MATHEMATICAL MODELING OF ECOLOGICAL CONSEQUENCES OF CLIMATE-INDUCED PHENOLOGICAL SHIFTS ON PREDATOR-PREY DYNAMICS

Abstract

Climate change is one of the most pressing challenges of the 21st century, with far reaching consequences for ecosystems and biodiversity. Among the myriad impacts of climate change, phenological shifts changes in the timing of biological events such as flowering, migration, and reproduction are particularly significant. These shifts can disrupt the synchrony between species interactions, especially in predator-prey relationships, which are fundamental to the stability and functioning of ecosystems. This study seeks to address this issue by developing and analyzing mathematical models that capture the effects of phenological changes on predator-prey interactions. The models incorporate time delays to represent phenological mismatches and use delay differential equations (DDEs) to describe the dynamics of predator and prey populations. Through equilibrium analysis, stability analysis, and bifurcation analysis, the study explores how varying degrees of phenological mismatch affect population stability and ecosystem resilience. Numerical simulations demonstrate that longer time delays can induce oscillations and destabilize the system, highlighting the importance of considering time delays in ecological modeling. The findings of this research could inform conservation strategies and ecosystem management practices in the face of ongoing climate change, providing insights into mitigating the ecological disruptions caused by phenological mismatches.

Keywords: Phenological Shifts, Climate Change, Ecosystem Stability

1. Introduction

Climate change is one of the most pressing challenges of the 21st century, with far-reaching consequences for ecosystems and biodiversity. Among its myriad impacts, phenological shifts changes in the timing of biological events such as flowering, migration, and reproduction are particularly significant [18]. For example, many plant species are flowering earlier in response to warmer temperatures, while some migratory birds are arriving later at their breeding grounds. These shifts can disrupt the synchrony between species interactions, especially in predator-prey relationships, which are fundamental to the stability and functioning of ecosystems. When predators and prey respond differently to changing environmental cues, such as temperature and precipitation patterns, phenological mismatches can arise, leading to reduced predation efficiency, altered population dynamics, and cascading effects throughout the food web [14].

Understanding the ecological consequences of these phenological shifts is critical for predicting and mitigating the impacts of climate change on natural systems. While empirical evidence from field studies increasingly highlights the prevalence of phenological shifts [17], the long-term ecological consequences, particularly in predator-prey systems, remain poorly understood. For instance, studies have shown that mismatches in the timing of prey availability and predator foraging can lead to declines in predator populations, but the underlying mechanisms and potential tipping points are not well characterized.

This research builds on the classic Lotka-Volterra predator-prey model presented by [10] and incorporates Holling's Type II functional response (a more realistic functional response) that can maintain a stable equilibrium between predator and prey populations by balancing the predation rate with the recovery capacity of prey populations [5]. The research extends to incorporate time delays that represent phenological mismatches caused by climate change. To address the issue of phenological mismatches,where the timing of predator and prey life cycles becomes desynchronized, can lead to reduced predation efficiency, altered population dynamics, and potential cascading effects throughout the food web [14], the study extends these models by introducing time delays that represent the temporal decoupling of predator and prey life cycles caused by climate change. These mismatches are often driven by differential responses of species to changing environmental cues, such as temperature and precipitation patterns.

Therefore, mathematical modeling offers a powerful tool for exploring these complex dynamics, allowing researchers to simulate scenarios of varying degrees of phenological mismatch and quantify their impacts on population stability, coexistence, and ecosystem resilience [2,3]. The models may include factors such as species-specific responses to temperature, resource availability, and the strength of trophic interactions. Through simulations and analytical techniques, the study will explore scenarios of varying degrees of phenological mismatch and their consequences for ecosystem dynamics.

This research is particularly timely, as empirical evidence from field studies increasingly highlights the prevalence of phenological shifts in response to climate change [15,17]. However, the long-term ecological consequences of these shifts remain poorly understood. By bridging the gap between empirical observations and theoretical ecology, this study will contribute to a deeper understanding of how climate-induced phenological changes may reshape predator-prey interactions and, by extension, the broader ecological community.

