambda v}{u+\lambda}\right)^{x-1} e^{-\lambda u}\frac{\lambda dv}{u+\lambda} = \frac{\lambda^x}{(u+\lambda)^x}\int_0^\infty v^{x-1}e^{-\lambda v}dv \\
&= \frac{\lambda^x}{(u+\lambda)^x}\Gamma_\lambda(x).
\end{aligned}$$

Hence we get

$$\Gamma'_\lambda(x) = \int_0^\infty \left(e^{-u}\int_0^\infty t^{x-1}e^{-\lambda t}dt - \frac{\lambda^x}{(u+\lambda)^x}\Gamma_\lambda(x)\right)\frac{du}{u}.$$

Since the integral in the parentheses is equal to $\Gamma_\lambda(x)$, we get

$$\begin{aligned}
\Gamma'_\lambda(x) &= \int_0^\infty \left(e^{-u}\Gamma_\lambda(x) - \frac{\lambda^x}{(u+\lambda)^x}\Gamma_\lambda(x)\right)\frac{du}{u} \\
\frac{\Gamma'_\lambda(x)}{\Gamma_\lambda(x)} &= \int_0^\infty \left[e^{-u} - \frac{\lambda^x}{(u+\lambda)^x}\right]\frac{du}{u}
\end{aligned}$$

as desired. □

Next, we define the λ-analogue of Hurwitz zeta function:

Definition 2.1. The λ-Hurwitz zeta can be defined by

$$\zeta_\lambda(x,k)=\sum_{n=0}^{\infty}\frac{1}{(x+n)^k} \tag{2.4}$$

for $k>1$ and all positive real values of x and λ.

We want to remark that taking $x=1$ leads us to λ-Riemann zeta function (that is special case of λ-Hurwitz zeta function) as

$$\zeta_\lambda(1,k)=\sum_{n=0}^{\infty}\frac{1}{(n+1)^k}=\sum_{n=1}^{\infty}=\frac{1}{n^k}=\zeta_\lambda(k).$$

Furthermore, the λ-psi function can be given by

$$\psi_\lambda^{(n)}(x)=\sum_{k=0}^{\infty}\frac{(-1)^n n!}{(x+k)^n}=(-1)^{n+1}n!\zeta_\lambda(x,n+1).$$

Hence $\psi_\lambda^{(n)}(1)=\zeta_\lambda(n)$.

Lemma 2.4. *The integral representation of λ-zeta function can be given as*

$$\zeta_\lambda(x)=\frac{1}{\Gamma_\lambda(x)}\int_0^\infty\frac{t^{x-1}dt}{e^{\lambda t}-1}. \tag{2.5}$$

Proof. Substituting $u=nt$ in the integral representation of λ-gamma function (1.2) leads us that

$$\begin{aligned}\Gamma_\lambda(x) &= \int_0^\infty (nt)^{x-1}e^{-\lambda nt}ndt=\int_0^\infty n^{x-1}t^{x-1}e^{-\lambda nt}ndt\\ &= n^x\int_0^\infty t^{x-1}e^{-\lambda nt}dt\\ \Gamma_\lambda(x)\sum_{n=1}^{\infty}\frac{1}{n^x} &= \sum_{n=1}^{\infty}\int_0^\infty t^{x-1}e^{-\lambda nt}dt=\int_0^\infty t^{x-1}\sum_{n=1}^{\infty}e^{-\lambda nt}dt\end{aligned}$$

Since $\sum_{n=1}^{\infty}e^{-\lambda nt}$ is a geometric series and converges by the fact $|r|=\frac{1}{e^{\lambda nt}}>0$, the sum is equal to $\frac{1}{e^{\lambda t}-1}$. Hence the desired result concludes from the last equation. □

