**Original Research Article**

**On Ultrafilters in Matroids: Structural Properties and Metric Considerations**

**Abstract:** The concept of ultrafilters is well known in set theory. An ultrafilter on a set is a collection of subsets that is closed under intersections and supersets, containing no empty set. It is maximal, meaning that for any subset of the set, either it is included in the ultrafilter or its complement is included, ensuring a strong form of decisiveness in set membership. Matroid theory naturally aligns with greedy algorithms, making it a fundamental tool in various algorithmic applications. Reference [1] introduced the notion of filters and filter bases in the context of matroids, suggesting the possibility of a matroid-specific ultrafilter. However, ultrafilters in matroids remain an unexplored concept. This paper aims to investigate and establish the framework of ultrafilters in the setting of matroids.  
**Keyword:** Matroid, Filter, Ultrafilter, Weak-Filter, Quasi-Filter

1. **Introduction**

**1.1. Linear Algebra and Matroid**  
Algebra is a fundamental branch of mathematics focused on symbolic manipulation and the unifying rules behind arithmetic operations [105]. Linear algebra, a subfield of algebra, emphasizes vector spaces and linear mappings, particularly addressing systems of linear equations, matrices, and transformations [106].

In this paper, we investigate the intricate realm of matroids, which discretely capture the notions of linear independence and dependence derived from linear algebra. A matroid generalizes the concept of linear independence in matrices, expanding it beyond rows or columns to a broader combinatorial framework. Matroids hold significant value in computer science, as their theory integrates seamlessly with greedy algorithms, making them indispensable for various algorithmic applications. Their profound mathematical importance has prompted extensive research in the field, as reflected by numerous studies [3–16, 35–50, 64–71].

**1.2. Filter and Ultrafilter**

In parallel, the concept of filters (specifically, Ultrafilters) finds its origins deeply embedded in disciplines such as topology and algebra. A filter is a collection of subsets capturing “large” structures, closed under supersets and finite intersections, essential for limit arguments. An ultrafilter is a maximal filter on a set, essential in set theory and topology for rigorously handling limits, convergence, and compactness. Its unique properties make it crucial in non-standard analysis, model theory, and first-order logic, providing powerful tools for both mathematical and logical applications [17-34, 72, 73].

Due to their versatility, various related concepts to ultrafilters have been proposed. One such concept is the Quasi-Ultrafilter, which offers an axiomatic analysis of incomplete social judgments [51-59]. The weak filter (or weak ultrafilter) was introduced by K. Schlechta in the 1990s as a relaxed version of an ultrafilter. It serves as a powerful tool for interpreting defaults using a generalized "most" quantifier in first-order logic [60-63].

* 1. **Our Contribution of this paper**

Building on the existing body of knowledge, reference [1] introduced the concept of filters and filter bases in the context of matroids. From this groundwork, it is natural to hypothesize the existence of a matroid-specific ultrafilter. However, research on ultrafilters in matroids is still in its infancy, and their properties remain largely unexplored.

This short paper aims to shed light on ultrafilters in the realm of matroids. Specifically, it provides definitions and brief examinations of ultrafilters, weak-ultrafilters, and quasi-ultrafilters, paving the way for deeper investigation into their structural properties. We sincerely hope that our examination of ultrafilters in the realm of matroids will help advance research in algebra, matroid theory, and set theory.

* 1. **Structure of this paper**

This short paper is structured as follows. Section 2 provides definitions of Matroids and Filters on Matroids, along with concrete examples. Section 3 examines the properties of Ultrafilters on Matroids. Section 4 explores the characteristics of Quasi-Ultrafilters and Weak-Ultrafilters on Matroids. Finally, Section 5 discusses future directions for this research.

1. **Definition in this paper**

In this section, we explain about definition in this paper. This definition is primarily based on references [1,2].  
  
**2.1. Basic Notation**  
 The basic notation used in this paper is outlined below. Readers interested in the fundamentals of set theory may refer to references [94–96] as needed.

