

Mellin-SBA method for solving nonlinear partial differential equations

Abstract :

In this paper, we present a new numerical method for solving nonlinear differential and partial differential equations. This method is the Mellin-SBA method. In the first part, we describe the SBA and Mellin-SBA methods. The second part is devoted to solving two nonlinear problems by the Mellin-SBA method.

Key words : Partial differential equations, Mellin transformation, SBA method. Mellin-SBA Method.

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1. Introduction

Solving nonlinear partial differential equations has always been a challenge for mathematicians. Several numerical methods have been presented. The method we present in this paper is a coupling between the Mellin transformation [15] [16] [17] and the SBA method [10] [11] [12] [13] [14] .

2. Mellin transformation and resolution methods**2.1 Mellin transformation :**

The Mellin transform [15] [16] [17] is closely related to the Fourier and Laplace transforms. It can be successfully applied to the resolution of a class of plane harmonic problems in an areal domain, to problems of elasticity theory..

2.2 Definition [6] [7]

Let f be a causal function. The Mellin transform of f is defined by :

$$F(s) = \mathcal{M}(f(t)) = \int_{t=0}^{\infty} f(t)t^{s-1}dt \quad (1)$$

the inverse Mellin transformation is given by the formula [7] :

$$f(t) = \mathcal{M}^{-1}F(s) = \frac{1}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} F(s)t^s ds \quad (2)$$

we also note

$$\mathcal{M}^{-1}[f(t); s] \quad (3)$$

properties [7] [6]

Let f be a numerical function and $\mathcal{M}(f(t)) = F(s)$ its Mellin transformation. We have :

a. Linearity :

$$\mathcal{M}[\alpha f(t) + \beta g(t)] = \alpha \mathcal{M}(f(t)) + \beta \mathcal{M}(g(t)) \quad (4)$$

$$\begin{aligned} \mathcal{M}[\alpha f(t) + \beta g(t)] &= \int_{t=0}^{\infty} (\alpha f(t) + \beta g(t))t^{s-1}dt \\ &= \int_{t=0}^{\infty} (\alpha f(t))t^{s-1}dt + \int_{t=0}^{\infty} (\beta g(t))t^{s-1}dt \\ &= \alpha \mathcal{M}f(t) + \beta \mathcal{M}g(t) \end{aligned}$$

b. Convolution Product :

Let f and g be two causal functions. We have :

$$\mathcal{M}(f(t) * g(t)) = \mathcal{M}(f(s))(\mathcal{M}(g(s))) \quad (5)$$

c. Mellin transformation of $f(at)$.

we have

$$\mathcal{M}(f(at)) = \frac{F(s)}{a^s} \quad (6)$$

d. Mellin transformation of $t^\alpha f(t)$

$$\mathcal{M}(t^\alpha f(t)) = F(a + s) \quad (7)$$

e. Mellin transformation of the first derivative

$$\begin{aligned} \mathcal{M}\{f'(t)\} &= \int_0^{+\infty} t^{s-1} f'(t) dt \\ &= [t^{s-1} f(t)]_0^\infty - \int_0^\infty (s-1)t^{s-2} f(t) dt \\ \mathcal{M}\{f'(t)\} &= [t^{s-1} f(t)]_0^\infty - (s-1)F(s-1) \end{aligned}$$

f. Mellin transformation of the n order derivative

$$\forall n \in \mathbb{N}; \mathcal{M}(f^{(n)}(t))(s) = (-1)^n \frac{\Gamma(s)}{\Gamma(s-n)} \mathcal{M}(f)(s-n) + \sum_{k=0}^{n-1} (-1)^k \frac{\Gamma(s)}{\Gamma(s-k)} [t^{s-k-1} f^{(n-k-1)}(t)]_0^\infty \quad (8)$$

3. Resolution methods

3.1 Introduction

The main purpose of this chapter is to recall and describe the principles of the SBA method [10] [11] [12] [13] [14] and the Mellin-SBA method.

3.2 SBA Method [2] [5] [8] [9] [1] [3] [4]

Consider the following functional equation :

$$\begin{cases} u_t(x, t) = R(u(x, t)) + N(u(x, t)), 0 < t < T \\ u(x, 0) = f \end{cases} \quad (9)$$

In a suitably chosen space V , with R a linear differential operator, N the nonlinear term, u the unknown function such that $u_t = \frac{\partial u}{\partial t}$.

let's ask $L_t = \frac{\partial}{\partial t}(\cdot)$ and $L_t^{-1}(\cdot) = \int_0^t (\cdot) ds$.

Applying L_t^{-1} to the equation, we obtain :

$$u(x, t) = u(x, 0) + L_t^{-1}R(u(x, t)) + L_t^{-1}N(u(x, t)) \quad (10)$$

Using the idea of the method of successive approximations, for any non-zero natural integer k , we obtain :

$$u^k(x, t) = u^k(x, 0) + L_t^{-1}R(u^k(x, t)) + L_t^{-1}N(u^{k-1}(x, t)) \quad (11)$$

We seek the solution $u^k(x, t)$ in the form :

$$u^k(x, t) = \sum_{k=0}^{+\infty} u_n^k(x, t) \quad (12)$$

By injecting 12 into 11 we obtain by identification the SBA algorithm :

$$\begin{cases} u_0^k(x, t) = u^k(x, 0) + L_t^{-1}N(u^{k-1}(x, t)), 0 < t < T \\ u_n^k(x, t) = L_t^{-1}R(u_{n-1}^k(x, t)); k = 1, 2, \dots \end{cases} \quad (13)$$