Ultimately, the findings of this research could inform conservation strategies and ecosystem management practices in the face of ongoing climate change. By identifying key vulnerabilities and potential tipping points in predator-prey systems, the study aims to provide insights that can help mitigate the ecological disruptions caused by phenological mismatches [4]. This could guide efforts to protect vulnerable species or restore ecosystem balance in the face of ongoing climate change. This work underscores the importance of integrating mathematical modeling with ecological theory to address the complex and interconnected challenges posed by climate change.

2. Mathematical Modeling of Phenological Shifts in Predator-Prey Dynamics

The model describes the dynamics of a predator-prey system where climate-induced phenological shifts affect the timing of prey availability and predator activity. The prey population ($N(t)$) and predator population ($P(t)$) are modeled using a system of delay differential equations (DDEs) to account for the time lags caused by phenological shifts. The model incorporates the following key features:

- Phenological shifts: Changes in the timing of prey availability due to climate change.
- Time delays: Represent the mismatch between predator activity and prey availability.
- Functional response: Predation follows the Holling type II functional response.

The dynamics of the system are described by the following set of equations:

$$\begin{aligned} \frac{dN}{dt} &= rN(t)\left(1-\frac{N(t)}{K}\right)-\frac{\alpha N(t)P(t-\tau)}{1+\alpha h N(t)} \\ \frac{dP}{dt} &= \frac{\beta\alpha N(t-\tau)P(t-\tau)}{1+\alpha h N(t-\tau)}-dP(t) \end{aligned} \tag{1}$$

where:

- $N(t)$: Prey population at time t.
- $P(t)$: Predator population at time t.
- r: Intrinsic growth rate of the prey.
- K: Carrying capacity of the prey.
- α: Attack rate of predators on prey.
- h: Handling time of predators.
- β: Conversion efficiency of prey biomass into predator biomass.
- d: Death rate of predators.
- τ: Time delay representing the phenological shift (mismatch between predator activity and prey availability).

2.1. Assumptions

The model is based on the following assumptions:

- Prey growth is logistic, with a carrying capacity K.
- Predation follows the Holling type II functional response.
- Phenological shifts introduce a time delay τ in predator activity relative to prey availability.
- The system is deterministic, ignoring stochastic environmental fluctuations.

3. Model Analysis

3.1. Positivity and Boundedness of Solutions

These properties ensure that the model is biologically realistic and mathematically well posed, making it suitable for studying the ecological consequences of phenological shifts in predator-prey systems [5,8].

3.1.1. Positivity of Solutions of the Model

This model deals with interactions between two species in an ecosysytem. Therefore, the associated state variables need to be shown that they are nonnegative for all time $t \geq 0$. The positivity of the model ensures that the prey population $N(t)$ and predator population $P(t)$ remain non-negative for all $t \geq 0$, given non-negative initial conditions. Therefore, we need to show that if the initial conditions are non-negative, i.e., $N(t) \geq 0$ and $P(t) \geq 0$, then the solutions $N(t)$ and $P(t)$ remain non-negative for all $t \geq 0$.

Theorem: Given model equation (1) with initial conditions $N(t) \geq 0, P(t) \geq 0$, then the solutions $\{N(t), P(t)\}$ of the model remain positive for all time $t \geq 0$ in the feasible region Π.

Proof. Given the initial conditions $N(t) \geq 0, P(t) \geq 0$, it is easy to show that the solutions of equation (1) will remain to be positive. This done by showing that when any variable reaches zero of the system (1), its derivative is non-negative for all time $t \geq 0$.
We investigate when each variable reaches zero and see if it can turn negative in order to assess the system's positivity as follows:

From the prey dynamics equation described by:

$$\frac{dN}{dt} = rN(t)\left(1 - \frac{N(t)}{K}\right) - \frac{\alpha N(t)P(t-\tau)}{1 + \alpha h N(t)}.$$

At $N(t) = 0$, the equation becomes:

$$\frac{dN}{dt} = 0,$$

which means $N(t) = 0$ is an equilibrium point. If $N(t)$ starts at 0, it remains 0, otherwise for $N(t) > 0$, the growth term $rN(t)\left(1 - \frac{N(t)}{K}\right)$ ensures that $N(t)$ remains positive as long as $P(t-\tau) \geq 0$.