**Definition 1 [94-96]:** In set theory, a set is a collection of distinct elements or objects, regarded as a single entity and typically represented using curly braces. In this paper, we consider only finite and undirected sets. And we consider only simple sets (i.e., sets that are not multisets).　And a subset is a set where all elements are also contained within another set. And the **power set** of a set is the collection of all possible subsets of that set, including the empty set and the set itself.  
**Notation 2 [94-96]:** Boolean algebra *(X, ∪, ∩)* is a mathematical structure with a set *X*, union *(∪*), and intersection (*∩*), satisfying specific axioms for operations. In this paper, we use expressions like *A ⊆ X* to indicate that *A* is a subset of *X, A ∪ B* to represent the union of two subsets *A* and *B* (both of which are subsets of *X*), and *A = ∅* to signify an empty set. Specifically, *A ∩ B* denotes the intersection of subsets *A* and *B*. Similarly, *A \ B* represents the difference between subsets *A* and *B*.

**2.2. Matroid and Filter**In this section, we introduce definitions related to matroids and filters, along with relevant concepts. Below, we present key definitions and examples. As mentioned in the introduction, a matroid is a combinatorial structure that generalizes linear independence in vector spaces, satisfying heredity and exchange properties. It is widely applied in optimization and graph theory. Within this framework, the concept of a filter is defined [1, 2]. Moreover, this paper deals with open sets, which have been extensively studied in various contexts (e.g., [100–103]).

**2.2.1 Definition of Matroids**First, we provide the general definition of a finite matroid. Notably, when axioms (MB1) and (MB2) are considered together, they define a combinatorial structure known as an independence system, which is also referred to as an abstract simplicial complex over a Boolean algebra. Note that in this paper, we consider only finite and undirected matroids (not ditroid [104]). **Definition 3 [4]:** In a finite Boolean algebra *(X,∪,∩)*, the set family *M ⊆2X* is called a (finite) matroid if the following axioms hold true:  
(MB1) *∅ ∈ M*,  
(MB2) if *A ∈ M* and *B ⊆ A* then *B ∈ M*,  
(MB3) if *A, B ∈ M, |A| < |B|* then there exists *e ∈ B \ A* such that *A ∪ {e} ∈ M*.

In this paper, we deal with a definition based on an open set different from the one described above. The definition is provided below.

**Definition 4 [1, 2]:** A matroid, denoted by M, is defined as an ordered pair *(X, O)*, where *X* is a finite and undirected set and *O* is a collection of subsets of *X,* called the open sets of *M*. These open sets satisfy the following properties:

1. The empty set is an open set.
2. The union of any open sets is an open set.
3. If O₁ and O₂ are open sets and an element e belongs to *O₁ ∪ O₂*, then there exists an open set *O₃* such that *(O₁ ∪ O₂) \ (O₁ ∩ O₂) ⊆ O₃ ⊆ (O₁ ∪ O₂) \ {e}.*

In other words, **open sets i**n *M* are precisely the members of *O,* which is a family of subsets required to include the empty set, remain closed under unions, and satisfy the additional condition involving *O3​.*  
We provide several concrete examples related to the above definition.

**Example 5:** Let X be any finite set and define the collection of open sets as  
*O = {∅}.*  
Here, the empty set is open by definition, the union of open sets (in this case*, ∅ ∪ ∅*) is *∅*, and property 3 of a matroid is vacuously satisfied because there is only one open set. Thus, *(X, {∅})* forms a (degenerate) matroid according to Definition.