The resolution of the scheme (13) by the method of successive approximations, consists in determining at each iteration ($k=1, 2, \dots$) approximate solutions $u^1(x, t)$, $u^2(x, t), \dots$

The solution at each step is :

$$u^k(x, t) = \sum_{n=0}^{+\infty} u_n^k(x, t) \quad (14)$$

For the calculation of u^1 , the choice of $u^0(x, t)$ is very important. According to the Picard principle, we choose $u^0(x, t)$ such that $N(u^0(x, t)) = 0$. The sought solution $u(x, t)$ of the problem (9) is obtained by :

$$u(x, t) = \lim_{k \rightarrow +\infty} u^k(x, t) \quad (15)$$

3.3 Convergence of the SBA method

The problems of the theoretical model are considered :

$$(P) \begin{cases} u_t = Lu(t) + N(u(t)), t_0 < t < t_1 \\ u(t_0) = u_0 \end{cases} \quad (16)$$

and the approximate model :

$$(P^k) \begin{cases} u_t^k = Lu^k + N(u^{k-1}, t), t_0 < t < t_1 \\ u^k(t_0) = u_0, k = 1, 2, 3, \dots \end{cases} \quad (17)$$

under the following hypotheses :

(H1) : N is a Lipschitzian of constant k on a ball $B(u_0, r)$ with center u_0 and radius r .

(H2) : L continuous on $C(t_0, t_1)$

Proposition : [9] [5]

if $u^0 \in B(u_0, r)$ (that's to say : $\|u^0 - u_0\|_{C(t_0, t_1)} \max_{t_0 < t < t_1}$) then there exists $\delta > 0$ such that $t_0 + \delta < t_1$ and the algorithm (P^k) admits a unique solution u^k such that $\max_{t_0 < t < t_1} |u^k(t) - u_0| \leq r$ for all k .

Preuve : for proof see [9] [5]

3.4 Mellin-SBA method

The Mellin-SBA method is a method that consists of combining the SBA method with the Mellin transformation.

We consider the problem :

$$\begin{cases} u_t(x, t) = R(u(x, t)) + N(u(x, t)), 0 < t < T \\ u(x, 0) = f \end{cases}$$

let us remember

$$\mathcal{M}\{u_t(x, t)\} = [t^{s-1}u(x, t)]_0^T - (s-1)\mathcal{M}[u(x, t); s-1]$$

We apply the Mellin transformation to (9) and we obtain :

$$[t^{s-1}u(x, t)]_0^T - (s-1)\mathcal{M}[u(x, t); s-1] = \mathcal{M}[R(u(x, t)); s] + \mathcal{M}[N(u(x, t)); s] \quad (18)$$

we obtain :

$$(s-1)\mathcal{M}[u(x,t);s-1] = T^{s-1}u(x,T) - \mathcal{M}[R(u(x,t));s] - \mathcal{M}[N(u(x,t));s] \quad (19)$$

we obtain

$$\mathcal{M}[u(x,t);s-1] = \frac{T^{s-1}}{(s-1)}u(x,T) - \frac{1}{(s-1)}\mathcal{M}[R(u(x,t));s] - \frac{1}{(s-1)}\mathcal{M}[N(u(x,t));s] \quad (20)$$

Applying the inverse Mellin transformation, we obtain :

$$t^{-1}u(x,t) = \mathcal{M}^{-1}\left[\frac{T^{s-1}}{(s-1)}u(x,T)\right] - \mathcal{M}^{-1}\left[\frac{1}{(s-1)}\mathcal{M}[R(u(x,t));s]\right] - \mathcal{M}^{-1}\left[\frac{1}{(s-1)}\mathcal{M}[N(u(x,t));s]\right]$$

We draw $u(x,t)$ and we obtain :

$$u(x,t) = t\mathcal{M}^{-1}\left[\frac{T^{s-1}}{(s-1)}u(x,T)\right] - t\mathcal{M}^{-1}\left[\frac{1}{(s-1)}\mathcal{M}[R(u(x,t));s]\right] - t\mathcal{M}^{-1}\left[\frac{1}{(s-1)}\mathcal{M}[N(u(x,t));s]\right]$$

we pose

$$\bar{N}(u(x,t)) = -t\mathcal{M}^{-1}\left[\frac{1}{(s-1)}\mathcal{M}[N(u(x,t));s]\right] \quad (21)$$

we obtain

$$u(x,t) = t\mathcal{M}^{-1}\left[\frac{T^{s-1}}{(s-1)}u(x,T)\right] - t\mathcal{M}^{-1}\left[\frac{1}{(s-1)}\mathcal{M}[R(u(x,t));s]\right] + \bar{N}(u(x,t))$$

Using the principle of successive approximations for $k \geq 1$, we obtain :

$$u^k(x,t) = t\mathcal{M}^{-1}\left[\frac{T^{s-1}}{(s-1)}u(x,T)\right] - t\mathcal{M}^{-1}\left[\frac{1}{(s-1)}\mathcal{M}[R(u^k(x,t));s]\right] + \bar{N}(u^{k-1}(x,t)) \quad (22)$$

