Efficient Numerical Solution of Nonlinear Systems of Fractional Partial Differential Equations Using the Conformable Laplace-Adomian Decomposition

Abstract

This study introduces an innovative numerical technique, the Conformable Laplace-Adomian Decomposition Method (CLDM), to address challenges in solving nonlinear fractional partial differential equation systems (FPDEs), where traditional methods like finite difference and Adomian Decomposition Method (ADM) struggle due to numerical instability and inefficiency. CLDM synergizes the advantages of the conformable fractional derivativewhich offers flexible algebraic rules (e.g., product and chain rules)with the Laplace-Adomian decomposition framework, yielding accurate, stable solutions while reducing computational costs. The methods efficacy was validated through applications in fluid mechanics and heat transfer, demonstrating superior accuracy and stability compared to Caputo-based or HPM approaches. This research contributes a novel methodology for handling complex fractional systems, a practical framework for scientific applications, and comparative insights into numerical method performance, paving the way for enhanced modeling of memory-driven phenomena in applied sciences and engineering.

keywords:Fractional calculus, Nonlinear fractional partial differential equations (FPDEs), Conformable fractional derivative, Laplace-Adomian Decomposition Method (LDM), Conformable Laplace-Adomian Decomposition Method (CLDM), Caputo derivative, Riemann-Liouville fractional integral, Numerical stability, Anomalous diffusion, Computational efficiency

1 Introduction

"'Fractional calculus, a generalization of classical calculus to non-integer orders, has emerged as a powerful tool for modeling complex systems that exhibit memory and hereditary properties [1]. Fractional partial differential equations (FPDEs) are widely used in various fields such as physics, engineering, biology, and finance [21, 31]. These equations are particularly useful for describing phenomena such as viscoelasticity, anomalous diffusion, and fractal geometry [24]. Despite the use of traditional numerical methods such as finite difference and finite element methods to solve FPDEs, these methods often suffer from limitations such as numerical instability and low accuracy, especially when dealing with nonlinear and singular problems [4, 33]. To address these challenges, various numerical and analytical methods have been developed, including the Adomian decomposition method (ADM) [2], the variational iteration method (VIM) [13], the homotopy perturbation method (HPM) [10], and the homotopy analysis method (HAM) [13]. However, these methods may not always be efficient or accurate, particularly for complex nonlinear problems [5].

In recent years, fractional calculus has gained significant attention, and various fractional derivative operators have been proposed, including the Riemann-Liouville, Caputo, and conformable derivatives [6, 33]. Each of these operators has its own advantages and limitations, and the choice of operator depends on the specific problem being studied [19].

To further enhance the efficiency and accuracy of solving FPDEs, this research proposes a novel numerical method called the Conformable Laplace-Adomian Decomposition Method (CLDM). This method combines the advantages of the conformable fractional derivative, which possesses properties such as the product rule, quotient rule, chain rule, power rule, and constant rule, with the Laplace-Adomian decomposition method [7, 32]. The conformable derivative offers enhanced accuracy and computational efficiency, while the Laplace-Adomian decomposition method provides a systematic approach for solving nonlinear equations [9].

The CLDM has been successfully applied to solve various types of nonlinear systems of FPDEs, including those arising in fluid mechanics, heat transfer, and control systems [23]. By comparing the CLDM with other well-established numerical techniques such as the ADM, MADM, HPM, Residual Power Series Method (RPSM) [4], and Abood Transform Decomposition Method (ATDM) [5], we aim to demonstrate its effectiveness and advantages [20, 31].

Through a comprehensive literature review, mathematical formulation, numerical implementation, error analysis, and comparative analysis, we aim to assess the advantages and limitations of the proposed method. By building upon the work of previous researchers such as Podlubny [25], Li and Chen [21], and recent advances in integro-differential equations [32], we seek to advance the state-of-the-art in solving nonlinear systems of FPDEs.

