**Short Research Article**

**Discussion on Maximal Edge-Ideal in Graph Theory**

**Abstract:** The exploration of width parameters within the fields of graph theory and algebra has garnered significant interest. Among these parameters, tree-cut decomposition stands out as a vital metric. The "Edge-Tangle" concept is intrinsically linked to the width parameter known as "tree-cut width" in graph theory. In this paper, we introduce a new definition termed Maximal Edge-Ideal for graphs and demonstrate their equivalence to Edge-Tangles.

**Keyword:** Ideal, tangle, edge-tangle, tree-cut-width, tree-cut-decomposition

1. **Introduction**

**1.1. Graph and Graph Parameters**

A **graph** consists of vertices and edges [63]. Graphs are studied for applications in networks, artificial intelligence, and various other fields [65, 66]. A **graph parameter** is a numerical invariant measuring structural properties. In recent years, research interest in width parameters within graph theory and algebra has significantly increased [1-33]. These width parameters are metrics derived from tree-like structures, commonly studied through graph decompositions. Well-known examples of width parameters include tree-width, branch-width [12], path-width [10], proper-path-width [86-88], hypertree-width [47,48], superhypertree-width [52,53], cut-width [67,68], linear-width [5,14], modular-width [54,55], Boolean-width [56,57], Band-width [58,59,61], Twin-width [82-85], and clique-width [60,89,90].

A key topic of study in this field is the concept of obstructions, which play a crucial role in determining width parameters. Notable examples include tangles [17], brambles [75, 76], ultrafilters [17, 77], blockages [64], and linear tangles [8]. In particular, the concept of an edge-tangle is closely related to the width parameter known as tree-cut width in graph theory [1]. These concepts are frequently used in the analysis of efficient algorithms and the exploration of mathematical structures in graph theory.

**1.2. Ideal in set theory and Graph Theory**

In set theory, an ideal is a collection of subsets of a given set that is closed under taking subsets and finite unions while excluding the entire set. The complementary structure to an ideal is called a filter, and similar concepts have been explored in graph theory as well. The concepts of Ideals and Filters have been widely studied across various fields [69-74].

The notion of maximality is frequently discussed in the context of ideals. A Maximal Ideal is an ideal that cannot be extended further without becoming the entire set. Its complement is called an Ultrafilter. These structures are known to possess various intriguing mathematical properties.

**1.3. Our Contribution of this paper**

This paper explains its contributions. As stated in the introduction, research on Ideals and Graph Width Parameters is significant. However, the relationship between Ideals and Graph Width Parameters has not been extensively explored.

In this paper, we introduce a new concept called Maximal-Edge-Ideal for graphs and demonstrate its connection to Edge-Tangles. Additionally, we briefly examine the maximality of Edge-Ideals and their structural similarities to traditional Ideals. Although the novelty of this work may be limited, we hope that it will make a modest contribution to the ongoing research on graph width parameters.

2. **Definitions and Notations in this paper**

This section provides the mathematical definitions of each concept. First, we briefly explain the notations used in this paper. Readers who wish to review the fundamentals of set theory and graph theory may refer to references [62,63] as needed.

**Notation 1[62,63]:** In this short paper, we use expressions like A ⊆ X to indicate that *A* is a subset of *X,* *A ∪ B* to represent the union of two subsets *A* and *B,* both of which are subsets of *X,* or *A = ∅*  to signify an empty set. Specifically, *A ∩ B* denotes the intersection of subsets *A* and *B*. A similar logic applies to *A \ B.* The powerset of a set *A*, denoted as *2A*, is the set of all possible subsets of *A*, including the empty set and *A* itself.  
  
 **Notation 2[62,63]:** Let *G* be a finite and undirected graph. The notation *V(G)* represents the set of vertices (nodes) in *G*, and *E(G)*  represents the set of edges in *G* . The expression *G = (V, E)* signifies that *G* is a graph defined by a pair of sets: *V* for vertices and *E* for edges. In this paper, we focus on the properties of undirected, finite, and simple graphs.

**Notation 3:** A natural number is a positive integer used for counting and ordering [91].In this paper, we utilize the natural number *k*.

Next, we briefly explain the definition of an edge-cut. This definition will be used later in the discussion of edge-tangles and edge-ideals.

**Definition 4 [1]:** An edge-cut *[A, B]* of a finite and undirected *G* is an ordered pair of disjoint subsets of *V(G)* such that *A ∪ B = V (G)*. And the order of an edge-cut *[A, B]* of G is the number of edges of *G* with one end in *A* and one end in *B* (cf: [1]).

For clarity, an example is provided below.

