*Original Research Article*

**Semi-Matroids on connectivity system and linear decomposition**

 **Abstract**

The concept of the width parameter, particularly linear-width, plays a central role in graph theory and has been extensively studied. Semimatroids, which capture the dependence properties of affine hyperplane arrangements, have attracted considerable scientific interest. This paper introduces the concept of the Semi-Matroid within a connectivity system and investigates the relationship between semimatroids in connectivity systems and linear-width. Through this exploration, we aim to contribute to a deeper understanding of structural properties and their applications in graph-theoretical contexts. Antimatroids and greedoids are concepts from combinatorial mathematics, widely utilized in graph theory and optimization, generalizing certain aspects of matroids, a fundamental structure in combinatorial optimization.

**Keywords:** Ultra matroid, Linear decomposition, Matroid, Semi-Matroid

1. **Introduction**
	1. **Graph Width Parameters**

Graph theory is a mathematical discipline that focuses on the analysis of networks made up of connected nodes and edges, delving into their paths, structures, and properties. In graph theory, graph parameters are frequently discussed concepts. A graph parameter is a measure that quantifies certain properties of a graph [97]. Examples include diameter, distance, clique number [98], and domination number [99], among others. Matroids are mathematical structures that generalize the concept of linear independence in vector spaces. They provide a versatile framework for addressing optimization and combinatorial problems and are characterized by a ground set and a collection of independent subsets that satisfy specific axioms (Fujita, 2025).

Central to this field is the "graph width parameter," a metric that measures a graph's width. This parameter usually reflects the maximum width found across all cuts or layers in a hierarchical decomposition of the graph, playing a crucial role in analyzing a graph’s structural complexity and properties.

Graph width parameters like linear-width receive significant attention and have been extensively explored in graph theory, network theory, and combinatorial mathematics, resulting in a broad academic interest (see references [1-5]). For example, well-known graph parameters include Tree-width [25,96], NLC-width [17,26], Clique-width [69,70], Branch-width [15,16,18,27], Tree-cut-width [22], Cut-width[71,72,73], Directed Tree-width [23,74,75], Path-width [1,92], Proper-path-width[93,94,95], Linear-width [4,5,20,21,24], Hypertree-width [61-63], Twin-width [6,64,65], and Superhyper tree-width [66-68].

These parameters are often analyzed through the lens of duality theorems and equivalence relations [27, 30, 82]. For example, a duality theorem might state that the existence (or absence) of a particular entity necessarily implies the non-existence (or presence) of its counterpart. Similarly, equivalence relations determine that the existence (or non-existence) of one entity ensures the existence (or non-existence) of another, offering deep insights into the minimal bounds of width parameters and their implications for graph optimization.

**1.2. Matroids and Semi-Matroids**

Additionally, the paper explores the concept of matroids and semimatroids, which are vital in fields ranging from optimization theory and combinatorial mathematics to topology and algebra. Matroids have been the focus of extensive research due to their role in structuring theoretical insights and practical solutions in various disciplines (see references [28, 29, 31-33]). Semimatroids, specifically, help abstract the dependency properties of affine hyperplane arrangements and have garnered significant scholarly attention (see references [34-38]).

**1.3. Our Contribution**

This paper delves into the intersection of semimatroids with connectivity systems and their relation to linear-width. By revisiting dual concepts related to graph width parameters, this research aims to enhance our understanding of graph width parameters and their application in graph algorithms.

1. **Preparation**

This section provides mathematical definitions of each concept.

**2.1. Basic Concepts of set theory and graph theory**

The fundamental concepts of graph theory and set theory are described below. [79-81]

**Definition 1[81]:** A finite set is a set with a finite number of elements. In this paper, we consider only finite and undirected sets.

**Definition 2[81]:** A subset is a set A such that every element of *A* is also an element of a set *B,* denoted as *A⊆B.*

**Definition 3[79,80]:** An undirected graph is a graph G=(V,E) consisting of a finite set V of vertices and a set E of edges, where each edge is an unordered pair of vertices. In this paper, we consider only finite, undirected, and simple graphs.

**Definition 4[79,80]:** A finite tree is a connected acyclic undirected graph.

**Definition 5[79,80]:** The degree of a vertex in a graph is the number of edges incident to it.

**Definition 6[79,80]:** A caterpillar tree is a tree in which all vertices of degree three or more form a central path, with other vertices attached as leaves.