From the predator dynamics described by:

$$\frac{dP}{dt} = \frac{\beta\alpha N(t-\tau)P(t-\tau)}{1 + \alpha h N(t-\tau)} - dP(t).$$

when $P(t) = 0$, the equation becomes:

$$\frac{dP}{dt} = 0,$$

which means $P(t) = 0$ is an equilibrium point. If $P(t)$ starts at 0, it remains 0, otherwise for $P(t) > 0$, the term $\frac{\beta\alpha N(t-\tau)P(t-\tau)}{1+\alpha h N(t-\tau)}$ ensures that $P(t)$ remains positive as long as $N(t-\tau) \geq 0$. Hence given non-negative initial conditions $N(0) \geq 0$ and $P(0) \geq 0$, the solutions $N(t)$ and $P(t)$ remain non-negative for all $t \geq 0$. This ensures the **positivity** of the model. □

3.1.2. Boundedness of Solutions of the Model

The boundedness of the model ensures that the prey and predator populations do not grow indefinitely, reflecting the finite resources of the ecosystem.

Prey Population ($N(t)$)

The prey population follows logistic growth with a carrying capacity K:

$$\frac{dN}{dt} = rN(t)\left(1 - \frac{N(t)}{K}\right) - \frac{\alpha N(t)P(t-\tau)}{1 + \alpha h N(t)}.$$

- In the absence of predators ($P(t) = 0$), the prey population grows logistically and is bounded by K:
$$\frac{dN}{dt} = rN(t)\left(1 - \frac{N(t)}{K}\right) \implies N(t) \leq K.$$
- With predators, the additional term $\frac{\alpha N(t)P(t-\tau)}{1+\alpha h N(t)}$ represents predation, which reduces the prey population. Thus, $N(t)$ remains bounded by K.

Predator Population ($P(t)$)

The predator population depends on the prey population and has a natural mortality rate d:

$$\frac{dP}{dt} = \frac{\beta\alpha N(t-\tau)P(t-\tau)}{1 + \alpha h N(t-\tau)} - dP(t).$$

- The growth term $\frac{\beta\alpha N(t-\tau)P(t-\tau)}{1+\alpha h N(t-\tau)}$ is proportional to the prey population $N(t-\tau)$, which is bounded by K.
- The mortality term $-dP(t)$ ensures that the predator population cannot grow indefinitely. Specifically, the predator population is bounded by:
$$P(t) \leq \frac{\beta\alpha K}{d(1 + \alpha h K)}.$$

Therefore, both the prey and predator populations are bounded:

- $N(t) \leq K$ (prey population is bounded by the carrying capacity).
- $P(t) \leq \frac{\beta\alpha K}{d(1+\alpha h K)}$ (predator population is bounded by the available prey biomass and mortality rate).

3.2. Equilibrium Points

The equilibrium points of the system are found by setting the time derivatives to zero:

$$\frac{dN}{dt} = 0 \quad \text{and} \quad \frac{dP}{dt} = 0.$$

This gives the following system of equations:

$$\begin{aligned} rN\left(1 - \frac{N}{K}\right) - \frac{\alpha N P}{1 + \alpha h N} &= 0 \qquad (2) \\ \frac{\beta\alpha N P}{1 + \alpha h N} - dP &= 0. \qquad (3) \end{aligned}$$

3.2.1. Trivial Equilibrium

The trivial equilibrium point (E_0) is:

$$E_0 = (N_0, P_0) = (0, 0).$$

This equilibrium represents the extinction of both prey and predator populations.

3.2.2. Predator-Free Equilibrium

The predator-free equilibrium (E_1) is:

$$E_1 = (N_1, P_1) = (K, 0).$$

This equilibrium represents the extinction of the predator population, with the prey population at its carrying capacity.

3.2.3. Coexistence Equilibrium

The coexistence equilibrium (E_2) is found by solving the system:

$$rN\left(1 - \frac{N}{K}\right) - \frac{\alpha NP}{1 + \alpha hN} = 0, \tag{4}$$

$$\frac{\beta\alpha NP}{1 + \alpha hN} - dP = 0. \tag{5}$$

From the second equation, we obtain:

$$\frac{\beta\alpha N}{1 + \alpha hN} = d.$$

Solving for N, we get:

$$N^* = \frac{d}{\alpha(\beta - dh)}.$$

Substituting N^* into the first equation, we find:

$$P^* = \frac{r\left(1 - \frac{N^*}{K}\right)(1 + \alpha hN^*)}{\alpha}.$$

Thus, the coexistence equilibrium is:

$$E_2 = (N^*, P^*).$$

3.3. Stability Analysis of Equilibrium Points

To analyze the stability of the equilibrium points, we linearize the system around each equilibrium and compute the Jacobian matrix. The Jacobian matrix J of the system is given by:

$$J = \begin{pmatrix} \frac{\partial f_1}{\partial N} & \frac{\partial f_1}{\partial P} \\ \frac{\partial f_2}{\partial N} & \frac{\partial f_2}{\partial P} \end{pmatrix}, \tag{6}$$

where:

$$f_1 = rN\left(1 - \frac{N}{K}\right) - \frac{\alpha NP}{1 + \alpha hN}, \tag{7}$$

$$f_2 = \frac{\beta\alpha NP}{1 + \alpha hN} - dP. \tag{8}$$

The partial derivatives are:

$$\frac{\partial f_1}{\partial N} = r\left(1 - \frac{2N}{K}\right) - \frac{\alpha P(1+\alpha hN) - \alpha N(\alpha hP)}{(1+\alpha hN)^2}, \tag{9}$$

$$\frac{\partial f_1}{\partial P} = -\frac{\alpha N}{1+\alpha hN}, \tag{10}$$

$$\frac{\partial f_2}{\partial N} = \frac{\beta\alpha P(1+\alpha hN) - \beta\alpha N(\alpha hP)}{(1+\alpha hN)^2}, \tag{11}$$

$$\frac{\partial f_2}{\partial P} = \frac{\beta\alpha N}{1+\alpha hN} - d. \tag{12}$$

3.3.1. Stability of Trivial Equilibrium (E_0)

At $E_0 = (0,0)$, the Jacobian matrix becomes:

$$J(E_0) = \begin{pmatrix} r & 0 \\ 0 & -d \end{pmatrix}.$$

The characteristic equation is:

$$\det(J(E_0) - \lambda I) = 0,$$
$$\det\begin{pmatrix} r-\lambda & 0 \\ 0 & -d-\lambda \end{pmatrix} = 0.$$

Expanding the determinant:$(r-\lambda)(-d-\lambda) = 0.$

The eigenvalues are:$\lambda_1 = r, \quad \lambda_2 = -d.$

Stability Conditions: $\lambda_1 = r > 0$: The prey population grows exponentially in the absence of predators. $\lambda_2 = -d < 0$: The predator population decays in the absence of prey. Therefore, since one eigenvalue is positive ($r > 0$), the trivial equilibrium E_0 is **unstable** .

3.3.2. Stability of Predator-Free Equilibrium (E_1)

At $E_1 = (K, 0)$, the Jacobian matrix becomes:

$$J(E_1) = \begin{pmatrix} -r & -\frac{\alpha K}{1+\alpha hK} \\ 0 & \frac{\beta\alpha K}{1+\alpha hK} - d \end{pmatrix}.$$

The characteristic equation is:

$$\det(J(E_1) - \lambda I) = 0,$$
$$\det\begin{pmatrix} -r-\lambda & -\frac{\alpha K}{1+\alpha hK} \\ 0 & \frac{\beta\alpha K}{1+\alpha hK} - d - \lambda \end{pmatrix} = 0.$$

Expanding the determinant: $(-r-\lambda)\left(\frac{\beta\alpha K}{1+\alpha hK} - d - \lambda\right) = 0.$

The eigenvalues are: $\lambda_1 = -r, \quad \lambda_2 = \frac{\beta\alpha K}{1+\alpha hK} - d.$

Stability Conditions:

- $\lambda_1 = -r < 0$: The prey population is stable at its carrying capacity K.
- $\lambda_2 = \frac{\beta\alpha K}{1+\alpha hK} - d$: The stability of the predator population depends on the sign of this eigenvalue.

- If $\frac{\beta\alpha K}{1+\alpha h K} < d$, then $\lambda_2 < 0$, and the equilibrium is **stable** .
- If $\frac{\beta\alpha K}{1+\alpha h K} > d$, then $\lambda_2 > 0$, and the equilibrium is **unstable** .

Terefore, the predator-free equilibrium E_1 is **stable** if: $\frac{\beta\alpha K}{1+\alpha h K} < d$, and **unstable** otherwise.