Applying the SBA algorithm (13) to (22), we obtain :

$$\begin{cases} u_0^k(x,t) = t\mathcal{M}^{-1}\left[\frac{T^{s-1}}{(s-1)}u(x,T)\right] + \bar{N}(u^{k-1}(x,t)) \\ u_n^k(x,t) = -t\mathcal{M}^{-1}\left[\frac{1}{(s-1)}\mathcal{M}[R(u_{n-1}^k(x,t));s]\right] \end{cases} \quad (23)$$

k^{th} iteration : Calculation of $u^k(x,t)$

$$\begin{cases} u_0^k(x,t) = t\mathcal{M}^{-1}\left[\frac{T^{s-1}}{(s-1)}u(x,T)\right] + \bar{N}(u^{k-1}(x,t)) \\ u_1^k(x,t) = -t\mathcal{M}^{-1}\left[\frac{1}{(s-1)}\mathcal{M}[R(u_0^k(x,t));s]\right] \\ \vdots \\ u_n^k(x,t) = -t\mathcal{M}^{-1}\left[\frac{1}{(s-1)}\mathcal{M}[R(u_{n-1}^k(x,t));s]\right] \end{cases} \quad (24)$$

The approximate solution of this kth iteration is :

$$u^k(x,t) = \sum_{n=0}^{+\infty} u_n^k(x,t)$$

The solution to the problem is given by :

$$u(x,t) = \lim_{k \rightarrow +\infty} u^k(x,t) \quad (25)$$

property

if

$$u^1(x, t) = u^2(x, t) \quad (26)$$

then

$$u(x, t) = u^1(x, t) = u^2(x, t) \quad (27)$$

4. Resolution of some partial differential equations by the Mellin-SBA method

The aim of this chapter is to apply the Mellin-SBA method to solving a partial differential equation.

Example 1 : Nonlinear diffusion equation

$$\begin{cases} u_t(x, t) = \varepsilon u_{xx} + u^3 + (u_{xx})^3 & 0 \leq t \leq 1 \\ u(x, 0) = \sin x \\ u(x, 1) = \sin x e^{-\varepsilon} \end{cases} \quad (28)$$

we pose $L_t = \frac{\partial}{\partial t}(\cdot)$ et $N(\cdot) = (\cdot)^3 + \frac{\partial^2}{\partial x^2}(\cdot)$.

The problem becomes :

$$\begin{cases} u_t(x, t) = \varepsilon u_{xx} + N(u) \\ u(x, 0) = \sin x \\ u(x, 1) = \sin x e^{-\varepsilon} \end{cases} \quad (29)$$

Applying the Mellin transformation, we obtain :

$$\mathcal{M}\{u_t(x, t)\} = [t^{s-1}u(x, t)]_0^\infty - (s-1)\mathcal{M}\{u(x, t); s-1\} \quad (30)$$

let

$$[t^{s-1}u(x, t)]_0^\infty - (s-1)[\mathcal{M}(u(x, t)); s-1] = \varepsilon\mathcal{M}(u_{xx}; s) + \mathcal{M}(N(u)) \quad (31)$$

Noting that $0 \leq t \leq 1$ we obtain :

$$[t^{s-1}u(x, t)]_0^1 - (s-1)[\mathcal{M}(u(x, t)); s-1] = \varepsilon\mathcal{M}(u_{xx}; s) + \mathcal{M}(N(u)) \quad (32)$$

we pose $a(x) = [t^{s-1}u(x, t)]_0^1 = u(x, 1) = \sin x e^{-\varepsilon}$

we obtain :

$$\mathcal{M}\{u(x, t); s-1\} = \frac{-\sin x e^{-\varepsilon}}{1-s} + \frac{1}{1-s}k\mathcal{M}(u_{xx}; s) + \frac{1}{1-s}\mathcal{M}(N(u)) \quad (33)$$

Let's apply the inverse transformation :

$$t^{-1}u(x, t) = \sin x e^{-\varepsilon} t^{-1} + \varepsilon\mathcal{M}^{-1}\left[\frac{1}{1-s}\mathcal{M}(u_{xx}; s)\right] + \mathcal{M}^{-1}\left[\frac{1}{1-s}\mathcal{M}(N(u))\right] \quad (34)$$

we obtain

$$u(x, t) = \sin x e^{-\varepsilon} + t\varepsilon\mathcal{M}^{-1}\left[\frac{-1}{s-1}\mathcal{M}(u_{xx}; s)\right] + t\mathcal{M}^{-1}\left[\frac{1}{1-s}\mathcal{M}(N(u))\right] \quad (35)$$

Using the idea of the principle of successive approximations we obtain the canonical form SBA :

$$u^k(x, t) = \sin x e^{-\varepsilon} + t\varepsilon\mathcal{M}^{-1}\left[\frac{-1}{s-1}\mathcal{M}(u_{xx}^k; s)\right] + t\mathcal{M}^{-1}\left[\frac{1}{1-s}\mathcal{M}(N(u^{k-1}))\right] \quad (36)$$

We obtain the SBA algorithm :

$$\begin{cases} u_0^k & = \sin x e^{-\varepsilon} + t \mathcal{M}^{-1} \left[\frac{1}{1-s} \mathcal{M}(N(u^{k-1})) \right] \\ u_n^k(x, t) & = \varepsilon t \mathcal{M}^{-1} \left[\frac{-1}{s-1} \mathcal{M} \left[\frac{\partial^2 u_{n-1}^k}{\partial x^2}(x, t) \right] \right] \end{cases} \quad (37)$$