The results of this research will have broad implications for various fields, including materials science, fluid mechanics, and biological systems. By developing more accurate and efficient numerical methods for solving systems of FPDEs, we can gain deeper insights into the behavior of complex systems and make significant contributions to scientific and technological advancement [26, 31]. "'

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2 Preliminaries

We'll introduce to the fundamental concepts and theories required for this study. These include the Caputo derivative, fractional conjugate derivative and the Laplace transform [1, 3, 7, 10, 19, 20, 22, 25, 26].

Definition 2.1. Let $\alpha \in \mathbb{R} \setminus \mathbb{N}$ and $\alpha \geq 0$. The Rieman- Liouville fractional partial integral by $\jmath_\tau^\alpha$ of order α for a function $\zeta(\xi,\tau)$ is defined as

$$\jmath_\tau^\alpha \zeta(\xi,\tau) = \begin{cases} \zeta(\xi,\tau) & \text{if} \quad \alpha = 0, \tau > 0, \\ \frac{1}{\Gamma(\alpha)} \int_0^\tau (\tau - \tau_0)^{\alpha-1} \zeta(\xi,\tau) d\tau & \text{if} \quad \alpha, \tau > 0, \end{cases}$$

where Γ is denoted as $\Gamma(\alpha) = \int_0^\infty e^{-\tau}\tau^{\alpha-1}d\tau, \quad \alpha \in C$.

Definition 2.2. For $\alpha \in \mathbb{R}, n-1 < \alpha < n \in \mathbb{N}$, the Rieman- Liouville fractional partial derivative of order α for $\zeta(\xi,\tau)$ is defined as
$\Xi_\tau^\alpha \zeta(\xi,\tau) = \frac{\partial^n}{\partial \tau^n} \int_0^\tau \frac{(\tau-\tau_0)^{n-\alpha-1}}{\Gamma(n-\alpha)} \zeta(\xi,\tau) d\tau$.

Definition 2.3. The fractional derivative of a function $\zeta(\xi,\tau)$ of order α in the Caputa sense is defined

$$c_{\Xi_\tau^\alpha \zeta(\xi,\tau)} = \begin{cases} \frac{\partial^n \zeta(\xi,\tau)}{\partial \tau^n} & \text{if} \quad \alpha = n \in \mathbb{N}, \\ \int_0^\tau \frac{(\tau-\tau_0)^{n-\alpha-1}}{\Gamma(n-\alpha)} \frac{\partial^n \zeta(\xi,\tau)}{\partial \tau^n} d\tau & \text{if} \quad n-1 < \alpha < n \in \mathbb{N}. \end{cases}$$

Definition 2.4. Given a function $\zeta : [0,\infty) \to \mathbb{R}$, the conformable derivative of ζ of order $\alpha \in (0,1]$ is defined as: $\Xi_\tau^\alpha \zeta(\tau) = \lim_{\epsilon \to 0} \frac{\zeta(\tau+\epsilon\tau^{1-\alpha})-\zeta(\tau)}{\epsilon}, \quad \tau > 0$.

Theorem 2.1. *Let $\alpha \in (0,1]$ and ϕ, φ be α-differentiable at a point $\tau > 0$, then*
$1.\Xi_\tau^\alpha(a\phi + b\varphi) = a\Xi_\tau^\alpha(\phi) + b\Xi_\tau^\alpha(\varphi)$ *for all $a, b \in R$,*
$2.\Xi_\tau^\alpha(\tau^m) = m\tau^{m-\alpha}$ *for all $m \in R$,*
$3.\Xi_\tau^\alpha(\zeta(\tau)) = 0$ *for all constant function $\zeta(\tau) = r$,*
$4.\Xi_\tau^\alpha(\phi\varphi) = \phi\Xi_\tau^\alpha(\varphi) + \varphi\Xi_\tau^\alpha(\phi)$*,*
$5.\Xi_\tau^\alpha(\frac{\phi}{\varphi}) = \frac{\varphi\Xi_\tau^\alpha(\phi)-\phi\Xi_\tau^\alpha(\varphi)}{\varphi^2}$*,*
$6.$*If $\zeta(\xi,\tau)$ is differentiable , then*
$\Xi_\tau^\alpha(\zeta(\xi,\tau)) = \tau^{1-\alpha}\frac{d}{d\tau}\zeta(\xi,\tau)$.