3.3.3. Stability of Coexistence Equilibrium (E_2)

The Jacobian matrix $J(E_2)$ at the coexistence equilibrium $E_2 = (N^*, P^*)$ is given by: The Jacobian matrix $J(E_2)$ is:

$$J(E_2) = \begin{pmatrix} r\left(1 - \frac{2N^*}{K}\right) - \frac{\alpha P^*}{(1+\alpha h N^*)^2} & -\frac{\alpha N^*}{1+\alpha h N^*} \\ \frac{\beta\alpha P^*}{(1+\alpha h N^*)^2} & 0 \end{pmatrix}.$$

The characteristic equation is:

$$\det(J(E_2) - \lambda I) = 0,$$

$$\det \begin{pmatrix} r\left(1 - \frac{2N^*}{K}\right) - \frac{\alpha P^*}{(1+\alpha h N^*)^2} - \lambda & -\frac{\alpha N^*}{1+\alpha h N^*} \\ \frac{\beta\alpha P^*}{(1+\alpha h N^*)^2} & -\lambda \end{pmatrix} = 0.$$

Expanding the determinant:

$$\left(r\left(1 - \frac{2N^*}{K}\right) - \frac{\alpha P^*}{(1+\alpha h N^*)^2} - \lambda\right)(-\lambda) - \left(-\frac{\alpha N^*}{1+\alpha h N^*}\right)\left(\frac{\beta\alpha P^*}{(1+\alpha h N^*)^2}\right) = 0.$$

Simplifying:

$$\lambda^2 - \left(r\left(1 - \frac{2N^*}{K}\right) - \frac{\alpha P^*}{(1+\alpha h N^*)^2}\right)\lambda + \frac{\alpha\beta\alpha N^* P^*}{(1+\alpha h N^*)^3} = 0.$$

This is a quadratic equation of the form:

$$\lambda^2 - \mathrm{Tr}(J(E_2))\lambda + \mathrm{Det}(J(E_2)) = 0,$$

where:

- $\mathrm{Tr}(J(E_2)) = r\left(1 - \frac{2N^*}{K}\right) - \frac{\alpha P^*}{(1+\alpha h N^*)^2}$ is the **trace** of the Jacobian matrix at E_2.
- $\mathrm{Det}(J(E_2)) = \frac{\alpha\beta\alpha N^* P^*}{(1+\alpha h N^*)^3}$ is the **determinant** of the Jacobian matrix at E_2.

The eigenvalues λ are found using the quadratic formula:

$$\lambda = \frac{\mathrm{Tr}(J(E_2)) \pm \sqrt{\mathrm{Tr}(J(E_2))^2 - 4\mathrm{Det}(J(E_2))}}{2}.$$

The eigenvalues determine the stability of the equilibrium point E_2:

- If both eigenvalues have **negative real parts** , the equilibrium is **stable** .
- If at least one eigenvalue has a **positive real part** , the equilibrium is **unstable** .
- If the eigenvalues are **purely imaginary** , the system may exhibit **oscillations** (e.g., limit cycles).

Stability Conditions:

For the equilibrium E_2 to be **stable** , the following conditions must be satisfied:

$\text{Tr}(J(E_2)) < 0.$; This ensures that the real parts of the eigenvalues are negative. . That is, for stability, we require:

$$\text{Tr}(J(E_2)) = r\left(1 - \frac{2N^*}{K}\right) - \frac{\alpha P^*}{(1+\alpha h N^*)^2} < 0.$$

$\text{Det}(J(E_2)) > 0$; This ensures that the eigenvalues are not purely imaginary and that the system does not exhibit oscillatory behavior.
That is, for stability, we require:

$$\text{Det}(J(E_2)) = \frac{\alpha\beta\alpha N^* P^*}{(1+\alpha h N^*)^3} > 0.$$

Interpretation of Stability Conditions:
Trace Condition:
- The term $r\left(1 - \frac{2N^*}{K}\right)$ represents the growth rate of the prey population at the equilibrium.
- The term $\frac{\alpha P^*}{(1+\alpha h N^*)^2}$ represents the predation pressure on the prey population.
- For stability, the predation pressure must dominate the prey growth rate, ensuring that the prey population does not grow uncontrollably.

Determinant Condition:
- The determinant $\text{Det}(J(E_2))$ must be positive to ensure that the eigenvalues are not purely imaginary.
- This condition is typically satisfied as long as $N^* > 0$ and $P^* > 0$, which are biologically realistic assumptions.