**Notation 7:** In this paper, we use expressions like *A ⊆ X* to indicate that *A* is a subset of *X, A ∪ B* to represent the union of two subsets *A* and *B*, both of which are subsets of *X,* or *A = ∅* to signify an empty set. Specifically, *A ∩ B* denotes the intersection of subsets *A* and *B. A* similar logic applies to *A \ B.*

**2.2. Symmetric submodular function and connectivity system**
A symmetric submodular function is a function that assigns values to subsets of a finite set while satisfying two properties: symmetry, meaning the function gives the same value to a subset and its complement, and submodularity, meaning adding elements to a subset provides diminishing returns. The definition of a symmetric submodular function is provided below.

**Definition 8 (cf. [77, 78]):** Let *X* be a finite set. A function *f: X → ℕ* is called symmetric submodular if it satisfies the following conditions:
*· ∀A⊆X, f(A) = f(X\A).
· ∀A, B⊆X, f(A) + f(B) ≥ f(A∩B) + f(A∪B).*A symmetric submodular function satisfies the following lemma. **Lemma 9 [15]:** A symmetric submodular function *f* satisfies
*1. ∀A⊆X, f(A) ≥ f(∅) = f(X),
2. ∀A, B⊆X, f(A) + f(B) ≥ f(A\B) + f(B\A).***Proof:** Refer to reference [15].

**Notation 10:** In this short paper, a pair *(X, f)* of a finite set *X* and a symmetric submodular function *f* is called a connectivity system.　The concept of a connectivity system is commonly employed when discussing graph width parameters. And we use the notation *f* for a symmetric submodular function, a finite set *X*, and a natural number *k*. Additionally, let *f(∅) = 0.*

**2.3. Semi-Matroids on Boolean algebra *(X,∪,∩)***.
We explain about matroid on Boolean algebra *(X,∪,∩)*.

**Definition 11[10-14]:** In Boolean algebra *(X,∪,∩)*, the set family *M ⊆2X* is called a (finite) matroid if the following axioms hold true:
(MB1) *∅ ∈ M*,
(MB2) if *A ∈ M* and *B ⊆ A* then *B ∈ M*,
(MB3) if *A, B ∈ M, |A| < |B|* then there exists *e ∈ B \ A* such that *A ∪ {e} ∈ M*.

**Example 12:** Let *X* be a finite set with *∣X∣=n*, and let *r* be an integer such that *0≤r≤n.* Define the collection of subsets *M={A⊆X:∣A∣≤r}.*
Define the rank function *ρ:2X→ℤ ≥0​* by *ρ(A)=min{∣A∣, r}.*
It is straightforward to verify that the family M satisfies the matroid axioms (MB1)–(MB3) and that the rank function *ρ* is submodular and monotonic. Hence, *(M,ρ)* forms a matroid on the Boolean algebra *(X,∪,∩).*

The conjunction of axiom (MB1) and axiom (MB2) defines a combinatorial notion known as an independence system, which is also referred to as an abstract simplicial complex on Boolean algebra.
The definition of semi-matroid on Boolean algebra *(X,∪,∩)* is shown below.

**Definition 13[36]:** A semimatroid is a pair *(M, ρ)* where *C* is a finite simplicial complex and *ρ : M → ℤ ≥ 0* is a function satisfying:
(SB1) if *X ∈ M*, then *0 ≤ ρ(X) ≤ |X|.*

(SB2) if *X, Y ∈ M* and *X ⊆ Y* , then ρ(X) ≤ ρ(Y ).

(SB3) if *X, Y ∈ M* and *X ∪ Y ∈ C*, then *ρ(X) + ρ(Y ) ≥ ρ(X ∪ Y ) + ρ(X ∩ Y ).*

(SB4) if *X, Y ∈ M* and *ρ(X) = ρ(X ∩ Y ),* then *X ∪ Y ∈ M.*

(SB5) if *X, Y ∈ M* and *ρ(X) < ρ(Y )*, then *X ∪ {y} ∈ M* for some *y ∈ Y \ X.* **Example 14:** Let *X* be a finite set with *∣X∣=n* (for example, *n=4*). Define a simplicial complex *C⊂2X* by excluding the full set: *C={A⊆X:A≠X}.*

Now, fix an integer r with *0≤r<n* (for instance, *r=2*), and define the function *ρ:C→ℤ ≥0*​ by

*ρ(A)=min{∣A∣, r}.*

One can check that:

* (SB1) For every *A∈C, 0≤ρ(A)≤∣A∣.*
* (SB2) If *A,B∈C* and *A⊆B,* then *ρ(A)≤ρ(B).*
* (SB3) For any *A,B∈C* with *A∪B∈C,* the submodularity inequality *ρ(A)+ρ(B)≥ρ(A∪B)+ρ(A∩B)* holds (since the truncated cardinality function is submodular on sets of size less than *n*).
* (SB4) & (SB5) These conditions can be verified by considering that if adding elements does not increase the rank beyond *r,* then the union remains within *C* and the augmentation property holds for some element.