Definition 2.5. The Laplace transform of the operator of Caputo fractional derivative $\Xi_\tau^\alpha\zeta(\xi,\tau)$ is defined as follows
$\mathcal{L}[\Xi_\tau^\alpha\zeta(\xi,\tau)] = s^\alpha\zeta(\xi,\tau) - \sum_{k=0}^{n-1} s^{\alpha-k-1}\zeta^{(k)}(\xi,0)$, $n-1<\alpha\leq n$.

Definition 2.6. Let $\zeta(\xi,\tau) : [\tau_0,\infty) \to R$ be a real valued function with $\tau_0 \in \mathbb{R}$ and $0<\alpha\leq 1$. Then conformable Laplace transform of the function ζ of order α is defined by

$$\mathcal{L}_\alpha[\Xi_\tau^\alpha\zeta(\xi,\tau)](s) = \int_{\tau_0}^{\infty} e^{-s\frac{(\tau-\tau_0)^\alpha}{\alpha}}\zeta(\xi,\tau)(\tau-\tau_0)^{\alpha-1}d\tau = F_\alpha(\xi,s). \tag{2.1}$$

If $\tau_0 = 0$, then
$\mathcal{L}_\alpha[\Xi_\tau^\alpha\zeta(\xi,\tau)](s) = \int_0^\infty e^{-s\frac{(\tau)^\alpha}{\alpha}}\zeta(\xi,\tau)d_\alpha = \int_0^\infty e^{-s\frac{\tau^\alpha}{\alpha}}\zeta(\xi,\tau)(\tau)^{\alpha-1}d\tau$
$= F_\alpha(\xi,s)$.
Particular, if $\alpha=1$, then equation(Eq) (2.1) is reduced the definition of the Laplace transform:
$\mathcal{L} = \int_0^\infty e^{-st}\zeta(\xi,\tau)d\tau = F(\xi,s)$.

Theorem 2.2. *Let ζ : $[0,\infty)\to R$ be a differentiable function and $0<\alpha\leq 1$. Then* $\mathcal{L}_\alpha[\Xi_\tau^\alpha\zeta(\xi,\tau)] = s\mathcal{L}_\alpha[\zeta(\xi,\tau)] - \zeta(\xi,0)$.

Theorem 2.3. *Let $c, m \in \mathbb{R}$ and $0<\alpha<1$, then*

$$\begin{aligned}
1 : \mathcal{L}_\alpha[m](s) &= \frac{m}{s}, \quad s>0,\\
2 : \mathcal{L}_\alpha[\tau^m](s) &= \alpha^{\frac{m}{\alpha}}\frac{\Gamma(1+\frac{m}{\alpha})}{s^{1+\frac{m}{\alpha}}}, \quad s>0,\\
3 : \mathcal{L}_\alpha[e^{\frac{c\tau^\alpha}{\alpha}}](s) &= \frac{1}{s-c}, \quad s>0.
\end{aligned}$$

3 Conformable Laplace-Adomian Decomposition Method

$$\begin{aligned}
\Xi_\tau^\alpha(z(\xi,\tau)) + R_1 z(\xi,\tau) + N_1(z,w) = h_1(\xi,\tau),\\
\Xi_\tau^\alpha(w(\xi,\tau)) + R_2 w(\xi,\tau) + N_2(z,w) = h_2(\xi,\tau).
\end{aligned} \tag{3.1}$$

Where $0<\alpha\leq 1$ and the initial conditions

$$\begin{aligned}
z(\xi,0) = \frac{\partial^{i-1}z(\xi,\tau)}{\partial\tau^{i-1}} = f_{i-1}(\xi), \quad i=1,2,\dots\\
w(\xi,0) = \frac{\partial^{i-1}w(\xi,\tau)}{\partial\tau^{i-1}} = g_{i-1}(\xi), \quad i=1,2,\dots
\end{aligned} \tag{3.2}$$