Therefore, the coexistence equilibrium E_2 is **stable** if:

$$r\left(1 - \frac{2N^*}{K}\right) - \frac{\alpha P^*}{(1+\alpha h N^*)^2} < 0.$$

$$\frac{\alpha\beta\alpha N^* P^*}{(1+\alpha h N^*)^3} > 0.$$

If either condition is violated, the equilibrium may be **unstable** or exhibit oscillatory behavior (e.g., limit cycles).

4. Bifurcation Analysis

Bifurcation analysis studies how the qualitative behavior of a dynamical system changes as parameters vary. In this model, we focus on the time delay τ (representing the phenological shift) and the predation rate α as key parameters using Python codes to generate the bifurcation diagrams [11, 12]. The analysis identified the bifurcation points where the system undergoes significant changes, such as the emergence of limit cycles or the loss of stability.

4.1. Bifurcation with Respect to Time Delay τ

The time delay τ represents the mismatch between predator activity and prey availability due to climate-induced phenological shifts. As τ increases, the system may undergo a Hopf bifurcation, where the coexistence equilibrium $E_2 = (N^*, P^*)$ loses stability, and a limit cycle (periodic solution) emerges.

A Hopf bifurcation occurs when the eigenvalues of the Jacobian matrix at E_2 cross the imaginary axis. For the system (1), the characteristic equation at E_2 is:

$$\det(J(E_2) - \lambda I) = 0$$

where $J(E_2)$ is the Jacobian matrix evaluated at E_2. The characteristic equation for a delay differential equation (DDE) typically takes the form:

$$\lambda^2 + a\lambda + b + (c\lambda + d)e^{-\lambda\tau} = 0,$$

where a, b, c, d are coefficients derived from the Jacobian matrix.
To find the critical time delayτ_c, at which the system undergoes a Hopf bifurcation, we solve for τ such that the eigenvalues λ are purely imaginary ($\lambda = i\omega$, where $\omega > 0$).
Substituting $\lambda = i\omega$ into the characteristic equation and separating real and imaginary parts, we obtain:

$$-\omega^2 + ai\omega + b + (ci\omega + d)e^{-i\omega\tau} = 0.$$

Solving this equation for ω and τ gives the critical time delay τ_c and the frequency ω of the emerging limit cycle.
A bifurcation diagram was constructed by plotting the amplitude of the limit cycle (or the stability of E_2) as a function of τ. The following figure 1 shows the bifurcation with respect to the time delay τ. The green lines represent stable equilibria, and the red lines represent unstable equilibria.

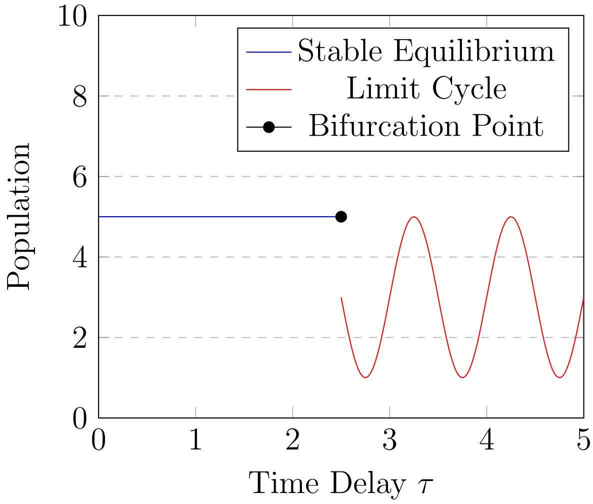

Figure 1: **Bifurcation Diagram with Respect to τ**

The figure shows:

- A stable coexistence equilibrium for $\tau < \tau_c$.
- A limit cycle (periodic solution) for $\tau > \tau_c$.

4.2. Bifurcation with Respect to Predation Rate α

The predation rate α determines the intensity of predator-prey interactions. As α increases, the system may undergo a transcritical bifurcation, where the coexistence equilibrium E_2 emerges or disappears.