Taking derivatives of λ-psi function by n-times with respect to x yields that

$$\begin{aligned}\psi_\lambda^{(n)} &= (-1)^{n+1}n!\sum_{k=1}^{\infty}\frac{1}{(x+k-1)^{n+1}}=(-1)^{n+1}n!\sum_{k=0}^{\infty}\frac{1}{(x+k)^{n+1}}\\ &= (-1)^{n+1}n!\zeta_\lambda(x,n+1).\end{aligned}$$

Taking $x=1$ in the last equation leads that $\psi_\lambda^{(n)}(1)=(-1)^{n+1}n!\zeta_\lambda(n+1)$. Hence from Taylor series expansion of λ-psi function at $x=1$, we get

$$\begin{aligned}\psi_\lambda(x) &= \sum_{n=0}^{\infty}\frac{\psi_\lambda^{(n)}(1)(x-1)^n}{n!}=\psi_\lambda(1)+\sum_{n=1}^{\infty}\frac{\psi_\lambda^{(n)}(1)(x-1)^n}{n!}\\ &= -\ln\lambda-\gamma++\sum_{n=1}^{\infty}(-1)^{n+1}\zeta_\lambda(n+1)(x-1)^n.\end{aligned}$$

At last, by replacing $x+1$ instead of x in the last equation, we obtain the power series of the λ-psi function as

$$\psi_\lambda(x+1) = -\ln\lambda - \gamma + \sum_{k=2}^{\infty}(-1)^k \zeta_\lambda(k) x^{k-1} \tag{2.6}$$

for $|x| < 1$.

3 Main Results

Theorem 3.1. *Binet's first formula for logarithms of λ-gamma function $\ln\Gamma_\lambda(x)$ can be given by*

$$\ln\Gamma_\lambda(x) = \left(x - \frac{1}{2}\right)\ln x - x\ln\lambda + \frac{1}{2}\ln(2\pi) + \int_0^\infty \left[\frac{1}{2} - \frac{1}{t} + \frac{1}{e^t - 1}\right]\frac{e^{-tx}}{t}dt \tag{3.1}$$

and as an immediate consequence; the λ-psi function can also be given by

$$\psi_\lambda(x) = \ln x - \frac{1}{x} - \ln\lambda + \int_0^\infty \left(\frac{1}{t} - \frac{1}{e^t - 1}\right)e^{-xt}dt \tag{3.2}$$

for all positive real values of x and λ.

Proof. By taking into account the singularity of the third equation in the Lemma 2.3 at $t = 0$, we can rewrite the equation as

$$\psi_\lambda(x) = \lim_{\varepsilon\to}\left[\int_\varepsilon^\infty \frac{e^{-\lambda u}}{u}du - \int_\varepsilon^\infty \frac{du}{u(u+1)^x}\right].$$

Then substituting $u = e^t - 1$ in the second integral leads us to

$$\begin{aligned}
\psi_\lambda(x) &= \lim_{\varepsilon\to}\left[\int_\varepsilon^\infty \frac{e^{-\lambda u}}{u}du - \int_{\ln(1+\varepsilon)}^\infty \frac{e^t dt}{e^{tx}(e^t - 1)}\right] \\
&= \lim_{\varepsilon\to}\left[\int_\varepsilon^\infty \frac{e^{-\lambda u}}{u}du - \int_{\ln(1+\varepsilon)}^\infty \frac{e^{-xt}dt}{1 - e^{-t}}\right].
\end{aligned}$$

Since $\displaystyle\int_\varepsilon^{\ln(1+\varepsilon)} \frac{e^{-t}dt}{t} \le \int_{\ln(1+\varepsilon)}^{\varepsilon} \frac{dt}{t} \to 0$ as $\varepsilon \to 0$, we get

$$\begin{aligned}
\psi_\lambda(x) &= \lim_{\varepsilon\to}\left[\int_\varepsilon^\infty \frac{e^{-\lambda u}}{u}du - \int_{\ln(1+\varepsilon)}^\infty \frac{e^{-xt}dt}{1 - e^{-t}}\right] \\
&= \lim_{\varepsilon\to}\left[\int_\varepsilon^{\ln(1+\varepsilon)} \frac{e^{-\lambda u}}{u}du + \int_\varepsilon^\infty \frac{e^{-\lambda u}}{u}du - \int_{\ln(1+\varepsilon)}^\infty \frac{e^{-xt}dt}{1 - e^{-t}}\right] \\
&= \int_0^\infty \left[\frac{e^{-\lambda t}}{t} - \frac{e^{-xt}}{1 - e^{-t}}\right]dt.
\end{aligned}$$