**Example 5:** Let finite and undirected graph *G = (V, E)* be defined as follows:

*V = {a, b, c, d}, E = {{a, b}, {b, c}, {c, d}, {d, a}, {a, c}}*

Define the subsets: *A = {a, b} B = {c, d}.*

These subsets satisfy:

*A ∩ B = ∅* (they are disjoint), *A ∪ B = V* (their union covers all vertices).

According to Definition 4, the edge-cut *[A, B]* is the ordered pair *(A, B)* and its order is the number of edges with one endpoint in *A* and the other in *B.*

Now, inspect each edge:

* Edge *{a, b}*: Both endpoints are in *A* — not part of the cut.
* Edge *{b, c}: b* *∈* A and c *∈* *B* — included in the cut.
* Edge *{c, d}:* Both endpoints are in *B* — not part of the cut.
* Edge *{d, a}: d ∈ B* and *a ∈ A* — included in the cut.
* Edge *{a, c}: a ∈ A* and *c ∈ B* — included in the cut.

Thus, the edge-cut *[A, B*] contains the edges: *{b, c}, {d, a}, {a, c},* and its order is 3 (since there are 3 edges crossing from *A* to B).

**2.1 Ideals on Boolean Algebras**

We provide an explanation of Ideals in Boolean Algebras.   
 **Definition 6:** In a Boolean algebra *(X,∪,∩)*, a set family *I ⊆ 2X* satisfying the following conditions is called an ideal on the carrier set *X*.

(IB1) *A, B ∈ I*  ⇒ *A ∪ B ∈ I (*Closure under Union*),*

(IB2) *B ∈ I, A ⊆ B ⊆ X* ⇒ *A∈ I (*Closure under Superset*),*

(IB3) *X i*s not belong to *I (*Exclusion of the Universal Set*)*.

In a Boolean algebras *(X,∪,∩)*, A maximal ideal satisfies the following axiom (IB4):

(IB4) *∀A ⊆ X, either A ∈ I or X / A ∈ I*

For clarity, an example is provided below.

**Example 7:** Let the finite carrier set be *X = {1, 2, 3}* and consider its power set *2X={∅,{1},{2},{3},{1,2},{1,3},{2,3},{1,2,3}}.* Define *I = {∅, {1}, {2}, {1,2}}.*

**Verification.**

* **(IB1) Closure under Union:**
  + For instance, *{1}∪{2}={1,2}∈I.*
  + Also, *∅∪{1}={1}∈I*
* **(IB2) Downward Closure:**
  + Any subset of an element of I is also in I. For example, since *{1,2}∈I* and *{1}⊂{1,2},* we have *{1}∈I.*
  + Clearly, *∅* is a subset of every element in I and is itself in *I.*
* **(IB3) Exclusion of the Universal Set:**
  + The full set *X = {1,2,3}* is not in *I.*

Thus, *I* satisfies all the ideal conditions on the Boolean algebra *(2X, ∪, ∩).*

**2.2　Edge*-*Tangle on the finite graph**  
We describe the concept of an edge-tangle in graphs. Below, we provide the definition of a tangle within graph theory. Edge-Tangles are renowned for their profound association with tree-cut decompositions [1]. These decompositions have been extensively studied by various researchers [24-31].  
 **Definition 8 [1, 23]:** Let *G* be a finite and undirected graph. A *edge-tang*le *E* of order *k* is a set of edge-cuts of G such that the following hold.   
(E1) For all edge-cut *[A, B]* of *G* of order less than *k*, either *[A, B] ∈ E* or *[B, A] ∈ E* .  
(E2) If *[A1, B1],[A2, B2],[A3, B3] ∈ E* then *A1 ∪ A2 ∪ A3 ≠ G.*  
(E3) If *[A, B] ∈ E*, then G has at least k edges incident with vertices in *B*.

For clarity, an example is provided below.  
**Example 9:** Let *G = K8​,* the complete graph on *8* vertices *V(G)={v1,v2,…,v8},* and choose *k=8.*  
Note that a complete graph *Kn* is a graph where every pair of distinct vertices is connected by an edge [79-82]. Select a fixed subset (our “central” region) as *R={v1, v2, v3}.*

We assume that in every edge-cut of *G* of order less than *8,* the set *R* is not split (this is a natural assumption in highly connected graphs such as *K8*).  
Define the edge-tangle of order *8* as:

*E={[A,B]  :  [A,B]*is an edge-cut of *G*with order*<8* and *R⊆B*}.