To simplify the writing, we designate by $T(\cdot)$, the operator defined by :

$$T(u) = \mathcal{M}^{-1} \left[\frac{-1}{s-1} \mathcal{M} \left[\frac{\partial^2 u_{n-1}}{\partial x^2}(x, t) \right] \right]$$

the algorithm becomes

$$\begin{cases} u_0^k & = \sin x e^{-\varepsilon} + t \mathcal{M}^{-1} \left[\frac{1}{1-s} \mathcal{M}(N(u^{k-1})) \right] \\ u_n^k(x, t) & = \varepsilon t T(u^k) \end{cases} \quad (38)$$

first iteration : $k = 1$

$$\begin{cases} u_0^1 & = \sin x e^{-\varepsilon} + t \mathcal{M}^{-1} \left[\frac{1}{1-s} \mathcal{M}(N(u^0)) \right] \\ u_n^1(x, t) & = \varepsilon t T(u^1) \end{cases} \quad (39)$$

By the Picard principle, there exists u^0 such that $N(u^0) = 0$.

we obtain : $u_0^1 = \sin x e^{-\varepsilon}$

We note that for : $u_0 = u_0^1 = \sin x e^{-\varepsilon}$, on a :

$\frac{\partial^2 u_0^1}{\partial x^2}(x, t) = -u_0^1$ and that :

$$P(t^n u_0) = \mathcal{M}^{-1} \left[\frac{-1}{s-1} \mathcal{M} \left[t^n \frac{\partial^2 u_0}{\partial x^2}(x, t) \right] \right] \quad (40)$$

$$= \mathcal{M}^{-1} \left[\frac{1}{s-1} \mathcal{M} \left[t^n u_0 \right] \right] \quad (41)$$

$$= u_0 \mathcal{M}^{-1} \left[\frac{1}{(s-1)(s+n)} \right] \quad (42)$$

$$= u_0 \mathcal{M}^{-1} \left[\frac{-1}{s+n} + \frac{1}{s-1} \right] \quad (43)$$

$$P(t^n u_0) = u_0 \left[-\frac{t^n}{n+1} + \frac{t^{-1}}{n+1} \right] \quad (44)$$

$$(45)$$

we obtain that :

$$tP(t^n u_0) = u_0 \left[-\frac{t^{n+1}}{n+1} + \frac{1}{n+1} \right] \quad (46)$$

calculation of u_1^1 we obtain

$$u_1^1(x, t) = \varepsilon t p(u_0) = \varepsilon t p(t^0 u_0) = \varepsilon u_0 (-t + 1)$$

$$= -\varepsilon t u_0 + \varepsilon u_0$$

$$u_1^1(x, t) = -\varepsilon u_0 (t - 1)$$

Calculation of u_2^1

$$\begin{aligned}
u_2^1(x, t) &= \varepsilon t T(u_1^1) = -\varepsilon^2 t T(tu_0) + \varepsilon^2 t T(u_0) \\
&= \varepsilon^2 u_0 \left(\frac{t^2}{2} - \frac{1}{2} \right) + \varepsilon^2 u_0 (-t + 1) \\
&= \frac{\varepsilon^2}{2} t^2 u_0 - t u_0 + \frac{1}{2} u_0 \\
u_2^1(x, t) &= u_0 \frac{\varepsilon^2 (t-1)^2}{2}
\end{aligned}$$

Calculation of u_3

$$\begin{aligned}
u_3^1(x, t) &= \varepsilon t T(u_2^1) = \frac{1}{2} \varepsilon^3 t T(t^2 u_0) - \varepsilon^3 t T(tu_0) + \frac{1}{2} \varepsilon^3 t T(u_0) \\
&= u_0 \varepsilon^3 \frac{1}{2} \left(-\frac{t^3}{3} + \frac{1}{3} \right) - u_0 \varepsilon^3 \left(-\frac{t^2}{2} + \frac{1}{2} \right) + \frac{1}{2} u_0 \varepsilon^3 (-t + 1) \\
&= -u_0 \varepsilon^3 \left(\frac{t^3}{6} - \frac{1}{6} - \frac{t^2}{2} + \frac{1}{2} + \frac{t}{2} - \frac{1}{2} \right) \\
&= \frac{-\varepsilon^3}{6} t^3 u_0 + \frac{\varepsilon^3}{2} t^2 + \frac{k^3}{2} t u_0 - \frac{\varepsilon^3}{2} t u_0 \\
&= -\frac{u_0 \varepsilon^3}{6} (t^3 - 3t^2 + 3t - 1) \\
u_3^1(x, t) &= -\frac{u_0 \varepsilon^3 (t-1)^3}{3!}
\end{aligned}$$

In recurring ways, we obtain :

$$u_n^1 = u_0 \frac{[-\varepsilon(t-1)]^n}{n!} \quad (47)$$

indeed

$$u_n^1 = u_0 \frac{(\varepsilon - \varepsilon t)^n}{n!} = \frac{u_0 \varepsilon^n}{n!} \sum_{i=0}^n C_n^i (-t)^i \quad (48)$$

At order $(n + 1)$ we have :