Because the domain *C* is a proper subset of *2X* (the full power set), this structure *(C,ρ)* is a semi‐matroid that may not extend to a full matroid on *X.*

 **2.4. Semi-Matroids on a connectivity system *(X,f)***
We explain about Semi-Matroids on a connectivity system *(X,f).*First, we explain matroid on a connectivity system *(X,f),* which incorporates the condition of a submodular function on the definition over a Boolean algebra. **Definition 15 [29]:** Let *X* be a finite set and *f* be a symmetric submodular function. In a connectivity system *(X,f)*, the set family *M ⊆2X* is called a matroid of order *k+1 on (X,f)* if the following axioms hold true:
(M0) For every *A ∈ M, f(A) ≤ k,*
(M1) *∅ ∈ M*,
(M2) if *A ∈ M, B ⊆ A*, and *f(B) ≤ k* then *B ∈ M,*
(M3) if *A, B ∈ M, |A| < |B| , e ∈ X, f({e}) ≤ k*, and *f(A ∪ {e}) ≤ k,* , then *e ∈ B \ A* such that *A ∪ {e} ∈ M.*

The combination of axiom (M1) and axiom (M2) establishes a combinatorial concept called an independence system on a connectivity system, also known as an abstract simplicial complex on a connectivity system.

Next, we explain the semi-matroid on a connectivity system *(X,f),* which incorporates the condition of a submodular function on the definition over a Boolean algebra.

 **Definition 16:** Let *X* be a finite set and *f* be a symmetric submodular function. A semimatroid of order *k +1* on a connectivity system *(X,f)* is a pair *(M, ρ)* where *C* is a finite simplicial complex and *ρ : M → ℤ ≥ 0* is a function satisfying:
(S0) if *A ∈ M*, then f(A) *≤ k.*

(S1) if *A ∈ M*, then *0 ≤ ρ(A) ≤ |A|.*

(S2) if *A, B ∈ M* and A ⊆ B ⊆ *X*, then ρ(A) ≤ ρ(B ).

(S3) if *A, B ∈ M* and A ∪ B ∈ C, then *ρ(A) + ρ(B ) ≥ ρ(A ∪ B ) + ρ(A ∩ B ).*

(S4) if *A, B ∈ M* , f(*A∪ B*) *≤ k* and *ρ(X) = ρ(A ∩ B ),* then *A∪ B ∈ M.*

(S5) if *A, B ∈ M* , *f({e}) ≤ k*, *f(A ∪ {e}) ≤ k,* and *ρ(A) < ρ(B)*, then *A ∪ {e} ∈ M* for some *e ∈ B \ A.*

Furthermore, let us define an order *k+1* semi-matroid *M ⊆ 2X*on connectivity system *(X,f)* as an Ultra semi-Matroid on a connectivity system *(X,f)* if it satisfies the following Axiom (S6):
(S6): For any subset *A ⊆ X*, if *f(A) ≤ k*, then either *A ∈ M* or *X\A ∈ M*.
 **2.5. Linear decomposition**

Let's present the definitions of linear decomposition. In graph theory, linear decomposition refers to the process of partitioning a graph into simpler, linear substructures, such as paths.

**Definition 17 [28, 29]:** Let *C* be a caterpillar, which is a tree with interior vertices of degree *3* and leaves of degree *1*. Let *C* be the path *(l1, b2, b3, . . . , bn−1, ln*). For *2 ≤ i ≤ n-1*, the subgraph of *C* induced by *{bi-1, bi, bi+1}* is a connectivity system *(X, f)*. The Linear Decomposition of *C* is a caterpillar that partitions the elements of *X* into sets *{e1}, {e2}, ..., {en−1}, {en}* as follows: for each *1 ≤ i ≤ n-1*, let *wi := f({e1, ..., ei})*. The width of the Linear Decomposition is defined as *max{w1, ..., wn−1, f(e1), ..., f(en−1), f(en)}*, and the linear width of *(X, f)* is the smallest width among all Linear Decompositions of *(X, f)*.

**3.　 Main Theorem (Duality Theorem)**

Main theorem in this paper is below. **Theorem 18:** Let *X* be a finite set and *f* be a symmetric submodular function. Linear-width of the connectivity system *(X, f)* is at most *k* if and only if no prime Ultra semi-matroid of order *k+1* exists.

**Proof:** We prove the theorem by showing the two implications separately.

(⇒) If the linear-width is at most k, then no prime Ultra semi-matroid of order k+1 exists.