That is, for every edge-cut *[A,B]* with fewer than *8* crossing edges, we “orient” it by declaring the side that contains *R* to be *B.*

**Verification of Edge-Tangle Properties.**

* **(E1) Orientation of Small Edge-Cuts:**  
  For every edge-cut *[A,B]* of order less than *8,* because *G* is highly connected, the cut cannot split *R* (i.e. either *R⊆A* or *R⊆B).* By our rule, we include the separation for which *R⊆B.* Thus, exactly one of *[A,B]* or *[B,A]* belongs to *E.*
* **(E2) Non-Covering Condition:**  
  Suppose *[A1,B1], [A2,B2],* and *[A3,B3]* are in *E*. Then each *Bi* contains *R*. Hence, *A1∪A2∪A3⊆V(G)∖R≠V(G),* ensuring that the union of the “small” sides does not cover the entire vertex set.
* **(E3) Connectivity Condition:**  
  For any *[A,B]∈E*, since *R⊆B* and *∣R∣=3*, consider a worst-case scenario when *B=R*. In *K8​,* each vertex has degree 7; hence, if *B={v1,v2,v3}* then the number of edges from *B* to *A=V(G)∖B* is *∣B∣×(8−∣B∣)=3×5=15*, which is at least *k=8.*

Thus, *E* is an edge-tangle of order *8* on *K8​.*

**3. Some Properties of Maximal Ideals of Edge-Cuts: Their Relation to Tree-Cut Decomposition**  
The definition of a Maximal Edge-ideal on the graph is given below. We naturally extend the definition from Boolean algebras to a set of edge-cuts.

**Definition 10:** Let *G* be a finite and undirected graph. An Edge-ideal *I* of order *k* is a set of edge-cuts of G such that the following hold.

(I1) *[A2, B2] ∈ I, A1 ⊆ A2 , [A1, B1]* of order less than *k*⇒ *[A1, B1] ∈ I*,

(I2) *[A1, B1]* *∈ I, [A2, B2] ∈ I*, *[A1∪A2, B1∩ B2 ]* of order less than *k*

⇒ *[A1 ∪ A2, B1∩ B2 ] ∈ I*,

(I3)If *[A, B] ∈ I,* then *G* has at least *k* edges incident with vertices in *B.*(I4) *If V(A) ＝ V(G), then [A, B] ∈I.*

A Maximal Edge-ideal satisfies the following additional axiom (I5):  
For all edge-cut *[A, B]*  of *G* of order less than *k*, either *[A, B] ∈ I* or *[B, A] ∈ I*

For clarity, an example is provided below.  
**Example 11:** Let *G = K8​,* the complete graph on 8 vertices *V(G)={v1,v2,…,v8},* and choose *k=8.*  
Note that a complete graph *Kn* is a graph where every pair of distinct vertices is connected by an edge [79-82]. Select a fixed subset (our “central” region) as *R = {v1, v2, v3}.*

Define *I={[A,B]  :  [A,B]*is an edge-cut of *G* with order*<8* and*R⊆B}*.

This is exactly the same collection as the tangle E above, but we now view it as an “ideal” with additional closure properties.

**Verification of Edge-Ideal Properties.**

* **(I1) Downward Closure:**  
  If *[A2,B2]∈I* (so *R⊆B2​*) and *A1⊆A2* ​, then the complementary side satisfies *B1=V(G)∖A1⊇V(G)∖A2=B2​.* Hence, *R⊆B1* ​ and so *[A1,B1]∈I.*
* **(I2) Closure under “Union”:**  
  Let *[A1,B1]∈I* and *[A2,B2]∈I.* Then *R⊆B1 ​* and *R⊆B2* imply *R⊆B1∩B2​.* If the edge-cut *[A1∪A2, B1∩B2]* has order less than *8,* then by definition it belongs to *I.*
* **(I3) Connectivity Condition:**  
  For any *[A,B]∈I,* since *R⊆B* and, as calculated earlier, even the smallest possible *B* (namely *R*) yields *∣R∣×(8−∣R∣)=3×5=15≥8,* the condition that “*G* has at least *k* edges incident with vertices in *B*” is satisfied.
* **(I4) Inclusion of Trivial Edge-Cuts:**  
  By convention, if an edge-cut is “trivial” (for example, if *A=V(G)* so that *B=∅)*, it is included in the ideal.
* **(I5) Maximality:**  
  For every edge-cut *[A,B]* of order less than *8,* the highly connected nature of *K8​* ensures that *R* is not split; that is, either *R⊆B* or *R⊆A.* In the first case, *[A,B]∈I;* in the second, its complement *[B,A]* satisfies *R⊆A* and would be in the ideal if we had chosen the opposite orientation. Thus, *I* is maximal in the sense that for every small edge-cut one of the two orientations is included.