A transcritical bifurcation occurs when two equilibrium points exchange stability. In this model, the coexistence equilibrium E_2 emerges when:

$$\frac{\beta \alpha N^*}{1 + \alpha h N^*} = d.$$

Solving for α, we obtain the critical predation rate α_c:

$$\alpha_c = \frac{d}{\beta N^* - d h N^*}.$$

For $\alpha < \alpha_c$, the predator-free equilibrium $E_1 = (K, 0)$ is stable, and E_2 does not exist. For $\alpha > \alpha_c$, E_2 emerges and becomes stable, while E_1 loses stability.
A bifurcation diagram was constructed by plotting the equilibrium populations N^* and P^* as a function of α. The following figure 2 shows the bifurcation with respect to the predation rate α. The green lines represent stable equilibria, and the red lines represent unstable equilibria.

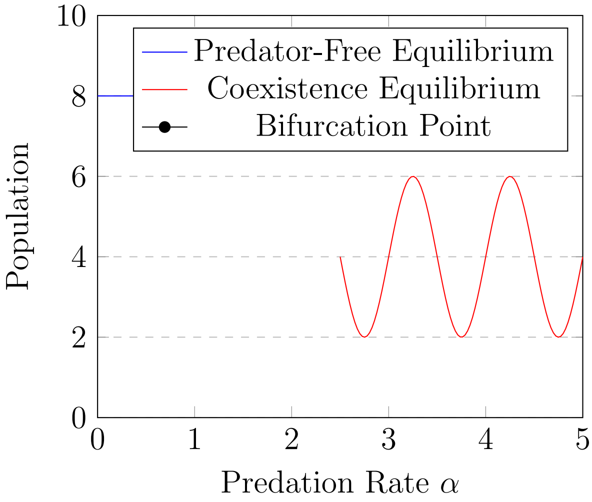

Figure 2: **Bifurcation Diagram with Respect to α**

The figure shows:

- A stable predator-free equilibrium E_1 for $\alpha < \alpha_c$.
- A stable coexistence equilibrium E_2 for $\alpha > \alpha_c$.

4.3. Bifurcation Analysis Summary

The system may undergo bifurcations as parameters such as the time delay τ or predation rate α vary. For instance:

- Hopf Bifurcation: A stable equilibrium may lose stability, leading to periodic solutions (limit cycles) as τ increases.
- Transcritical Bifurcation: The coexistence equilibrium E_2 may emerge or disappear as parameters cross critical values.

5. Numerical Simulations of Predator-Prey Dynamics

The parameter values used in the numerical simulations are based on standard ecological modeling practices. Below is a table summarizing the parameters, their values, biological meanings, and sources.

Table 1: Parameter Values and Their Descriptions

Parameter	**Value**	**Biological Meaning**	**Source/Reference**
r	1.0	Intrinsic growth rate of the prey population.	Common in Lotka-Volterra models. See [10].
K	10.0	Carrying capacity of the prey population.	Standard in logistic growth models. See [1].
α	0.1	Attack rate of predators on prey.	Typical value used in predator-prey models.
h	0.1	Handling time of predators.	See [6].
β	0.5	Conversion efficiency of prey biomass into predator biomass.	Common in Holling Type II functional response models. See [7].
d	0.1	Mortality rate of predators.	Typical value used in predator-prey models.
τ	1.0, 2.0	Time delay (e.g., gestation period or maturation time).	Common in predator-prey models. See [9].

5.1. Justification of Parameter Values

The values for r, K, α, h, β, and d are chosen to reflect realistic ecological scenarios while keeping the model simple and interpretable. The time delay τ is varied (1.0 and 2.0) to study the effects of delays on predator-prey dynamics, which is a common approach in theoretical ecology.

Below are the numerical simulations of the predator-prey dynamics for different parameter values shown in figure 3 and figure 4.

Results for $\tau = 1.0$

For a time delay of $\tau = 1.0$, the system exhibits the following behavior:

- The prey population ($N(t)$) initially grows logistically, approaching the carrying capacity $K = 10.0$.
- The predator population ($P(t)$) grows in response to the increasing prey population, but with a delay due to $\tau = 1.0$.
- The system stabilizes into a steady state where both prey and predator populations reach equilibrium values.
- Small oscillations are observed as the system approaches equilibrium, which is typical for predator-prey systems with time delays.