Ξ_τ^α is the Conformable fractional derivatives of the functions $z(\xi,\tau), w(\xi,\tau)$, R_1 and R_2 are the linear derivative operator, N_1, N_2 are the nonlinear terms and $h_1(\xi,\tau), h_2(\xi,\tau)$

are the nonhomogeneous parts.
By applying Laplace transform $\mathcal{L}_\alpha$ and its property to both sides of (3.1), yields

$$\begin{aligned}\mathcal{L}_\alpha[z(\xi,\tau)] &= \frac{1}{s^i}\sum_{k=0}^{i-1}\frac{z^{(k)}(\xi,0)}{s^{1-i+k}} + \frac{1}{s^i}\mathcal{L}_\alpha[h_1(\xi,\tau) - R_1 z(\xi,\tau) - N_1(z,w)],\\ \mathcal{L}_\alpha[w(\xi,\tau)] &= \frac{1}{s^i}\sum_{k=0}^{i-1}\frac{w^{(k)}(\xi,0)}{s^{1-i+k}} + \frac{1}{s^i}\mathcal{L}_\alpha[h_2(\xi,\tau) - R_2 w(\xi,\tau) - N_2(z,w)].\end{aligned} \tag{3.3}$$

By taking the inverse Laplace transform ${\mathcal{L}_\alpha}^{-1}$ of both sides of the Eq. (3.3) in system and then applying initial conditions given in (3.2), we arrive at

$$\begin{aligned}z(\xi,\tau) &= H_1(\xi,\tau) - {\mathcal{L}_\alpha}^{-1}\left[\frac{1}{s^i}\mathcal{L}_\alpha[R_1 z(\xi,\tau) + N_1(z,w)]\right],\\ w(\xi,\tau) &= H_2(\xi,\tau) - {\mathcal{L}_\alpha}^{-1}\left[\frac{1}{s^i}\mathcal{L}_\alpha[R_2 w(\xi,\tau) + N_2(z,w)]\right],\end{aligned} \tag{3.4}$$

where $H_1(\xi,\tau)$ and $H_2(\xi,\tau)$ represent terms due to homogeneous terms and given initial conditions. Using Adomian decomposition method, we assume solution as an infinite series given by

$$z_n(\xi,\tau) = \sum_{n=0}^{\infty} z_n, \quad w_n(\xi,\tau) = \sum_{n=0}^{\infty} w_n. \tag{3.5}$$

The nonlinear operator is decomposed

$$N_1(z_n,w_n) = \sum_{n=0}^{\infty}\theta_n, \quad N_2(z_n,w_n) = \sum_{n=0}^{\infty}\vartheta_n, \tag{3.6}$$

$$\theta_n = \frac{1}{n!}\left[\frac{d^n}{d\lambda^n}N_1\left(\sum_{j=0}^{n}(z_j,w_j)\lambda^j\right)\right]_{\lambda=0}, \quad \vartheta_n = \frac{1}{n!}\left[\frac{d^n}{d\lambda^n}N_2\left(\sum_{j=0}^{n}(z_j,w_j)\lambda^j\right)\right]_{\lambda=0}, \quad n = 0,1,2,\dots$$

where θ_n and ϑ_n are Adomian polynomials.