$$\begin{aligned}
u_{n+1}^1 &= \varepsilon t T(u_n^1) = \varepsilon t \frac{u_0 \varepsilon^n}{n!} \sum_{i=0}^n C_n^i (-1)^i T(t^i u_0) \\
&= \varepsilon \frac{u_0 \varepsilon^n}{n!} \sum_{i=0}^n C_n^i (\varepsilon^{n-i}) (-1)^i \left(\frac{t^{i+1}}{i+1} + \frac{1}{i+1} \right) \\
&= \varepsilon^{n+1} \frac{u_0}{n!} \sum_{i=0}^n \frac{C_n^i}{(i+1)} (-1)^i (1 - t^{i+1}) \\
&= \varepsilon^{n+1} \frac{u_0}{n!} \sum_{i=0}^n \frac{n!}{(n-i)! i! (i+1)} (-1)^{i+1} (t^{i+1} - 1) \\
&= \varepsilon^{n+1} \frac{u_0}{(n+1)!} \sum_{i=0}^n \frac{(n+1)!}{(n-i)! i! (i+1)} (-1)^{i+1} (t^{i+1} - 1) \\
&= \varepsilon^{n+1} \frac{u_0}{(n+1)!} \sum_{i=0}^n \frac{(i+1)! C_{n+1}^{i+1}}{i! (i+1)} (-1)^{i+1} (t^{i+1} - 1) \\
&= \varepsilon^{n+1} \frac{u_0}{(n+1)!} \sum_{i=0}^n C_{n+1}^{i+1} (-1)^{i+1} (t^{i+1} - 1) \\
&= \varepsilon^{n+1} \frac{u_0}{(n+1)!} \sum_{i=1}^{n+1} C_{n+1}^i (-1)^i (t^i - 1) \\
&= \varepsilon^{n+1} \frac{u_0}{(n+1)!} \left[\sum_{i=1}^{n+1} C_{n+1}^i (-1)^i t^i - \sum_{i=1}^{n+1} C_{n+1}^i (-1)^i \right] \\
&= \varepsilon^{n+1} \frac{u_0}{(n+1)!} \left[\sum_{i=1}^{n+1} C_{n+1}^i (-1)^i t^i + 1 \right] \\
&= \varepsilon^{n+1} \frac{u_0}{(n+1)!} \left[\sum_{i=0}^{n+1} C_{n+1}^i (-1)^i t^i \right] \\
&= u_0 \frac{(\varepsilon - \varepsilon t)^{n+1}}{(n+1)!}
\end{aligned}$$

Conclusion :

$$u_n^1 = u_0 \frac{(\varepsilon - \varepsilon t)^n}{n!} \quad (49)$$

$$u^1(x, t) = \sum_{n=0}^{+\infty} u_n^1 = \sin x e^{-\varepsilon t} \quad (50)$$

algorithm at step $k = 2$:

$$\begin{cases} u_0^2 &= \sin x e^{-\varepsilon} + t \mathcal{M}^{-1} \left[\frac{1}{1-s} \mathcal{M}(N(u^1)) \right] \\ u_n^2(x, t) &= \varepsilon t T(u_n^2) \end{cases} \quad (51)$$

$$\begin{aligned}
N(u^1) &= (u^1)^3 + \left[\frac{\partial^2}{\partial x^2} (u^1) \right]^3 \\
&= \sin^3 x e^{-3\varepsilon t} - \sin^3 x e^{-3\varepsilon t} = 0
\end{aligned}$$

we obtain $u_0^2 = \sin x e^{-\varepsilon}$
we calculate the u_i^2 for $i \in \{1, 2, \dots\}$

$$\begin{cases} u_0^2 & = \sin x e^{-\varepsilon} \\ u_1^2(x, t) & = -\varepsilon u_0(t-1) \\ u_2^2(x, t) & = u_0 \frac{\varepsilon^2 (t-1)^2}{2} \\ u_3^2(x, t) & = -\frac{u_0 \varepsilon^3 (t-1)^3}{3!} \\ \vdots & \\ u_n^2 & = u_0 \frac{(\varepsilon - \varepsilon t)^n}{n!} \frac{u_0 \varepsilon^n}{n!} \sum_{i=0}^n C_n^i (-t)^i \end{cases} \quad (52)$$

$$u^2(x, t) = \sum_{n=0}^{+\infty} u_n^2 = \sin x e^{-\varepsilon t} \quad (53)$$

$$u^1(x, t) = u^2(x, t) \quad (54)$$

We deduce the general solution to the problem :

$$u(x, t) = \sin x e^{-\varepsilon t} \quad (55)$$

Example 2 : nonlinear evolution equation

$$\begin{cases} \frac{\partial u(t, x, y)}{\partial t} = \beta u(t, x, y) + (\Delta u(t, x, y))^{2m} - (\lambda \omega^2 e^{\beta t} - \omega^2 u(t, x, y))^{2m} \\ u(0, x, y) = \lambda + \cos(\omega x) + \sin(\omega y); \omega > 0 \\ u(1, x, y) = (\lambda + \cos(\omega x) + \sin(\omega y)) e^{\beta}; \omega > 0 \end{cases} \quad (56)$$

with $\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$

we pose $L_t = \frac{\partial}{\partial t}(\cdot)$ et $N(\cdot) = [\Delta(\cdot)]^{2m} - [\lambda \omega^2 e^{\beta t} - \omega^2(\cdot)]^{2m}$.
the equation becomes

$$L_t u(t, x, y) = \beta u(t, x, y) + N(u(t, x, y)) \quad (57)$$

Let us apply the Mellin transformation to the equation. We obtain :

$$[t^{s-1} u(t, x, y)]_0^\infty - (s-1) \mathcal{M}\{(u(t, x, y); s-1)\} = \beta \mathcal{M}(u(t, x, y)) + \mathcal{M}(N(u(t, x, y))) \quad (58)$$

we obtain :

$$\mathcal{M}\{(u(t, x, y); s-1)\} = -\frac{1}{1-s} [t^{s-1} u(t, x, y)]_0^\infty + \frac{\beta}{1-s} \mathcal{M}(u(t, x, y)) + \frac{1}{1-s} \mathcal{M}(N(u(t, x, y))) \quad (59)$$

Noting that $0 < t < 1$, we find

$$\mathcal{M}\{(u(t, x, y); s-1)\} = -\frac{1}{1-s} [t^{s-1} u(t, x, y)]_0^1 + \frac{\beta}{1-s} \mathcal{M}(u(t, x, y)) + \frac{1}{1-s} \mathcal{M}(N(u(t, x, y))) \quad (60)$$