Therefore, *I* is a maximal edge-ideal of order *8* in *G.*  
Proving the Main Theorem of this paper, which establishes the　equivalence between Maximal Edge-ideal and Edge-Tangles.  
**Theorem 12:** Let *G* be a finite and undirected graph. *E* is anedge-Tangle of separations of order *k* in graph if and only if *E* is a Maximal Edge-ideal of separations of order *k* in graph.  
**Proof.** (⇒) Assume that *E* is an edge-tangle of separations of order *k* in *G.*

We show that *E* satisfies the axioms (I1)–(I4) for an edge-ideal and, moreover, the maximality condition (I5).

* (I1) Downward Closure: Let *[A2, B2] ∈ E* and suppose *A1 ⊆ A2* with *[A1, B1]* an edge-cut of order less than k. By the tangle property (E1), either *[A1, B1] ∈ E* or *[B1, A1] ∈ E.*
* (I2) Closure under 'Union': Let *[A1, B1] ∈ E* and *[A2, B2] ∈ E*. If *[A1 ∪ A2, B1 ∩ B2]* is an edge-cut of order less than *k,* then *[A1 ∪ A2, B1 ∩ B2] ∈ E.*
* (I3) Connectivity Condition: If *[A, B] ∈ E,* then *G* has at least *k* edges incident with vertices in B.
* (I4) Inclusion of Trivial Edge-Cuts: If *V(A) = V(G)*, then *[A, B]* is included in *E.*
* (I5) Maximality: For every edge-cut *[A, B]* of order less than *k,* either *[A, B] ∈ E* or *[B, A] ∈ E.*

Thus, if *E* is an edge-tangle, then *E* is a maximal edge-ideal.

(⇐) Conversely, assume that *E* is a maximal edge-ideal, that is, *E* satisfies axioms (I1)–(I4) and the maximality condition (I5). We show that *E* is an edge-tangle.

* Edge-Cut Orientation (E1): Axiom (I5) states that for every edge-cut *[A, B]* of order less than *k*, either *[A, B] ∈ E* or *[B, A] ∈ E.*
* Non-Covering Condition (E2): Suppose for contradiction that there exist edge-cuts *[A1, B1], [A2, B2], [A3, B3] ∈ E* such that *A1 ∪ A2 ∪ A3 = V(G).* This contradicts the connectivity condition (I3).
* Connectivity (E3): Axiom (I3) of the edge-ideal directly provides the connectivity requirement (E3).

Thus, *E* satisfies all the conditions for being an edge-tangle. This completes the proof of the Theorem.

**Theorem 13:**  Let *G* be a finite and undirected graph. The following holds.  
(i) An Edge-Ideal *I* of order *k* in a graph *G* possesses the structure of an ideal analogous to ideals in Boolean algebras.  
(ii) Moreover, a maximal Edge-Ideal satisfies the additional axiom (I5).  
  
**Proof:** Each can be proved as follows.

(i) Edge-Ideal as an Ideal Structure:  
Downward Closure (I1): If *[A2, B2] ∈ I* and *A1 ⊆ A2,* then *[A1, B1] ∈ I.*  
Closure under 'Union' (I2): If *[A1, B1]* and *[A2, B2]* are in *I,* then *[A1 ∪ A2, B1 ∩ B2] ∈ I.*Connectivity Condition (I3): If *[A, B] ∈ I,* then *G* has at least *k* edges incident with vertices in *B.*Trivial Edge-Cut Inclusion (I4): If *V(A) = V(G),* then *[A, B] ∈ I.*(ii) Maximal Edge-Ideals Satisfy Axiom (I5):  
By definition, a maximal Edge-Ideal is one that cannot be extended further without violating (I1)–(I4). For every edge-cut *[A, B]* of order less than *k*, if *[A, B]* is not in *I,* then its complement *[B, A]* must be in *I.*  
Thus, a maximal Edge-Ideal not only has the ideal structure but also satisfies the additional maximality condition (I5). This completes the proof of the Theorem.  
  
**4. Conclusion and Future Direction of This Research**

In this paper, we explored the characteristics of Edge-Ideals as an extension of traditional Ideals. As a future research direction, we aim to extend this concept to Hypergraphs[37-39], SuperHyperGraphs[34-36], Fuzzy Graphs[40-42], Neutrosophic Graphs[43-46], and Plithogenic Graphs[49-51], further investigating their properties and potential applications.

**Data Availability**

This research is purely theoretical, involving no data collection or analysis. We encourage future researchers to pursue empirical investigations to further develop and validate the concepts introduced here.

**Ethical Approval**

As this research is entirely theoretical in nature and does not involve human participants or animal subjects, no ethical approval is required.