Using Eq. (3.5) and Eq. (3.6) in Eq. (3.4), the system (3.4) can be rewritten as

$$\begin{aligned}z(\xi,\tau) = \sum_{n=0}^{\infty} z_n &= H_1(\xi,\tau) - {\mathcal{L}_\alpha}^{-1}\left[\frac{1}{s^i}\mathcal{L}_\alpha\left(R_1\sum_{n=0}^{\infty} z_n + \sum_{n=0}^{\infty}\theta_n\right)\right],\\ w(\xi,\tau) = \sum_{n=0}^{\infty} w_n &= H_2(\xi,\tau) - {\mathcal{L}_\alpha}^{-1}\left[\frac{1}{s^i}\mathcal{L}_\alpha\left(R_2\sum_{n=0}^{\infty} w_n + \sum_{n=0}^{\infty}\vartheta_n\right)\right].\end{aligned} \tag{3.7}$$

A comparison of both sides of the Eq (3.7), we find

$$z_0(\xi,\tau) = H_1(\xi,\tau),$$
$$w_0(\xi,\tau) = H_2(\xi,\tau),$$

$$z_1(\xi,\tau) = -\mathcal{L}_\alpha{}^{-1}\left[\frac{1}{s^i}\mathcal{L}_\alpha\left(R_1(z_0)+\theta_0\right)\right],$$
$$w_1(\xi,\tau) = -\mathcal{L}_\alpha{}^{-1}\left[\frac{1}{s^i}\mathcal{L}_\alpha\left(R_2(w_0)+\vartheta_0\right)\right].$$

$$\vdots$$

Through this iterative process, we obtain the general recursive relations.

$$z_{n+1}(\xi,\tau) = -\mathcal{L}_\alpha{}^{-1}\left[\frac{1}{s^i}\mathcal{L}_\alpha\left(R_1(z_n)+\theta_n\right)\right], \quad n \geq 0,$$
$$w_{n+1}(\xi,\tau) = -\mathcal{L}_\alpha{}^{-1}\left[\frac{1}{s^i}\mathcal{L}_\alpha\left(R_2(w_n)+\vartheta_n\right)\right], \quad n \geq 0. \tag{3.8}$$

By applying MADM, we fined

$$z_0(\xi,\tau) = H_1(\xi,\tau) = z(\xi,0),$$
$$w_0(\xi,\tau) = H_2(\xi,\tau) = w(\xi,0),$$

$$z_1(\xi,\tau) = \mathcal{L}_\alpha{}^{-1}\left[\frac{1}{s^i}\mathcal{L}_\alpha\left(h_1(\xi,\tau)-R_1(z_0)-\theta_0\right)\right],$$
$$w_1(\xi,\tau) = \mathcal{L}_\alpha{}^{-1}\left[\frac{1}{s^i}\mathcal{L}_\alpha\left(h_2(\xi,\tau)-R_2(w_0)-\vartheta_0\right)\right].$$

$$\vdots$$

Through this iterative process, we obtain the general recursive relations.

$$z_{n+1}(\xi,\tau) = -\mathcal{L}_\alpha{}^{-1}\left[\frac{1}{s^i}\mathcal{L}_\alpha\left(R_1(z_n)+\theta_n\right)\right], \quad n \geq 1,$$
$$w_{n+1}(\xi,\tau) = -\mathcal{L}_\alpha{}^{-1}\left[\frac{1}{s^i}\mathcal{L}_\alpha\left(R_2(w_n)+\vartheta_n\right)\right], \quad n \geq 1. \tag{3.9}$$

4 Example and Graphical Result

Three nonlinear, nonhomogeneous fractional PDE systems were solved in this section. Maple 2020 was used for computations and visualizations

Example 4.1. *Let us consider a system of nonlinear partial differential equations with derivatives of fractional order in time [11, 14]*

$$\begin{aligned} \frac{\partial^\alpha z(\xi,\tau)}{\partial\tau^\alpha} + z(\xi,\tau) + w(\xi,\tau)\frac{\partial z(\xi,\tau)}{\partial\xi} &= 1, \quad 0<\alpha\le 1, \\ \frac{\partial^\alpha w(\xi,\tau)}{\partial\tau^\alpha} - w(\xi,\tau) - z(\xi,\tau)\frac{\partial w(\xi,\tau)}{\partial\xi} &= 1, \quad 0<\alpha\le 1. \end{aligned} \tag{4.1}$$

Initial Conditions

$$\begin{aligned} z(\xi,0) &= e^{\xi}, \\ w(\xi,0) &= e^{-\xi}. \end{aligned} \tag{4.2}$$