We finally obtain :

$$\mathcal{M}\{(u(t, x, y); s-1)\} = -\frac{1}{1-s} u(1, x, y) + \frac{\beta}{1-s} \mathcal{M}(u(t, x, y)) + \frac{1}{1-s} \mathcal{M}(N(u(t, x, y))) \quad (61)$$

Let us apply the inverse transformation to the equation :

$$t^{-1}u(t, x, y) = t^{-1}u(1, x, t) + \mathcal{M}^{-1}\left[\frac{\beta}{1-s}\mathcal{M}(u(t, x, y))\right] + \mathcal{M}^{-1}\left[\frac{1}{1-s}\mathcal{M}(N(u(t, x, y)))\right] \quad (62)$$

we obtain :

$$u(t, x, y) = u(1, x, t) + t\mathcal{M}^{-1}\left[\frac{\beta}{1-s}\mathcal{M}(u(t, x, y))\right] + t\mathcal{M}^{-1}\left[\frac{1}{1-s}\mathcal{M}(N(u(t, x, y)))\right] \quad (63)$$

let's ask

$$N(\bar{u}(t, x, y)) = t\mathcal{M}^{-1}\left[\frac{1}{1-s}\mathcal{M}(N(u(t, x, y)))\right] \quad (64)$$

we obtain :

$$u(t, x, y) = u(1, x, t) + t\mathcal{M}^{-1}\left[\frac{\beta}{1-s}\mathcal{M}(u(t, x, y))\right] + N(\bar{u}(t, x, y)) \quad (65)$$

We use the idea of the principle of successive approximations and we obtain for any non-zero natural integer k :

$$u^k(t, x, y) = u(1, x, t) + t\mathcal{M}^{-1}\left[\frac{\beta}{1-s}\mathcal{M}(u^k(t, x, y))\right] + N(\bar{u}^k(t, x, y)) \quad (66)$$

We deduce the canonical form SBA below :

$$\begin{cases} u_0^k(t, x, y) = u(1, x, t) + N(\bar{u}^{k-1}(t, x, y)) \\ u_n^k(t, x, y) = t\mathcal{M}^{-1}\left[\frac{\beta}{1-s}\mathcal{M}(u_{n-1}^k(t, x, y))\right] \end{cases} \quad (67)$$

firsts iteration : $k = 1$

$$\begin{cases} u_0^1(t, x, y) = u(1, x, t) + N(\bar{u}^0(t, x, y)) \\ u_n^1(t, x, y) = t\mathcal{M}^{-1}\left[\frac{\beta}{1-s}\mathcal{M}(u_{n-1}^1(t, x, y))\right] \end{cases} \quad (68)$$

According to Picard's principle, there exists u^0 such that $Nu^0 = 0$. We obtain :

$$\begin{cases} u_0^1(t, x, y) = (\lambda + \cos(\omega x) + \sin(\omega y))e^\beta \\ u_n^1(t, x, y) = t\mathcal{M}^{-1}\left[\frac{\beta}{1-s}\mathcal{M}(u_{n-1}^1(t, x, y))\right] \end{cases} \quad (69)$$

calculation of u_n^1

$$\begin{aligned} u_1^1(t, x, y) &= t\mathcal{M}^{-1}\left[\frac{\beta}{1-s}\mathcal{M}(u_0^1(t, x, y))\right] \\ &= t(\lambda + \cos(\omega x) + \sin(\omega y))e^\beta \mathcal{M}^{-1}\left[\frac{\beta}{s(1-s)}\right] \\ &= t\beta(\lambda + \cos(\omega x) + \sin(\omega y))e^\beta \mathcal{M}^{-1}\left[\frac{1}{s} + \frac{1}{1-s}\right] \\ &= t\beta(\lambda + \cos(\omega x) + \sin(\omega y))e^\beta \left[1 - \frac{1}{t}\right] \\ u_1^1(t, x, y) &= (\lambda + \cos(\omega x) + \sin(\omega y))e^\beta \beta(t - 1) \end{aligned}$$

$$\begin{aligned}
u_2^1(t, x, y) &= t\mathcal{M}^{-1}\left[\frac{\beta}{1-s}\mathcal{M}(u_1^1(t, x, y))\right] \\
&= t\beta^2(\lambda + \cos(\omega x) + \sin(\omega y))e^\beta\mathcal{M}^{-1}\left[\frac{1}{1-s}\mathcal{M}(t-1)\right] \\
&= t\beta^2(\lambda + \cos(\omega x) + \sin(\omega y))e^\beta\mathcal{M}^{-1}\left[-\frac{1}{s(1-s)} + \frac{1}{(s+1)(1-s)}\right] \\
&= t\beta^2(\lambda + \cos(\omega x) + \sin(\omega y))e^\beta\mathcal{M}^{-1}\left[-\frac{1}{s} - \frac{1}{1-s} + \frac{\frac{1}{2}}{s+1} + \frac{\frac{1}{2}}{1-s}\right] \\
&= t\beta^2(\lambda + \cos(\omega x) + \sin(\omega y))e^\beta\mathcal{M}^{-1}\left[-\frac{1}{s} + \frac{\frac{1}{2}}{s+1} + \frac{\frac{1}{2}}{s-1}\right] \\
&= \beta^2(\lambda + \cos(\omega x) + \sin(\omega y))e^\beta\left[-t + \frac{t^2}{2} + \frac{1}{2}\right] \\
u_2^1(t, x, y) &= (\lambda + \cos(\omega x) + \sin(\omega y))e^\beta\frac{(\beta(t-1))^2}{2}
\end{aligned}$$