**Example 6:** Let *X = {a}.* The power set of *X* is *{∅, {a}},* but note that if we try to take *O = {∅, {a}},* then property 3 fails. To see this, consider *O₁ = ∅* and *O₂ = {a}:*

* *O₁ ∪ O₂ = {a}* and *O₁ ∩ O₂ = ∅*, so the symmetric difference is *{a}.*
* For the element *e = a* (which belongs to *O₁ ∪ O₂*), property 3 would require an open set *O₃* satisfying  
   *{a} ⊆ O₃ ⊆ {a} \ {a} = ∅,*  
  which is impossible.  
  Thus, for *X = {a}*, the only valid choice is to let  
    *O = {∅},*  
  yielding the trivial matroid *(X, {∅}).*

**Example 7:** Let *X = ∅.* Since the only subset of the empty set is *∅* itself, define  
  *O = {∅}.*All conditions in Definition are satisfied (in a vacuous sense), so *(∅, {∅})* is a valid matroid.

**2.2.2. Metric for matroid**

We now introduce the definition of a metric, following the approach in [1,2]. Note that a **metric** is a function that defines distances between elements in a set, satisfying non-negativity, symmetry, and the triangle inequality (cf. [97-99]). throughout this paper, we assume X to be the underlying finite set.

**Definition 8 [1,2] (Metric on *X*):** Let *M = (X, S)* be a matroid (where *S* denotes the collection of open sets). A metric on *X* is a function *d : X × X → ℝ* that satisfies the following properties:

1. **Non-negativity:** For all *x, y ∈ X, d(x, y) ≥ 0*, and *d(x, y) = 0* if and only if *x = y.*
2. **Symmetry:** For all *x, y ∈ X, d(x, y) = d(y, x).*
3. **Triangle Inequality:** For all *x, y, z ∈ X, d(x, y) + d(y, z) ≥ d(x, z).*

Here are the examples for a metric on *X.*

**Example 9:** Consider a set *X = {a, b, c}*. Let d be a function defined as follows:

*d(a, a) = 0,d(a, b) = 2,d(a, c) = 3,d(b, b) = 0,d(b, c) = 1,d(c, c) = 0*

This function d is a metric on X because:

1. Non-negativity: *d(x, y) ≥ 0* for all *x, y ∈ X* and *d(x, y) = 0* if and only if *x = y.*
2. Symmetry: *d(x, y) = d(y, x)* for all *x, y ∈ X.*
3. Triangle Inequality: For all *x, y, z ∈ X, d(x, y) + d(y, z) ≥ d(x, z)*. For example, *d(a, b) + d(b, c) = 2 + 1 = 3 ≥ d(a, c) = 3.*

**2.2.3. Filter in matroids**  
Before introducing filters and ultrafilters in matroids, we first present the general definitions of filters and ultrafilters. The definition of a filter in a Boolean algebra *(X,∪,∩)* is given below. As mentioned in the introduction, Filters and Ultrafilters are fundamental concepts in mathematics.　The complement of an filter in a Boolean algebra *(X,∪,∩)* is referred to as an ideal in a Boolean algebra *(X,∪,∩)*.  
**Definition 10[24]:** In a Boolean algebra *(X,∪,∩)*, a set family *F ⊆ 2X* satisfying the following conditions is called a filter on the carrier set *X*.

(FB1) *A, B ∈ F* ⇒ *A ∩ B ∈ F,*

(FB2) *A ∈ F, A ⊆ B ⊆ X* ⇒ *B ∈ F,*

(FB3) *∅* is not belong to *F*.

In a Boolean algebras *(X,∪,∩)*, A maximal filter is called an ultrafilter and satisfies the following axiom (FB4):

(FB4) *∀A ⊆ X, either A ∈ F or X / A ∈ F.*

In this paper, we examine the extension of the above general definition of a filter to matroids. We define a filter in matroid. This definition is primarily based on references [1,2]. Note that Co-filter in matroid is ideal in matroid.

**Definition 11 [1,2]:** Let *M = (X,S)* be a matroid and *d* be a metric on *X*. For a given real number *r≥ 0*, the set *Bd(e,r) = { x ∈ X | d(x,e) < r }*  is termed as the *r*-ball centered at *e*. A filter in *M = (X, S)* is a collection *F = { Aα |α∈ Δ}* of subsets of a finite set X that satisfy these properties:

(F1) *Aα≠ ∅* for every *α∈ Δ*

(F2) For all *α, β ∈ Δ* , *Aα ∩ Aβ∈ F*.