Exact Solution for $\alpha = 1$

$$\begin{aligned} z(\xi,\tau) &= e^{\xi-\tau}, \\ w(\xi,\tau) &= e^{-\xi+\tau}. \end{aligned} \tag{4.3}$$

By rewriting the Eq. (4.1)

$$\begin{aligned} \frac{\partial^\alpha z(\xi,\tau)}{\partial\tau^\alpha} + z(\xi,\tau) &= 1 - w(\xi,\tau)\frac{\partial z(\xi,\tau)}{\partial\xi}, \quad 0<\alpha\le 1, \\ \frac{\partial^\alpha w(\xi,\tau)}{\partial\tau^\alpha} - w(\xi,\tau) &= 1 + z(\xi,\tau)\frac{\partial w(\xi,\tau)}{\partial\xi}, \quad 0<\alpha\le 1. \end{aligned} \tag{4.4}$$

Applying the Laplace transform on both sides of Eq. (4.4) , we obtain

$$\begin{aligned} \mathcal{L}_\alpha[z(\xi,\tau)] &= \frac{e^{\xi}}{s+1} + \frac{1}{s+1}\mathcal{L}_\alpha[1 - w(\xi,\tau)z_\xi(\xi,\tau)], \\ ,\mathcal{L}_\alpha[w(\xi,\tau)] &= \frac{e^{-\xi}}{s-1} + \frac{1}{s-1}\mathcal{L}_\alpha[1 + z(\xi,\tau)w_\xi(\xi,\tau)]. \end{aligned} \tag{4.5}$$

By taking the inverse laplace transform on both sides of Eq (4.5), we get

$$\begin{aligned} z(\xi,\tau) &= e^{\xi-\frac{\tau^\alpha}{\alpha}} - e^{-\frac{\tau^\alpha}{\alpha}} + 1 - \mathcal{L}_\alpha{}^{-1}\left[\frac{1}{s+1}\mathcal{L}_\alpha[w(\xi,\tau)z_\xi(\xi,\tau)]\right], \\ w(\xi,\tau) &= e^{-\xi+\frac{\tau^\alpha}{\alpha}} + e^{\frac{\tau^\alpha}{\alpha}} - 1 + \mathcal{L}_\alpha{}^{-1}\left[\frac{1}{s-1}\mathcal{L}_\alpha[z(\xi,\tau)w_\xi(\xi,\tau)]\right]. \end{aligned} \tag{4.6}$$

The approximate solution can be written as the following infinite series

$$z(\xi,\tau) = \sum_{n=0}^{\infty} z_n, \quad w(\xi,\tau) = \sum_{n=0}^{\infty} w_n. \tag{4.7}$$

The nonlinear terms wz_ξ and zw_ξ are represented by Adomian polynomials:

$$wz_\xi = \sum_{n=0}^{\infty} \theta_n, \quad zw_\xi = \sum_{n=0}^{\infty} \vartheta_n. \tag{4.8}$$

The first few components of the θ_n and ϑ_n polynomials are given by

$$\begin{aligned} &\theta_0 = w_0 z_{0_\xi}, && \vartheta_0 = z_0 w_{0_\xi},\\ &\theta_1 = w_1 z_{0_\xi} + w_o z_{1_\xi}, && \vartheta_1 = z_1 w_{0_\xi} + z_0 w_{1_\xi},\\ &\theta_2 = w_2 z_{0_\xi} + w_1 z_{1_\xi} + w_0 z_{2_\xi}, && \vartheta_2 = z_2 w_{0_\xi} + z_1 w_{1_\xi} + z_0 w_{2_\xi}. \end{aligned}$$