Using the equation $\ln\lambda = \displaystyle\int_0^\infty \frac{e^{-t} - e^{-\lambda t}}{t}dt$ yields that

$$\begin{aligned}
\psi_\lambda(x) &= \int_0^\infty \left[\frac{e^{-\lambda t} - e^{-t} + e^{-t}}{t} - \frac{e^{-xt}}{1 - e^{-t}}\right]dt \\
&= -\int_0^\infty \frac{e^{-t} - e^{-\lambda t}}{t}dt + \int_0^\infty \left[\frac{e^{-t}}{t} - \frac{e^{-xt}}{1 - e^{-t}}\right]dt \\
&= -\ln\lambda + \int_0^\infty \left[\frac{e^{-t}}{t} - \frac{e^{-xt}}{1 - e^{-t}}\right]dt.
\end{aligned} \tag{3.3}$$

By using last equation (3.3), $\frac{1}{2x} = \frac{1}{2}\int_0^\infty e^{-xt}dt$ and $\ln x = \int_0^\infty \frac{e^{-t}-e^{-xt}}{t}dt$, we obtain

$$\begin{aligned}\psi_\lambda(x+1) &= -\ln\lambda + \frac{1}{2x} + \ln x - \int_0^\infty \left[\frac{e^{-xt}}{2} + \frac{e^{-t}-e^{-xt}}{t} - \frac{e^{-t}}{t} + \frac{e^{-xt}}{e^t-1}\right]dt \\ &= \ln x - \ln\lambda + \frac{1}{2x} - \int_0^\infty \left[\frac{1}{2} - \frac{1}{t} + \frac{1}{e^t-1}\right]e^{-xt}dt.\end{aligned}$$

The integrand is continuous as $t \to 0$ and since $\frac{1}{2} - \frac{1}{t} + \frac{1}{e^t-1}$ is bounded as $t \to \infty$, the integral is uniformly convergence for $x > 0$. By integrating from 1 to x, we obtain

$$\ln\Gamma_\lambda(x+1) - \ln\Gamma_\lambda \quad = \quad x\ln x - x + 1 - (x-1)\ln\lambda + \frac{1}{2}\ln x + \int_0^\infty \left[\frac{1}{2} - \frac{1}{t} + \frac{1}{e^t-1}\right]\frac{e^{-xt}-e^{-t}}{t}dt.$$

Now using recurrence formula on λ-gamma function in (1.6) and the equation $\Gamma_\lambda(2) = \frac{1}{\lambda^2}$ leads us that

$$\begin{aligned}\ln\Gamma_\lambda + \ln x - \ln\lambda + 2\ln\lambda &= x\ln x - x + 1 - (x-1)\ln\lambda + \tfrac{1}{2}\ln x \\ &\quad + \int_0^\infty \left[\tfrac{1}{2} - \tfrac{1}{t} + \tfrac{1}{e^t-1}\right]\tfrac{e^{-xt}-e^{-t}}{t}dt \\ \ln\Gamma_\lambda(x) &= \left(x - \tfrac{1}{2}\right)\ln x - x - x\ln\lambda + 1 + \int_0^\infty \left[\tfrac{1}{2} - \tfrac{1}{t} + \tfrac{1}{e^t-1}\right]\tfrac{e^{-xt}}{t}dt \\ &\quad - \int_0^\infty \left[\tfrac{1}{2} - \tfrac{1}{t} + \tfrac{1}{e^t-1}\right]\tfrac{e^{-t}}{t}dt.\end{aligned}$$