(F3) If *Aα∈ F* and *Aα ⊆ A ⊆ X*, then *A ∈ F*.  
  
Here are the examples for filter on a matroids.

**Example 12:** Consider the set *X = {a, b, c}* and the metric d defined as above. Let *r = 2.* The r-balls centered at each element are:

*Bd(a, 2) = { a, b },Bd (b, 2) = { b, c },Bd (c, 2) = { c }*

Now, consider a collection *F = { A1, A2 }* where:

*A1 = { a, b },A2 = { b, c }*

This collection F is a filter in *M = (X, S)* because:

1. *A1*≠ ∅ and *A2* ≠ ∅ (each *Aα ≠ ∅* for every *α ∈ Δ*).
2. *A1 ∩ A2 = { b } ∈ F* (the intersection of any two sets in *F* is also in *F*).
3. If A*α ∈ F* and *Aα ⊆ A ⊆ X*, then *A ∈ F.* For instance, *A1 ⊆ { a, b, c } ⊆ X* and *{ a, b, c }∈ F.*
4. **Main result: Ultrafilter on Matroid**

In this section, we present the results of this paper. We consider about new notion of ultrafilter in *M = (E, S)*. An ultrafilter in *M = (E, S)* is a maximal filter in *M = (E, S)*. We prove following theorem.

**Theorem 13:** Let *M = (X,S)* be a finite matroid and *d* be a metric on a finite set *X*. Ultrafilter *F = { Aα |α∈ Δ}*  in *M = (X, S)* satisfies following axiom

(F4) For all *A⊆ X,* either *A∈ F* or *X\ A∈ F*

**Proof.** Let *F* be an ultrafilter in *M = (X, S)*. We want to show that for any subset *A ⊆ X,* either *A* is in *F* or the complement *X \ A*  is in *F*.

Let's consider an arbitrary subset *A ⊆ X*.

**Case 1: *A ∈ F*** If *A* is already in *F,* then the condition of axiom (F4) is directly satisfied.

**Case 2: *A ∉ F***  Assume, for the sake of contradiction, that neither A nor its complement *X \ A*  belong to *F.*

Given that *F* is a filter, we want to consider another filter *G* that contains F as a subset and includes *A.*

To construct such a G, start with the collection: *G' = F* ∪ {A}. Extend G' by including all possible intersections of its elements until no more intersections can be added while maintaining the properties of a filter. This results in a filter *G* such that  *F ⊆ G* and *A ∈ G.*

The existence of such a *G i*s assured since *F* is not maximal with respect to *A* (from our assumption). However, this contradicts the definition of *F* as an ultrafilter (a maximal filter). Hence, our assumption is false.

Thus, if *A ∉ F*, then *X\A* must be in *F*. This concludes the proof. ■

From Theorem above, the following Definition can be redefined.

**Definition 14:** Let *M = (X,S)* be a matroid and *d* be a metric on *X*. For a given real number *r≥ 0*, the set *Bd(e,r) = { x ∈ X | d(x,e) < r }*  is termed as the *r*-ball centered at *e*. A ultrafilter in *M = (X, S)* is a collection *F = { Aα |α∈ Δ}* of subsets of a finite set X that satisfy these properties:

(F1) *Aα≠ ∅* for every *α∈ Δ*

(F2) For all *α, β ∈ Δ* , *Aα ∩ Aβ∈ F*.

(F3) If *Aα∈ F* and *Aα ⊆ A ⊆ X*, then *A ∈ F*.  
(F4) For all *A⊆ X,* either *A∈ F* or *X\ A∈ F*

**4. Other concepts: Quasi Ultrafilter and weak Ultrafilter in matroid**  
In this section, we introduce two concepts: the Quasi-Ultrafilter and the Weak Ultrafilter.