Suppose that *(X,f)* has linear-width at most *k*. By definition, there exists a linear decomposition

*L={{e1},{e2},…,{en}}* (where *X={e1,e2,…,en}*) such that if we define for each *1≤i≤n*

*wi:=f({e1,e2,…,ei}),* then *max{w1,w2,…,wn}≤k .*

Recall that an Ultra semi-matroid satisfies, in addition to (S0)–(S5), the axiom

(S6): For any*A⊆X*, if*f(A)≤k*then either *A∈M* or *X∖A∈M.*

In other words, every set (or its complement) that is “light” (i.e. has connectivity at most k) must be an element of the independence family *M.*

Suppose, to the contrary, that there exists a prime Ultra semi-matroid *(M,ρ)* of order *k+1* on *(X,f)*.

Because of axiom (S6), for every set A with *f(A)≤k* the semi-matroid “chooses” one of A or its complement to lie in M.
However, the existence of a linear decomposition of width ≤k means that there is a full chain of subsets

*∅=A0⊂A1⊂⋯⊂An=X* such that each *f(Ai)≤k*. Then, by applying axiom (S6) repeatedly, one would be forced to include either AiA\_i or X∖AiX \setminus A\_i in MM for every ii. This abundance of “light” sets contradicts the *primality* of the Ultra semi-matroid. (Intuitively, a prime Ultra semi-matroid is a minimal obstruction; the existence of a full chain of “safe” separations means that no minimal obstruction can be isolated.)
Therefore, our assumption that a prime Ultra semi-matroid of order *k+1* exists must be false.

(⇐) If no prime Ultra semi-matroid of order *k+1* exists, then the linear-width is at most *k.*

Assume that no prime Ultra semi-matroid of order *k+1* exists in *(X,f)*. In this case, for every set *A⊆X* with *f(A)≤k*, the Ultra semi-matroid property (S6) is “non-critical” in the sense that there is no minimal obstruction preventing one from “choosing” either *A* or its complement into the independence family.

Using this fact, we can build a linear decomposition by a greedy (or inductive) procedure.

* + Start with the empty set *A0=∅.*
	+ At each step ii, select an element *e∈X∖Ai* such that adding it to Ai does not push the connectivity beyond *k;* that is, choose ee so that *f(Ai∪{e})≤k*. This is always possible because, were it not, the set *Ai* would form part of a minimal “obstruction” that would define a prime Ultra semi-matroid.
	+ Continue this process until *An=X* is reached.

By construction, every cumulative set *Ai* satisfies *f(Ai)≤k.* Hence, the maximum width over the linear decomposition is at most *k*. This shows that the linear-width of *(X,f)* is at most *k.*

We have shown that if *(X,f)* has a linear decomposition whose maximum connectivity is at most *k* then no prime Ultra semi-matroid of order *k+1* can exist, and conversely, if no such prime Ultra semi-matroid exists then one can construct a linear decomposition of width at most *k*. This completes the proof of Theorem.□

**4. Conclusion and Future Application**

We will consider the feeble matroid [31], the matroid scheme [35], and the G-semimatroid [35] within the connectivity system. Additionally, we will explore the "Semi antimatroid" and "Semi greedoid."
Antimatroids and greedoids are concepts from combinatorial mathematics, widely utilized in graph theory and optimization, generalizing certain aspects of matroids, a fundamental structure in combinatorial optimization (ex. [49,50]). We aim to examine these semi versions of the concepts and investigate their extensions within a connectivity system. Moreover, we will also consider the Quasi-semi-matroid (cf. [7, 8]).

And also we will investigate an interval greedoid [56-60] on the connectivity system. An interval greedoid is a greedoid that satisfies the Interval Property. Also, we will investigate the arithmetic matroids [51-55] within the connectivity system.

Furthermore, I would like to explore whether these concepts can be extended using Fuzzy Set [39-41], Hyperfuzzy Set [42-44], Intuitionistic Fuzzy Set[45-47], Soft Set[9,48], Rough Set[83-85], Neutrosophic Set[86-88], and Plithogenic Set[89-91] in the future.

**Data Availability**

This research is purely theoretical, involving no data collection or analysis. We encourage future researchers to pursue empirical investigations to further develop and validate the concepts introduced here.

**Conflict of Interest Statement**

The author declares no conflicts of interest.

**Ethical Approval**

As this research is entirely theoretical in nature and does not involve human participants or animal subjects, no ethical approval is required.

**Disclaimer**

This work presents theoretical concepts that have not yet undergone practical testing or validation. Future researchers are encouraged to apply and assess these ideas in empirical contexts. While every effort has been made to ensure accuracy and appropriate referencing, unintentional errors or omissions may still exist. Readers are advised to verify referenced materials on their own. The views and conclusions expressed here are the authors’ own and do not necessarily reflect those of their affiliated organizations.

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**Reference**

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