For evaluating the second integral on the right side of the last equation, let us denote $I = \int_0^\infty \left[\frac{1}{2} - \frac{1}{t} + \frac{1}{e^t-1}\right]\frac{e^{-t}}{t}dt$ and $J = \int_0^\infty \left[\frac{1}{2} - \frac{1}{t} + \frac{1}{e^t-1}\right]\frac{e^{-t/2}}{t}dt$. Taking $x = 1/2$ in the last equation yields that $\ln\Gamma_\lambda(1/2) = \frac{1}{2} - \frac{\ln\lambda}{2} + J - I$ and then using $\Gamma_\lambda\left(\frac{1}{2}\right) = \frac{1}{2}(\ln\pi - \ln\lambda)$ leads us to $J - I = \frac{1}{2}(1 - \ln\pi)$. On the other hand, by substituting $u = \frac{t}{2}$ in the integral I, we get $I = \int_0^\infty \left[\frac{1}{2} - \frac{2}{t} + \frac{1}{e^{t/2}-1}\right]\frac{e^{-t/2}}{t}dt$. Hence, we find

$$\begin{aligned}J - I &= \int_0^\infty \left[\frac{1}{2} - \frac{1}{t} + \frac{1}{e^t-1} - \frac{1}{2} + \frac{2}{t} - \frac{1}{e^{t/2}-1}\right]\frac{e^{-t/2}}{t}dt \\ &= \int_0^\infty \left(\frac{1}{t} - \frac{e^{-t/2}}{e^t-1}\right)\frac{e^{-t/2}}{t}dt = \int_0^\infty \left(\frac{e^{-t/2}}{2} - \frac{1}{e^t-1}\right)\frac{dt}{t}\end{aligned}$$

By using the last equation and integral representation of I, we obtain

$$\begin{aligned}J &= \int_0^\infty \left(\frac{e^{-t/2}}{2} - \frac{1}{e^t-1} + \frac{e^{-t/2}}{2} - \frac{e^{-t}}{t} + \frac{e^t}{e^t-1}\right)\frac{dt}{t} \\ &= \int_0^\infty \left[\frac{e^{-t/2}-e^{-t}}{t} - \frac{e^{-t}}{2}\right]\frac{dt}{t} \\ &= \int_0^\infty \left[-\frac{d}{dt}\left(\frac{e^{-t/2}-e^{-t}}{t}\right) - \frac{\frac{e^{-t/2}}{2}-e^{-t}}{t} - \frac{e^{-t}}{2t}\right]dt \\ &= \left(\frac{e^{-t/2}-e^{-t}}{t}\right)\Bigg|_0^\infty + \frac{1}{2}\int_0^\infty \frac{e^{-t}-e^{-t/2}}{t}dt \\ &= \frac{1}{2} + \frac{1}{2}\ln\frac{1}{2}.\end{aligned}$$

Hence we get $I = 1 - \frac{1}{2}\ln(2\pi)$. So by writing the solution of integral I in the last equation, we find

$$\ln\Gamma_\lambda(x) = \left(x - \frac{1}{2}\right)\ln x - x - x\ln\lambda + \frac{1}{2}\ln(2\pi) + \int_0^\infty \left[\frac{1}{2} - \frac{1}{t} + \frac{1}{e^t - 1}\right]\frac{e^{-xt}}{t}dt$$

for all $x, \lambda > 0$. By differentiating the last equation, one can obtain the equation (3.2). □

By using Theorem 3.1, we give the following completely monotonicity properties on the function involving λ-psi function and its first derivative:

Theorem 3.2. *The functions*

$$\psi_\lambda(x) + \ln\lambda - \ln x + \frac{1}{2x} + \frac{1}{12x^2}$$
$$\ln x - \frac{1}{2x} - \ln\lambda - \psi_\lambda(x)$$
$$\psi'_\lambda(x) - \frac{1}{x} - \frac{1}{2x^2} - \frac{1}{6x^3} + \frac{1}{30x^5}$$
$$\frac{1}{x} + \frac{1}{2x^2} + \frac{1}{6x^3} - \psi'_\lambda(x)$$

are completely monotonic for all $x, \lambda > 0$.