$$\begin{aligned}
u_3^1(t, x, y) &= t\mathcal{M}^{-1}\left[\frac{\beta}{1-s}\mathcal{M}(u_2^1(t, x, y))\right] \\
&= t\beta^3(\lambda + \cos(\omega x) + \sin(\omega y))e^\beta\mathcal{M}^{-1}\left[\frac{1}{1-s}\mathcal{M}\left[-t + \frac{t^2}{2} + \frac{1}{2}\right]\right] \\
&= t\beta^3(\lambda + \cos(\omega x) + \sin(\omega y))e^\beta\mathcal{M}^{-1}\left[\frac{1}{1-s}\left[-\frac{1}{s+1} + \frac{1}{2(s+2)} + \frac{1}{2s}\right]\right] \\
&= t\beta^3(\lambda + \cos(\omega x) + \sin(\omega y))e^\beta\mathcal{M}^{-1}\left[\frac{1}{(s+1)(s-1)} - \frac{1}{2(s+2)(s-1)} - \frac{1}{2s(s-1)}\right]
\end{aligned}$$

we obtain :

$$u_3^1(t, x, y) = t\beta^3(\lambda + \cos(\omega x) + \sin(\omega y))e^\beta\mathcal{M}^{-1}\left[\frac{\frac{1}{2}}{s-1} + \frac{-\frac{1}{2}}{s+1} - \frac{1}{2}\left(\frac{-\frac{1}{3}}{s+2} + \frac{\frac{1}{3}}{s-1}\right) - \frac{1}{2}\left(\frac{-1}{s} + \frac{1}{s-1}\right)\right].$$

either finally

$$\begin{aligned}
u_3^1(t, x, y) &= t\beta^3(\lambda + \cos(\omega x) + \sin(\omega y))e^\beta\left(\frac{1}{2}t^{-1} - \frac{1}{2}t + \frac{1}{6}(t^2 - t^{-1}) + \frac{1}{2}(1 - t^{-1})\right) \\
&= \beta^3(\lambda + \cos(\omega x) + \sin(\omega y))e^\beta\left(\frac{3 - 3t^2 + t^3 - 1 + 3t - 3}{6}\right) \\
&= \beta^3(\lambda + \cos(\omega x) + \sin(\omega y))e^\beta\left(\frac{t^3 - 3t^2 + 3t - 1}{3!}\right) \\
u_3^1(t, x, y) &= (\lambda + \cos(\omega x) + \sin(\omega y))e^\beta\frac{[\beta(t-1)]^3}{3!}
\end{aligned}$$

It has been shown repeatedly that

$$u_n^1(x, t) = (\lambda + \cos(\omega x) + \sin(\omega y))e^\beta\frac{[\beta(t-1)]^n}{n!} \quad (70)$$

$$u^1(x, t) = \sum_{n=0}^{+\infty} u_n^1 = (\lambda + \cos(\omega x) + \sin(\omega y))e^\beta e^{(\beta(t-1))} = (\lambda + \cos(\omega x) + \sin(\omega y))e^{\beta t} \quad (71)$$

Calculation of $u^2(t, x, y)$

$$\begin{cases} u_0^2(t, x, y) = u(1, x, t) + N(\bar{u}^1(t, x, y)) \\ u_n^2(t, x, y) = t\mathcal{M}^{-1}\left[\frac{\beta}{1-s}\mathcal{M}(u_{n-1}^2(t, x, y))\right] \end{cases} \quad (72)$$

$$N(\bar{u}^1(t, x, y)) = t\mathcal{M}^{-1}\left[\frac{1}{1-s}\mathcal{M}(N(u(t, x, y)))\right]$$

with

$$\begin{aligned} N(u^1(t, x, y)) &= [\Delta u^1(t, x, y)]^{2m} - [\lambda\omega^2 e^{\beta t} - \omega^2 u^1(t, x, y)]^{2m} \\ &= [(-\omega^2 \cos(\omega x))e^{\beta t} - \omega^2 \sin(\omega x)]e^{\beta t}]^{2m} - [\lambda\omega^2 e^{\beta t} - \omega^2(\lambda + \cos(\omega x) + \sin(\omega y))e^{\beta t}]^{2m} \\ &= [(\cos(\omega x) + \sin(\omega y))e^{\beta t}]^{2m} - [(\cos(\omega x) + \sin(\omega y))e^{\beta t}]^{2m} \\ &= 0 \end{aligned}$$

We deduce that $N(\bar{u}^1(t, x, y)) = 0$.