**Disclaimer**

This work presents theoretical concepts that have not yet undergone practical testing or validation. Future researchers are encouraged to apply and assess these ideas in empirical contexts. While every effort has been made to ensure accuracy and appropriate referencing, unintentional errors or omissions may still exist. Readers are advised to verify referenced materials on their own. The views and conclusions expressed here are the authors’ own and do not necessarily reflect those of their affiliated organizations.

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Author(s) hereby declare that NO generative AI technologies such as Large Language Models (ChatGPT, COPILOT, etc.) and text-to-image generators have been used during the writing or editing of this manuscript.

## **Reference**

1. Liu, Chun-Hung. "A global decomposition theorem for excluding immersions in graphs with no edge-cut of order three." Journal of Combinatorial Theory, Series B 154 (2022): 292-335.
2. Fujita, T. (2025). Superhypertree-length and superhypertree-breadth in superhypergraphs. *Advancing Uncertain Combinatorics through Graphization, Hyperization, and Uncertainization: Fuzzy, Neutrosophic, Soft, Rough, and Beyond*, 41.
3. Amini, Omid, et al. "Submodular partition functions." Discrete Mathematics 309.20 (2009): 6000-6008.
4. Diestel, Reinhard, et al. "Duality and tangles of set separations." arXiv preprint arXiv:2109.08398 (2021).
5. Fujita, T. (2025). Bounding Linear-width and Distance-width Using Feedback Vertex Set and MM-width for Graph. *Journal of Fundamental Mathematics and Applications (JFMA)*, *8*(1), 33-50.
6. Fedor V Fomin and Dimitrios M Thilikos. On the monotonicity of games generated by symmetric submodular functions. Discrete Applied Mathematics, Vol. 131, No. 2, pp. 323–335, 2003.
7. Fujita, T. (2025). Short note of supertree-width and n-superhypertree-width. *Neutrosophic Sets and Systems*, *77*, 54-78.
8. Fujita, Takaaki. "Alternative Proof of Linear Tangle and Linear Obstacle: An Equivalence Result." Asian Research Journal of Mathematics 19.8 (2023): 61-66.
9. Yolov, N. (2018). Minor-matching hypertree width. In *Proceedings of the Twenty-Ninth Annual ACM-SIAM Symposium on Discrete Algorithms* (pp. 219-233). Society for Industrial and Applied Mathematics.
10. Daniel Bienstock. Graph searching, path-width, tree-width and related problems (a survey). In Reliability of Computer and Communication Networks, Vol. 5 of DIMACS. Series in Discrete Mathematics and Theoretical Computer Science, pp. 33–50, 1989.
11. Neil Robertson and Paul D Seymour. Graph minors. X. Obstructions to tree-decomposition. Journal of Combinatorial Theory, Series B, Vol. 52, No. 2, pp. 153–190, 1991.
12. Sang-il Oum and Paul Seymour. Testing branch-width. Journal of Combinatorial Theory, Series B, Vol. 97, No. 3, pp. 385–393, 2007.
13. Fujita, T. (2025). The Interplay between Quasi-Matroid and Connectivity Systems. *Asian Research Journal of Mathematics*, *21*(1), 115-129.
14. Dimitrios M Thilikos. Algorithms and obstructions for linear-width and related search parameters. Discrete Applied Mathematics, Vol. 105, No. 1-3, pp. 239–271, 2000.
15. Bodlaender, Hans L., and Arie MCA Koster. "Treewidth computations I. Upper bounds." *Information and Computation* 208.3 (2010): 259-275.
16. Jim Geelen, Bert Gerards, Neil Robertson, and Geoff Whittle. Obstructions to branch-decomposition of matroids. Journal of Combinatorial Theory, Series B, Vol. 96, No. 4, pp. 560–570, 2006.
17. Fujita, T. (2024). Reconsideration of Tangle and Ultrafilter using Separation and Partition. *International Journal of Mathematics Trends and Technology-IJMTT*, *70*.
18. Bodlaender, Hans L., et al. "Approximating treewidth, pathwidth, frontsize, and shortest elimination tree." *Journal of Algorithms* 18.2 (1995): 238-255.
19. Bodlaender, Hans L. "A tourist guide through treewidth." (1992).
20. Courcelle, Bruno, and Stephan Olariu. "Upper bounds to the clique width of graphs." *Discrete Applied Mathematics* 101.1-3 (2000): 77-114.