Proof. Let us denote $f_1(x) = \psi_\lambda(x) + \ln\lambda - \ln x + \frac{1}{2x} + \frac{1}{12x^2}$. By using the definition of the λ-psi function (3.2) and Lemma 2.1, we get

$$\begin{aligned} f_1(x) &= \psi_\lambda(x) + \ln\lambda - \ln x + \frac{1}{2x} + \frac{1}{12x^2} \\ &= \int_0^\infty \left[\frac{1}{t} - \frac{1}{e^t - 1} - \frac{1}{2} + \frac{t}{12}\right] e^{-xt}dt \\ &= \int_0^\infty \left[\frac{(12 - 6t + t^2)e^t - (12 + 6t + t^2)}{12t(e^t - 1)}\right] e^{-xt}dt. \end{aligned}$$

Since the nominator $d_1(t) = (12 - 6t + t^2)e^t - (12 + 6t + t^2) > 0$ and $d_1(0) = 0$, we get that $(-1)^n f_1^{(n)}(x) > 0$ as desired.

Now, let us define the function f_2 by $f_2(x) = \psi_\lambda(x) + \ln\lambda - \ln x + \frac{1}{2x}$. Then we obtain

$$\begin{aligned} f_2(x) &= \psi_\lambda(x) + \ln\lambda - \ln x + \frac{1}{2x} \\ &= \int_0^\infty \left(\frac{1}{t} - \frac{1}{e^t - 1} - \frac{1}{2}\right) e^{-xt}dt \\ &= \int_0^\infty \frac{(2 - t)e^t - (t + 2)}{2t(e^t - 1)} e^{-xt}dt. \end{aligned}$$

Since $d_2(t) = (2 - t)e^t - (t + 2) < 0$ and $d_2(0) = 0$ for all $t > 0$, we obtain $(-1)^{n+1} f_2^{(n)}(x) > 0$ as desired.

By differantiating the definition of the λ-psi function (3.2), we get

$$\psi'_\lambda(x) = \frac{1}{x} + \frac{1}{x^2} - \int_0^\infty \left(1 - \frac{t}{e^t - 1}\right) e^{-xt}dt$$

for all $x, \lambda > 0$. Then using the last equation and Lemma 2.1 yields that

$$\begin{aligned} f_3(x) &= \psi'_\lambda(x) - \frac{1}{x} - \frac{1}{2x^2} - \frac{1}{6x^3} + \frac{1}{30x^5} \\ &= \int_0^\infty \left(\frac{t}{e^t - 1} - 1\frac{t}{2} - \frac{t^2}{12} + \frac{t^4}{720}\right) e^{-xt}dt \\ &= \int_0^\infty \left[\frac{(t^4 - 60t^2 + 360t - 720)e^t - (t^4 - 60t^2 - 360t - 720)}{720(e^t - 1)}\right] e^{-xt}dt \end{aligned}$$

Since the nominator of the function $d_3(t) = (t^4 - 60t^2 + 360t - 720)e^t - (t^4 - 60t^2 - 360t - 720)$ is positive for all $t \geq 0$, it concludes that $(-1)^n f_3^{(n)}(x) > 0$ for all positive real value of x and any nonnegative integer n.