**4.1. Quasi-ultrafilter on a matroid**  
In reference [51], the idea of a Quasi-Ultrafilter is presented. This literature also delves into an axiomatic analysis of incomplete social judgments. The Quasi-Ultrafilter plays a crucial role in the proofs found in reference [51]. Furthermore, the Quasi-Ultrafilter serves as a fundamental tool for examining incomplete social judgments axiomatically (see [53-59,107]).

First, we provide the general definition of a Quasi-Ultrafilter on finite Boolean algebras below.

**Definition 15** [51]**:** In a finite Boolean algebra *(X,∪,∩)*, a set family *Q ⊆ 2X* satisfying the following conditions is called a Quasi-Ultrafilter on the carrier set *X*.

(QB1) A*⊆ X*, B*⊆ X , A∉ Q , B ∉ Q* ⇒ *A ∪ B ∉ Q,*

(QB2) *A ∈ Q, A ⊆ B ⊆ X* ⇒ *B ∈ Q,*

(QB3) *∅* is not belong to *Q*.

(QB4) *∀A ⊆ X, either A ∈ Q or X / A ∈ Q.*

In this paper, we extend the above concept to a quasi-ultrafilter on a matroid. The definition is presented below.  
**Definition 16:** Let *M = (X,S)* be a matroid and *d* be a metric on *X*. For a given real number *r≥ 0*, the set *Bd(e,r) = { x ∈ X | d(x,e) < r }*  is termed as the *r*-ball centered at *e*. Quasi filter *F = { Aα |α∈ Δ}* satisfies axioms (F1), (QF2), (F3). Quasi Ultrafilter satisfies axioms (F1), (QF2), (F3), (F4). Axiom (QF2) is following:  
(QF2)*α, β ∈ Δ*, *Aα ⊆ X , Aβ ⊆ X , Aα∉ F, Aβ∉ F → Aα∪Aβ ∉ F.*

**4.2. Weak-ultrafilter on a matroid**  
A weak filter represents a relaxed form of the traditional filter definition and is utilized in domains like logic, as cited in reference [52,60-63].

First, we present the general definition of a weak ultrafilter in set theory and algebra. The definition of a weak ultrafilter in a finite Boolean algebra (X,∪,∩) is provided below.  
**Definition 17 [52]:** In a finite Boolean algebra *(X,∪,∩)*, a finite set family *F ⊆ 2X* satisfying the following conditions is called a weak filter on the carrier set X.(WBF1) *A, B ∈ F* ⇒ *A ∩ B ≠ ∅*(WBF2) *A ∈ F, A ⊆ B ⊆ X* ⇒ *B ∈ F,*(WBF3) *∅* is not belong to *F*.

In a finite Boolean algebra *(X,∪,∩)*, A weak ultrafilter is called an weak ultrafilter and satisfies the following axiom (WBF4):

(WBF4) *∀A ⊆ X, either A ∈ F or X / A ∈ F.*

We now extend the above definition of a weak ultrafilter to matroids. The definition of a weak ultrafilter with respect to a matroid is presented below.  
**Definition 18:** Let *M = (X,S)* be a finite matroid and *d* be a metric on *X*. For a given real number *r≥ 0*, the set *Bd(e,r) = { x ∈ X | d(x,e) < r }*  is termed as the *r*-ball centered at *e*. Weak filter *F = { Aα |α∈ Δ}* satisfies axioms (F1), (WF2), (F3). Weak Ultrafilter satisfies axioms (F1), (WF2), (F3), (F4). Axiom (WF2) is following:  
(WF2) For all *α, β ∈ Δ* , *Aα ∩ Aβ ≠ ∅*.

**4.3. Theorem of Maximal Quasi-filters and Maximal Weak-filters**

We briefly examine the properties of Maximal Quasi-filters and Maximal Weak-filters in the following theorem.