21. YRobertson, Neil, Paul Seymour, and Robin Thomas. "Quickly excluding a planar graph." *Journal of Combinatorial Theory, Series B* 62.2 (1994): 323-348.
22. Fujita, T. (2024). Various properties of various ultrafilters, various graph width parameters, and various connectivity systems (with survey). *arXiv preprint arXiv:2408.02299*.
23. Fujita, Takaaki. "Novel Idea on Edge-Ultrafilter and Edge-Tangle." *Asian Research Journal of Mathematics* 20.4 (2024): 18-22.
24. Giannopoulou, Archontia C., et al. "Lean tree-cut decompositions: obstructions and algorithms." STACS 2019-36th International Symposium on Theoretical Aspects of Computer Science. Vol. 126. 2019.
25. Bożyk, Łukasz, et al. "On objects dual to tree-cut decompositions." Journal of Combinatorial Theory, Series B 157 (2022): 401-428.
26. Cenek, Lisa, et al. "Scramble number and tree-cut decompositions." arXiv preprint arXiv:2209.01459 (2022).
27. Sau, Ignasi, and Dimitrios M. Thilikos. "An FPT 2-Approximation for Tree-cut Decomposition." *Approximation and Online Algorithms*: 35.
28. Ganian, Robert, Eun Jung Kim, and Stefan Szeider. "Algorithmic applications of tree-cut width." *International Symposium on Mathematical Foundations of Computer Science*. Berlin, Heidelberg: Springer Berlin Heidelberg, 2015.
29. Giannopoulou, Archontia C., et al. "A Menger-like property of tree-cut width." *Journal of Combinatorial Theory, Series B* 148 (2021): 1-22.
30. Brand, Cornelius, et al. "Edge-cut width: An algorithmically driven analogue of treewidth based on edge cuts." *International Workshop on Graph-Theoretic Concepts in Computer Science*. Cham: Springer International Publishing, 2022.
31. Ganian, Robert, and Viktoriia Korchemna. "Slim tree-cut width." *arXiv preprint arXiv:2206.15091* (2022).
32. Fujita, T. Relation between ultra matroid and linear decomposition. *Italian Journal of Pure and Applied Mathematics*, 18.
33. Bonnet, Édouard, et al. "Twin-width IV: ordered graphs and matrices." *Journal of the ACM* 71.3 (2024): 1-45.
34. Fujita, T., & Smarandache, F. (2025). *A concise study of some superhypergraph classes*. Infinite Study.
35. Fujita, T. (2024). Review of some superhypergraph classes: Directed, bidirected, soft, and rough. *Advancing Uncertain Combinatorics through Graphization, Hyperization, and Uncertainization: Fuzzy, Neutrosophic, Soft, Rough, and Beyond (Second Volume)*.
36. Fujita, T. Exploration of Graph Classes and Concepts for SuperHypergraphs and n-th Power Mathematical Structures. *Advancing Uncertain Combinatorics through Graphization, Hyperization, and Uncertainization: Fuzzy, Neutrosophic, Soft, Rough, and Beyond*, *3*(4), 512.
37. Bretto, A. (2013). Hypergraph theory. *An introduction. Mathematical Engineering. Cham: Springer*, *1*, 209-216.
38. Antelmi, A., Cordasco, G., Polato, M., Scarano, V., Spagnuolo, C., & Yang, D. (2023). A survey on hypergraph representation learning. *ACM Computing Surveys*, *56*(1), 1-38.
39. Feng, Y., You, H., Zhang, Z., Ji, R., & Gao, Y. (2019, July). Hypergraph neural networks. In *Proceedings of the AAAI conference on artificial intelligence* (Vol. 33, No. 01, pp. 3558-3565).
40. Pal, M., Samanta, S., & Ghorai, G. (2020). *Modern trends in fuzzy graph theory* (pp. 7-93). Berlin: Springer.
41. Mathew, S., & Sunitha, M. S. (2009). Types of arcs in a fuzzy graph. *Information sciences*, *179*(11), 1760-1768.
42. Mordeson, J. N., & Nair, P. S. (2012). *Fuzzy graphs and fuzzy hypergraphs* (Vol. 46). Physica.
43. Smarandache, F., & Broumi, S. (Eds.). (2019). *Neutrosophic graph theory and algorithms*. IGI Global.
44. Fujita, T., & Smarandache, F. (2025). *Survey of planar and outerplanar graphs in fuzzy and neutrosophic graphs*. Infinite Study.
45. Fujita, T., & Smarandache, F. (2025). Interval graphs and proper interval graphs in fuzzy and neutrosophic graphs. *Information Sciences with Applications*, *5*, 11-32.
46. Tripathy, A., Panada, A. C., Behera, S. P., & Mohanty, B. S. (2025). Edge connectivity of a neutrosophic graph. *Neutrosophic Sets and Systems*, *81*, 729-740.