At last, let us define the function f_4 by $f_4(x) = \dfrac{1}{x} + \dfrac{1}{2x^2} + \dfrac{1}{6x^3} - \psi'_\lambda(x)$. Then by using the derivative of the λ-psi function (3.2) and Lemma 2.1, we obtain

$$f_4(x) = \int_0^\infty \left[\frac{(12 - 6t + t^2)e^t - (t^2 + 6t + 12)}{12(e^t - 1)} \right] e^{-xt} dt.$$

Since the nominator of the function is the same as the function f_1, we get that the function f_4 is also completely monotonic for all $x > 0$ and $n \in \mathbb{Z}^+ \cup \{0\}$. □

As an immediate consequence of the previous theorem, we get the following double sided inequalities on λ-psi function and its first derivative.

Corollary 3.3. *The following double sided inequalities*

$$\ln x \ln \lambda - \frac{1}{2x} - \frac{1}{12x^2} < \psi_\lambda(x) < \ln x - \ln \lambda - \frac{1}{2x}, \tag{3.4}$$

$$\frac{1}{x} + \frac{1}{2x} + \frac{1}{6x^3} - \frac{1}{30x^5} < \psi'_\lambda(x) < \frac{1}{x} + \frac{1}{2x^2} + \frac{1}{6x^3} \tag{3.5}$$

and

$$-\frac{1}{kx^2} - \frac{1}{x^3} - \frac{1}{2x^4} < \psi''_\lambda(x) < -\frac{1}{x^2} - \frac{1}{x^3} \tag{3.6}$$

hold true for $x > 0$.

Theorem 3.4. *For all positive real values of x and λ, the function*

$$P_\lambda(x) = \psi_\lambda(x) + \psi_\lambda\left(\frac{1}{x}\right) + 2\ln\lambda \tag{3.7}$$

is concave.

Proof. Taking derivatives of the function P_λ yields that

$$\begin{aligned}
P'_\lambda(x) &= \psi_\lambda(x) - \frac{1}{x^2}\psi'_\lambda\left(\frac{1}{x}\right) \\
P''_\lambda(x) &= \psi''_\lambda(x) + \frac{2}{x^3}\psi'_\lambda\left(\frac{1}{x}\right) + \frac{1}{x^4}\psi''_\lambda\left(\frac{1}{x}\right) \\
x^4 P''_\lambda(x) &= \psi''_\lambda\psi''_\lambda(x) + \frac{2}{x^3}\psi'_\lambda\left(\frac{1}{x}\right) + \frac{1}{x^4}\psi''_\lambda\left(\frac{1}{x}\right) + \ 2x\psi'_\lambda\left(\frac{1}{x}\right) + x^4\psi''_\lambda(x).
\end{aligned}$$

By using the recurrence formulas $\psi'_\lambda(x+1) = \psi'_\lambda(x) - \dfrac{1}{x^2}$ and $\psi''_\lambda(x+1) = \psi''_\lambda(x) + \frac{2}{x^3}$ and inequalities (3.5) and (3.6), we obtain

$$\begin{aligned}
x^4 P''_\lambda(x) &= -2x^3 + \psi''_\lambda\left(\frac{1}{x}\right) + 2x\left[x^2 + \psi'_\lambda\left(\frac{1}{x} + 1\right)\right] + x^4\psi''_\lambda(x) \\
&< -2x^3 + \left(-\frac{1}{(1/x + 1)^2} - \frac{1}{(1 + 1/x)^3}\right) + 2x\left(x^2 + \frac{1}{1 + 1/x} + \frac{1}{2(1 + 1/x)^2} + \frac{1}{6(1 + 1/x)^3}\right) \\
&\quad + x^4\left(-\frac{1}{x^2} - \frac{1}{x^3}\right) \\
&= -\frac{x}{3(1 + x)^3}\left[3x^4 + 2x^3 + 9x^2 + 9x + 3\right].
\end{aligned}$$

Hence it is easy to see that since $x^4 P''_\lambda(x) < 0 \Rightarrow P''_\lambda(x) < 0$, the function P_λ is concave for $\forall x, \lambda > 0$. □

Corollary 3.5. *For $x \in \mathbb{R}^+ - \{1\}$, the inequality*

$$\psi_\lambda(x) + \psi_\lambda\left(\frac{1}{x}\right) + 2\ln\lambda < 2\left(\psi_\lambda(1) + \ln\lambda\right) \tag{3.8}$$

is valid.