We obtain :

$$\begin{cases} u_0^2(t, x, y) = u(1, x, t) \\ u_n^2(t, x, y) = t\mathcal{M}^{-1}\left[\frac{\beta}{1-s}\mathcal{M}(u_{n-1}^2(t, x, y))\right] \end{cases} \quad (73)$$

calculation of u_n^2

$$\begin{aligned} u_1^2(t, x, y) &= t\mathcal{M}^{-1}\left[\frac{\beta}{1-s}\mathcal{M}(u_0^2(t, x, y))\right] \\ &= t(\lambda + \cos(\omega x) + \sin(\omega y))e^{\beta} \mathcal{M}^{-1}\left[\frac{\beta}{s(1-s)}\right] \\ &= t\beta(\lambda + \cos(\omega x) + \sin(\omega y))e^{\beta} \mathcal{M}^{-1}\left[\frac{1}{s} + \frac{1}{1-s}\right] \\ &= t\beta(\lambda + \cos(\omega x) + \sin(\omega y))e^{\beta}\left[1 - \frac{1}{t}\right] \\ u_1^2(t, x, y) &= (\lambda + \cos(\omega x) + \sin(\omega y))e^{\beta}\beta(t-1) \end{aligned}$$

$$\begin{aligned} u_2^2(t, x, y) &= t\mathcal{M}^{-1}\left[\frac{\beta}{1-s}\mathcal{M}(u_1^2(t, x, y))\right] \\ &= t\beta^2(\lambda + \cos(\omega x) + \sin(\omega y))e^{\beta} \mathcal{M}^{-1}\left[\frac{1}{1-s}\mathcal{M}(t-1)\right] \\ &= t\beta^2(\lambda + \cos(\omega x) + \sin(\omega y))e^{\beta} \mathcal{M}^{-1}\left[-\frac{1}{s(1-s)} + \frac{1}{(s+1)(1-s)}\right] \\ &= t\beta^2(\lambda + \cos(\omega x) + \sin(\omega y))e^{\beta} \mathcal{M}^{-1}\left[-\frac{1}{s} - \frac{1}{1-s} + \frac{\frac{1}{2}}{s+1} + \frac{\frac{1}{2}}{1-s}\right] \\ &= t\beta^2(\lambda + \cos(\omega x) + \sin(\omega y))e^{\beta} \mathcal{M}^{-1}\left[-\frac{1}{s} + \frac{\frac{1}{2}}{s+1} + \frac{\frac{1}{2}}{s-1}\right] \\ &= \beta^2(\lambda + \cos(\omega x) + \sin(\omega y))e^{\beta}\left[-t + \frac{t^2}{2} + \frac{1}{2}\right] \\ u_2^1(t, x, y) &= (\lambda + \cos(\omega x) + \sin(\omega y))e^{\beta}\frac{(\beta(t-1))^2}{2} \end{aligned}$$

$$\begin{aligned}
u_3^2(t, x, y) &= t\mathcal{M}^{-1}\left[\frac{\beta}{1-s}\mathcal{M}(u_2^2(t, x, y))\right] \\
&= t\beta^3(\lambda + \cos(\omega x) + \sin(\omega y))e^\beta\mathcal{M}^{-1}\left[\frac{1}{1-s}\mathcal{M}\left[-t + \frac{t^2}{2} + \frac{1}{2}\right]\right] \\
&= t\beta^3(\lambda + \cos(\omega x) + \sin(\omega y))e^\beta\mathcal{M}^{-1}\left[\frac{1}{1-s}\left[-\frac{1}{s+1} + \frac{1}{2(s+2)} + \frac{1}{2s}\right]\right] \\
&= t\beta^3(\lambda + \cos(\omega x) + \sin(\omega y))e^\beta\mathcal{M}^{-1}\left[\frac{1}{(s+1)(s-1)} - \frac{1}{2(s+2)(s-1)} - \frac{1}{2s(s-1)}\right]
\end{aligned}$$

we obtain :

$$u_3^2(t, x, y) = t\beta^3(\lambda + \cos(\omega x) + \sin(\omega y))e^\beta\mathcal{M}^{-1}\left[\frac{\frac{1}{2}t^{-1} + \frac{-\frac{1}{2}}{s+1} - \frac{1}{2}\left(\frac{-\frac{1}{3}}{s+2} + \frac{\frac{1}{3}}{s-1}\right) - \frac{1}{2}\left(\frac{-1}{s} + \frac{1}{s-1}\right)}{1}\right] \text{ we obtain}$$

$$\begin{aligned}
u_3^2(t, x, y) &= t\beta^3(\lambda + \cos(\omega x) + \sin(\omega y))e^\beta\left(\frac{1}{2}t^{-1} - \frac{1}{2}t + \frac{1}{6}(t^2 - t^{-1}) + \frac{1}{2}(1 - t^{-1})\right) \\
&= \beta^3(\lambda + \cos(\omega x) + \sin(\omega y))e^\beta\left(\frac{3 - 3t^2 + t^3 - 1 + 3t - 3}{6}\right) \\
&= \beta^3(\lambda + \cos(\omega x) + \sin(\omega y))e^\beta\left(\frac{t^3 - 3t^2 + 3t - 1}{3!}\right) \\
u_3^2(t, x, y) &= (\lambda + \cos(\omega x) + \sin(\omega y))e^\beta\frac{[\beta(t-1)]^3}{3!}
\end{aligned}$$

It has been shown repeatedly that

$$u_n^2(x, t) = (\lambda + \cos(\omega x) + \sin(\omega y))e^\beta\frac{[\beta(t-1)]^n}{n!} \quad (74)$$

$$u^2(t, x, y) = \sum_{n=0}^{+\infty} u_n^2 = (\lambda + \cos(\omega x) + \sin(\omega y))e^\beta e^{\beta(t-1)} = (\lambda + \cos(\omega x) + \sin(\omega y))e^{\beta t} \quad (75)$$

we obtain :

$$u^1(t, x, y) = u^2(t, x, y) \quad (76)$$

We deduce the general solution to the problem :

$$u(t, x, y) = (\lambda + \cos(\omega x) + \sin(\omega y))e^{\beta t} \quad (77)$$

5. Conclusion :

In this paper, we have successfully tested the solution of some nonlinear problems by the Mellin-SBA method. In the future, we plan to use the method to solve fractional order partial differential equations.

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