47. Fujita, T. (2025). Obstruction for hypertree width and superhypertree width. *Advancing Uncertain Combinatorics through Graphization, Hyperization, and Uncertainization: Fuzzy, Neutrosophic, Soft, Rough, and Beyond*, 26.
48. Marx, D. (2010). Approximating fractional hypertree width. *ACM Transactions on Algorithms (TALG)*, *6*(2), 1-17.
49. Fujita, T., & Smarandache, F. (2024). Mixed graph in fuzzy, neutrosophic, and plithogenic graphs. *Neutrosophic Sets and Systems*, *74*, 457-479.
50. Fujita, T., & Smarandache, F. A Review of the Hierarchy of Plithogenic, Neutrosophic, and Fuzzy Graphs: Survey and Applications. *Advancing Uncertain Combinatorics through Graphization, Hyperization, and Uncertainization: Fuzzy, Neutrosophic, Soft, Rough, and Beyond*, 9.
51. Smarandache, F. (2022). Plithogeny, plithogenic set, logic, probability and statistics: a short review. *Journal of Computational and Cognitive Engineering*, *1*(2), 47-50.
52. Fujita, T., & Florentin, S. (2024). Some graph parameters for superhypertree-width and neutrosophictree-width. *Advancing Uncertain Combinatorics through Graphization, Hyperization, and Uncertainization: Fuzzy, Neutrosophic, Soft, Rough, and Beyond (Third Volume)*.
53. Fujita, T., & Smarandache, F. (2024). Fundamental computational problems and algorithms for superhypergraphs. *Advancing Uncertain Combinatorics through Graphization, Hyperization, and Uncertainization: Fuzzy, Neutrosophic, Soft, Rough, and Beyond (Second Volume)*.
54. Gajarský, J., Lampis, M., & Ordyniak, S. (2013). Parameterized algorithms for modular-width. In *Parameterized and Exact Computation: 8th International Symposium, IPEC 2013, Sophia Antipolis, France, September 4-6, 2013, Revised Selected Papers 8* (pp. 163-176). Springer International Publishing.
55. Abu-Khzam, F. N., Li, S., Markarian, C., Meyer auf der Heide, F., & Podlipyan, P. (2017). Modular-width: An auxiliary parameter for parameterized parallel complexity. In *Frontiers in Algorithmics: 11th International Workshop, FAW 2017, Chengdu, China, June 23-25, 2017, Proceedings 11* (pp. 139-150). Springer International Publishing.
56. Bui-Xuan, B. M., Telle, J. A., & Vatshelle, M. (2011). Boolean-width of graphs. *Theoretical computer science*, *412*(39), 5187-5204.
57. Rabinovich, Y., Telle, J. A., & Vatshelle, M. (2013). Upper bounds on boolean-width with applications to exact algorithms. In *Parameterized and Exact Computation: 8th International Symposium, IPEC 2013, Sophia Antipolis, France, September 4-6, 2013, Revised Selected Papers 8* (pp. 308-320). Springer International Publishing.
58. Chinn, P. Z., Chvátalová, J., Dewdney, A. K., & Gibbs, N. E. (1982). The bandwidth problem for graphs and matrices—a survey. *Journal of Graph Theory*, *6*(3), 223-254.
59. Diamant, N. L., Tseng, A. M., Chuang, K. V., Biancalani, T., & Scalia, G. (2023, July). Improving graph generation by restricting graph bandwidth. In *International Conference on Machine Learning* (pp. 7939-7959). PMLR.
60. Lozin, V., & Rautenbach, D. (2004). On the band-, tree-, and clique-width of graphs with bounded vertex degree. *SIAM Journal on Discrete Mathematics*, *18*(1), 195-206.
61. Diamant, N. L., Tseng, A. M., Chuang, K. V., Biancalani, T., & Scalia, G. (2023, July). Improving graph generation by restricting graph bandwidth. In *International Conference on Machine Learning* (pp. 7939-7959). PMLR.
62. Bondy, J. A., & Murty, U. S. R. (2008). *Graph theory*. Springer Publishing Company, Incorporated.
63. Bollobás, B. (2012). *Graph theory: an introductory course* (Vol. 63). Springer Science & Business Media.
64. Bienstock, D., Robertson, N., Seymour, P. D., & Thomas, R. (1991). Quickly excluding a forest. *J. Comb. Theory, Ser. B*, *52*(2), 274-283.
65. Xu, J. (2003). *Theory and application of graphs* (Vol. 10). Springer Science & Business Media.
66. Bondy, J. A., & Murty, U. S. R. (1976). *Graph theory with applications* (Vol. 290). London: Macmillan.