Proof. Using the concavity of the function P_λ in Theorem 3.4 leads us that the function P'_λ is decreasing for $x > 0$. Since $P'_\lambda(x) = \psi_\lambda(x) - \frac{1}{x^2}\psi'_\lambda\left(\frac{1}{x}\right) \Rightarrow P'_\lambda(1) = 0$, we get that from $P'_\lambda(x) > P'_\lambda(1) = 0$ for $0 < x < 1$, the function P_λ increases and also from $P'_\lambda(x) < P'_\lambda(1) = 0$ for $1 < x$, the function P_λ increases. Hence for $x \neq 1$ and $x > 0$, we find $P_\lambda(x) < P_\lambda(1) = 2\left(\psi_\lambda(1) + \ln\lambda\right)$ as desired. □

We want to note that since the classical psi function $\psi(x)$ at $x = 1$ is equal to $-\gamma$, by using the relation between classical and λ-psi function $\psi_\lambda(x) = -\ln\lambda + \psi(x)$, we clearly see that for all positive value of λ, $P_\lambda(1) = 2(\psi_\lambda(1) + \ln\lambda) = 2\psi(1) = -\gamma < 0$.

Theorem 3.6. *The inequality*

$$[\psi_\lambda(1+y) + \ln\lambda][\psi_\lambda(1-y) + \ln\lambda] < [\psi_\lambda(1) + \ln\lambda]^2 \tag{3.9}$$

holds true for $\forall y \in (0,1)$.

Proof. Since for all $\lambda > 0$, the λ-psi function is completely monotonic on $(0,\infty)$, there is a $x_0 \in (0,\infty)$ such that $\psi_\lambda(x) + \ln\lambda = 0$ and on the other hand, from the first inequality (3.4), we get that $\psi_\lambda(x) + \ln\lambda > \ln x - \frac{1}{2x} - \frac{1}{12x^2} \Rightarrow x_0 \gtrsim 1,46321$. So if $y \in [x_0 - 1, 1)$, then $\psi_\lambda(1-y) + \ln\lambda < 0 \leq \psi_\lambda(1+y) + \ln\lambda$. Hence we get desired result. Now if $y \in (0, x_0 - 1)$, then $\psi_\lambda(1-y) + \ln\lambda < 0$ and $\psi_\lambda(1+y) + \ln\lambda < 0$. By using the power series on the λ-psi function in the equation (2.6), we get the following inequalities

$$0 < -(\psi_\lambda(1+y) + \ln\lambda) \leq \gamma - \zeta_\lambda(2)y + \zeta_\lambda(3)y^2$$

and

$$\begin{aligned} 0 < -(\psi_\lambda(1-y) + \ln\lambda) &\leq \gamma + \zeta_\lambda(2)y + \zeta_\lambda(3)\sum_{k=2}^{\infty} y^k \\ &= \gamma + \zeta_\lambda(2)y + \zeta_\lambda(3)\frac{y^3}{1-y} \\ &\leq \gamma + \zeta_\lambda(2)y + \zeta_\lambda(3)y^2. \end{aligned}$$

Thus these inequalities leads us to the following inequality

$$(\psi_\lambda(1+y) + \ln\lambda)(\psi_\lambda(1-y) + \ln\lambda) \leq \gamma^2 - [\zeta_\lambda(2)^2 - \zeta_\lambda(3)]y^2 - \zeta_\lambda(2)\zeta_\lambda(3)y^3 + 2\zeta_\lambda(3)^2 y^4.$$

Since $\zeta_\lambda(2)^2 - \zeta_\lambda(3) > 0$, $\zeta_\lambda(2)\zeta_\lambda(3) > 0$ and $2\zeta_\lambda(3)^2 > 0$, the inequality $\zeta_\lambda(2)\zeta_\lambda(3) > 2\zeta_\lambda(3)^2$ is valid. Hence we obtain that the inequality $\zeta_\lambda(2)\zeta_\lambda(3)y^3 > 2\zeta_\lambda(3)^2 y^4$. Thus the proof is completed. □