**Theorem 19:** Let *M = (X,S)* be a matroid and *d* be a metric on a finite set *X*. Maximal Quasi-filter/Maximal Weak-filter *F = { Aα |α∈ Δ}* in *M = (X, S)* satisfies following axiom (F4) For all *A⊆ X*, either *A∈ F* or *X\ A∈ F*  
**Proof.** Let *M=(X,S)* be a matroid and let *d* be a metric on the finite set *X*. Assume that *F* is a maximal quasi-filter (or maximal weak-filter) on *M;* that is, *F* satisfies the filter axioms (F1) and (F3) together with either (QF2) (in the quasi-filter case) or (WF2) (in the weak-filter case), and no proper extension of *F* satisfies these axioms. We want to prove that *F* also satisfies axiom (F4):

(F4) For all *A⊆X,* either *A∈F o*r *X∖A∈F*

Let *A⊆X* be an arbitrary subset. There are two cases to consider:

* **Case 1:** *A∈F*  
  In this case, (F4) is trivially satisfied.
* **Case 2:** *A∉F*We now show that under this assumption, it must follow that X∖A∈F.

Assume, for the sake of contradiction, that *A∉F* and also *X∖A∉F.*Since *F* is a filter (satisfying (F1) and (F3)) but neither *A* nor *X∖A* is in *F,* consider the possibility of extending *F* by adding *A* to it.  
Define a new collection *G′=F∪{A}.*

Then, by applying the closure property required by axiom (F3) (i.e., if *B∈F* and *B⊆C⊆X*, then *C* must be in the filter), along with the properties (QF2) or (WF2) as appropriate, one can generate a new quasi-filter (or weak-filter) *G* that includes all sets from *G′* and is closed under the necessary operations (for example, taking intersections or unions as required).  
By construction, *G* is a filter that properly extends *F* because *A∈G* while *A∉F.* This contradicts the maximality of *F,* which by definition cannot be extended any further while preserving the filter properties.  
The contradiction shows that our assumption in **Case 2** (that both *A∉F* and *X∖A∉F*) must be false. Therefore, if *A∉F,* it must be that *X∖A∈F.*

Thus, for every subset *A⊆X,* either *A∈F* or *X∖A∈F.* This completes the proof of axiom (F4) for any maximal quasi-filter (or maximal weak-filter) *F* on *M.* This concludes the proof. ■  
  
**5. Conclusion and Future Work**

This paper investigates the role of ultrafilters in the context of matroids, analyzing their structural properties and characterizing their behavior within the matroid framework. Specifically, we extended the concepts of ultrafilters, weak-ultrafilters, and quasi-ultrafilters to matroids and examine their properties, including their maximality.

For future research, we plan to explore the concept of ideals in matroids, which serve as co-filters in this setting. Additionally, we aim to investigate potential applications of ultrafilters on matroids in computer science, graph theory, and linear algebra.

Furthermore, we intend to extend the notion of ultrafilters in matroids to various advanced set structures, including Fuzzy Sets [75–77], Hyperfuzzy Sets [78–80], Neutrosophic Sets [81–83], HyperNeutrosophic Sets [84–86], Plithogenic Sets [87–90], and HyperPlithogenic Sets [91–93], examining their fundamental properties and possible applications. These set-theoretic frameworks are particularly effective in handling real-world phenomena characterized by uncertainty. By extending the concepts studied in this paper to these generalized set structures, we believe it may be possible to develop new applications and modeling approaches that better capture the complexity of uncertain systems.

**Data Availability**

This research is purely theoretical, involving no data collection or analysis. We encourage future researchers to pursue empirical investigations to further develop and validate the concepts introduced here.

**Conflict of Interest Statement**

The author declares no conflicts of interest.

**Ethical Approval**

As this research is entirely theoretical in nature and does not involve human participants or animal subjects, no ethical approval is required.

**Disclaimer**

This work presents theoretical concepts that have not yet undergone practical testing or validation. Future researchers are encouraged to apply and assess these ideas in empirical contexts. While every effort has been made to ensure accuracy and appropriate referencing, unintentional errors or omissions may still exist. Readers are advised to verify referenced materials on their own. The views and conclusions expressed here are the authors’ own and do not necessarily reflect those of their affiliated organizations.

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