67. Korach, E., & Solel, N. (1993). Tree-width, path-width, and cutwidth. *Discrete Applied Mathematics*, *43*(1), 97-101.
68. Giannopoulou, A. C., Pilipczuk, M., Raymond, J. F., Thilikos, D. M., & Wrochna, M. (2019). Cutwidth: Obstructions and algorithmic aspects. *Algorithmica*, *81*, 557-588.
69. Kist, J. (1963). Minimal prime ideals in commutative semigroups. *Proceedings of the London Mathematical Society*, *3*(1), 31-50.
70. Sun, S. H. (1991). Noncommutative rings in which every prime ideal is contained in a unique maximal ideal. *Journal of Pure and Applied Algebra*, *76*(2), 179-192.
71. Mordeson, J. N. (2001). Rough set theory applied to (fuzzy) ideal theory. *Fuzzy Sets and Systems*, *121*(2), 315-324.
72. Bell, J. L., & Fremlin, D. H. (1972). The maximal ideal theorem for lattices of sets. *Bulletin of the London Mathematical Society*, *4*(1), 1-2.
73. Kuratowski, K. (2014). *Introduction to set theory and topology*. Elsevier.
74. Monk, J. D. (2008). Maximal irredundance and maximal ideal independence in Boolean algebras. *The Journal of Symbolic Logic*, *73*(1), 261-275.
75. Lucena, B. (2007). Achievable sets, brambles, and sparse treewidth obstructions. *Discrete applied mathematics*, *155*(8), 1055-1065.
76. Birmelé, E., Bondy, J. A., & Reed, B. A. (2007). Brambles, prisms and grids. In *Graph Theory in Paris: Proceedings of a Conference in Memory of Claude Berge* (pp. 37-44). Birkhäuser Basel.
77. Fujita, Takaaki. 2025. “Ultrafilters and Their Dual Relationship to Tree-Width in Graph Theory”. *Asian Research Journal of Mathematics* 21 (1):98-114. <https://doi.org/10.9734/arjom/2025/v21i1886>.
78. Behzad, M., Chartrand, G., & Cooper Jr, J. K. (1967). The colour numbers of complete graphs. Journal of the London Mathematical Society, 1(1), 226-228.
79. Bloom, G. S., & Golomb, S. W. (2006). Numbered complete graphs, unusual rulers, and assorted applications. In Theory and Applications of Graphs: Proceedings, Michigan May 11–15, 1976 (pp. 53-65). Berlin, Heidelberg: Springer Berlin Heidelberg.
80. Winkler, P. M. (1984). Isometric embedding in products of complete graphs. Discrete Applied Mathematics, 7(2), 221-225.
81. Stewart, B. M. (1967). Supermagic complete graphs. Canadian Journal of Mathematics, 19, 427-438.
82. Lamaison, A. (2025). Planar graph with twin-width seven. *European Journal of Combinatorics*, *123*, 103749.
83. Bonnet, É., Geniet, C., Kim, E. J., Thomassé, S., & Watrigant, R. (2024). Twin-width III: max independent set, min dominating set, and coloring. *SIAM Journal on Computing*, *53*(5), 1602-1640.
84. Bonnet, É., Nešetřil, J., de Mendez, P. O., Siebertz, S., & Thomassé, S. (2024). Twin-width and permutations. *Logical Methods in Computer Science*, *20*.
85. Horev, Y., Shay, S., Cohen, S., Friedrich, T., Issac, D., Kamma, L., ... & Simonov, K. (2024, April). A Contraction Tree SAT Encoding for Computing Twin-Width. In *Pacific-Asia Conference on Knowledge Discovery and Data Mining* (pp. 444-456). Singapore: Springer Nature Singapore.
86. Takahashi, A., Ueno, S., & Kajitani, Y. (1995). Mixed searching and proper-path-width. *Theoretical Computer Science*, *137*(2), 253-268.
87. TAKAHASHI, A., UENO, S., & KAJITANI, Y. (1995). On the proper-path-decomposition of trees. *IEICE transactions on fundamentals of electronics, communications and computer sciences*, *78*(1), 131-136.
88. Takahashi, A., Ueno, S., & Kajitani, Y. (1995). Minimal forbidden minors for the family of graphs with proper-path-width at most two. *IEICE transactions on fundamentals of electronics, communications and computer sciences*, *78*(12), 1828-1839.
89. Chakraborty, S., Jo, S., Sadakane, K., & Satti, S. R. (2024). Succinct data structures for bounded clique-width graphs. *Discrete Applied Mathematics*, *352*, 55-68.
90. Ganian, R., Hamm, T., Korchemna, V., Okrasa, K., & Simonov, K. (2024). The fine-grained complexity of graph homomorphism parameterized by clique-width. *ACM Transactions on Algorithms*, *20*(3), 1-26.
91. Bancerek, G. (1990). The fundamental properties of natural numbers. *Formalized Mathematics*, *1*(1), 41-46.