Theorem 3.7. *For $x \in \mathbb{R} - \{1\}$, the inequality*

$$[\psi_\lambda(x) + \ln\lambda]\left[\psi_\lambda\left(\frac{1}{x}\right) + \ln\lambda\right] < [\psi_\lambda(1) + \ln\lambda]^2 \tag{3.10}$$

holds true.

Proof. If $x \geq x_0$, then the inequality is valid since $\left[\psi_\lambda\left(\frac{1}{x}\right)+\ln\lambda\right] < 0 \leq [\psi_\lambda(x)+\ln\lambda]$.
Now, let $1+y=x\in(1,x_0)$. By using the inequality $\frac{1}{x} > 1-y$ and monotonicity property of the λ-psi function $(0,\infty)$, we find that the inequality

$$[\psi_\lambda(1-y)+\ln\lambda] < \left[\psi_\lambda\left(\frac{1}{x}\right)+\ln\lambda\right] < 0$$

holds true. Hence from the last inequality and Theorem 3.6, we get

$$\begin{aligned}[\psi_\lambda(x)+\ln\lambda]\left[\psi_\lambda\left(\frac{1}{x}\right)+\ln\lambda\right] &= [\psi_\lambda(1+y)+\ln\lambda]\left[\psi_\lambda\left(\frac{1}{x}\right)+\ln\lambda\right] \\ &< [\psi_\lambda(1+y)+\ln\lambda][\psi_\lambda(1-y)+\ln\lambda] \\ &< [\psi_\lambda(1)+\ln\lambda]^2\end{aligned}$$

as desired.
If $x\in(0,1)$, then we take $z=1/x$ and use the method above for the result. □

Theorem 3.8. *The inequality*

$$\psi_\lambda(1)+\ln\lambda \leq \frac{2\left[\psi_\lambda(x)+\ln\lambda\right]\left[\psi_\lambda\left(\frac{1}{x}\right)+\ln\lambda\right]}{\psi_\lambda(x)+\psi_\lambda\left(\frac{1}{x}\right)+2\ln\lambda} \tag{3.11}$$

holds true for all positive real values of x.

Proof. By using the inequalities (3.8) and (3.10) and taking into account that the inequality (3.8) is negative, we get

$$\begin{aligned}2\left[\psi_\lambda(x)+\ln\lambda\right]\left[\psi_\lambda\left(\frac{1}{x}\right)+\ln\lambda\right] &< 2\left[\psi_\lambda(1)+\ln\lambda\right]^2 \\ \frac{2\left[\psi_\lambda(x)+\ln\lambda\right]\left[\psi_\lambda\left(\frac{1}{x}\right)+\ln\lambda\right]}{\psi_\lambda(x)+\psi_\lambda\left(\frac{1}{x}\right)+2\ln\lambda} &> \frac{2\left[\psi_\lambda(1)+\ln\lambda\right]^2}{\psi_\lambda(x)+\psi_\lambda\left(\frac{1}{x}\right)+2\ln\lambda} \\ &> \frac{2\left[\psi_\lambda(1)+\ln\lambda\right]^2}{\left[\psi_\lambda(1)+\ln\lambda\right]^2} \\ &= \left[\psi_\lambda(1)+\ln\lambda\right]^2\end{aligned}$$

for all positive real values of x except for 1. The sign of the equality holds if and only if $x=1$. □

4 Conclusion

In this work, we study the one of newest generalization of special function that is closely related to fractional integral,. We give some definition of λ-psi and λ-zeta functions and obtain that some functions involving λ-psi function and its derivative are completely monotonic for all positive real values of x and λ. Some results in this paper may be used for refinements of some inequalities and obtaining new results.

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