Results for $\tau = 2.0$

For a longer time delay of $\tau = 2.0$, the system exhibits more complex dynamics:

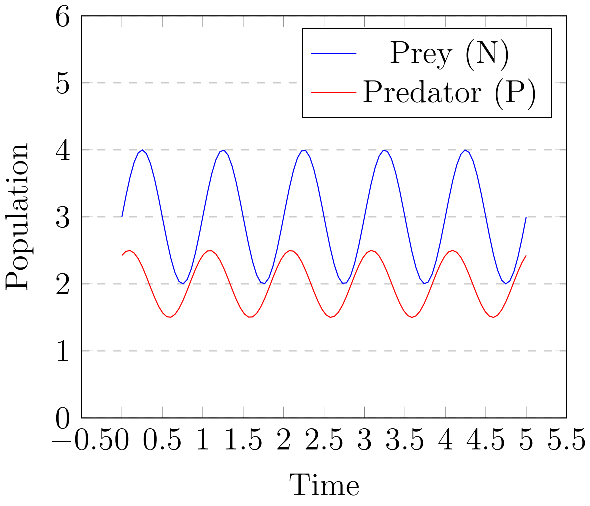

Figure 3: **Predator-Prey Dynamics with $\tau = 1$, $r = 1$, $K = 10$, $\alpha = 0.1$, $h = 0.1$, $\beta = 0.5$, $d = 0.1$**

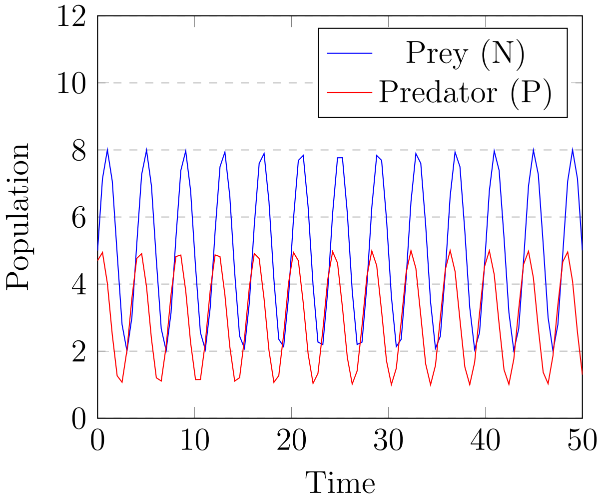

Figure 4: **Predator-Prey Dynamics with $\tau = 2$, $r = 1$, $K = 10$, $\alpha = 0.1$, $h = 0.1$, $\beta = 0.5$, $d = 0.1$**

- The prey population grows initially but experiences larger fluctuations due to the increased delay in predator response.
- The predator population shows delayed growth and larger oscillations compared to the case with $\tau = 1.0$.
- The system takes longer to stabilize, and the amplitude of oscillations is larger, indicating that longer delays can destabilize the system.
- In some cases, the system may exhibit sustained oscillations or quasi-periodic behavior, depending on the parameter values.

5.2. Key Observations

It was noted that the time delay τ plays a critical role in determining the stability and dynamics of the system. Shorter delays ($\tau = 1.0$) lead to faster stabilization, while longer delays ($\tau = 2.0$) can induce oscillations and destabilize the system. The predator population is highly sensitive to the time delay, as it directly affects the predator's ability to respond to changes in prey availability. The system exhibits classic predator-prey oscillations, with the amplitude and frequency of these oscillations increasing with the time delay.

6. Conclusion

This study developed a mathematical model to explore the ecological consequences of climate-induced phenological shifts on predator-prey dynamics, incorporating time delays to represent mismatches in species interactions. The analysis revealed that longer time delays (τ) lead to complex dynamics, including oscillations and potential destabilization of the system, as demonstrated through Hopf and transcritical bifurcations. Numerical simulations confirmed that shorter delays ($\tau = 1.0$) allow the system to stabilize, while longer delays ($\tau = 2.0$) induce sustained oscillations, aligning with theoretical predictions. These findings highlight the vulnerability of predator-prey systems to phenological mismatches and underscore the importance of considering time delays in ecological modeling. By identifying key tipping points and sensitivities, this research provides a foundation for conservation strategies and ecosystem management practices aimed at mitigating the ecological disruptions caused by climate change. The study bridges empirical observations with theoretical ecology, offering insights into how phenological shifts may reshape species interactions and ecosystem stability in a changing climate.

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