and so on.
Substituting Eq. (4.7) and Eq. (4.8) into Eq. (4.6), we obtain

$$\begin{aligned} z(\xi,\tau) &= \sum_{n=0}^{\infty} z_n = e^{\xi-\frac{\tau^\alpha}{\alpha}} - e^{-\frac{\tau^\alpha}{\alpha}} + 1 - \mathcal{L}_\alpha{}^{-1}\left[\frac{1}{s+1}\mathcal{L}_\alpha[\sum_{n=0}^{\infty}\theta_n]\right],\\ w(\xi,\tau) &= \sum_{n=0}^{\infty} w_n = e^{-\xi+\frac{\tau^\alpha}{\alpha}} + e^{\frac{\tau^\alpha}{\alpha}} - 1 + \mathcal{L}_\alpha{}^{-1}\left[\frac{1}{s-1}\mathcal{L}_\alpha[\sum_{n=0}^{\infty}\vartheta_n]\right], \end{aligned} \tag{4.9}$$

Comparing both sides of equation (4.9), directly gives us the recursive relations

$$z_0 = e^{\xi-\frac{\tau^\alpha}{\alpha}} - e^{-\frac{\tau^\alpha}{\alpha}} + 1, \quad w_0 = e^{-\xi+\frac{\tau^\alpha}{\alpha}} + e^{\frac{\tau^\alpha}{\alpha}} - 1,$$

$$\vdots$$

$$z_{n+1} = -\mathcal{L}_\alpha{}^{-1}\left[\frac{1}{s+1}\mathcal{L}_\alpha[\theta_n]\right], \quad w_{n+1} = \mathcal{L}_\alpha{}^{-1}\left[\frac{1}{s-1}\mathcal{L}_\alpha[\vartheta_n]\right], \quad n \geq 0.$$

The first three approximate terms of $z(\xi,\tau)$ and $w(\xi,\tau)$ are given as follows

$$\begin{aligned} z_0 &= e^{\xi-\frac{\tau^\alpha}{\alpha}} - e^{-\frac{\tau^\alpha}{\alpha}} + 1,\\ w_0 &= e^{-\xi+\frac{\tau^\alpha}{\alpha}} + e^{\frac{\tau^\alpha}{\alpha}} - 1, \end{aligned}$$

$$\begin{aligned} z_1 &= e^{\xi-\frac{\tau^\alpha}{\alpha}} + e^{-\frac{\tau^\alpha}{\alpha}} + \frac{\tau^\alpha}{\alpha}e^{\xi-\frac{\tau^\alpha}{\alpha}} - e^{\xi} - 1,\\ w_1 &= e^{-\xi+\frac{\tau^\alpha}{\alpha}} - e^{\frac{\tau^\alpha}{\alpha}}, - \frac{\tau^\alpha}{\alpha}e^{-\xi+\frac{\tau^\alpha}{\alpha}} - e^{-\xi} + 1, \end{aligned}$$

$$\begin{aligned} z_2 &= -\frac{1}{2}e^{\xi-\frac{\tau^\alpha}{\alpha}} + \frac{1}{2}e^{\xi+\frac{\tau^\alpha}{\alpha}} + \frac{\tau^{2\alpha}}{2\alpha^2}e^{\xi-\frac{\tau^\alpha}{\alpha}} - e^{\xi}\frac{\tau^\alpha}{\alpha} + \frac{\tau^\alpha}{\alpha}e^{-\frac{\tau^\alpha}{\alpha}} + \frac{3}{2}e^{-\frac{\tau^\alpha}{\alpha}} + \frac{1}{2}e^{\frac{\tau^\alpha}{\alpha}} - 2,\\ w_2 &= -\frac{1}{2}e^{-\xi+\frac{\tau^\alpha}{\alpha}} + \frac{1}{2}e^{-\xi-\frac{\tau^\alpha}{\alpha}} + \frac{\tau^{2\alpha}}{2\alpha^2}e^{-\xi+\frac{\tau^\alpha}{\alpha}} + e^{-\xi}\frac{\tau^\alpha}{\alpha} + \frac{\tau^\alpha}{\alpha}e^{\frac{\tau^\alpha}{\alpha}} - \frac{3}{2}e^{\frac{\tau^\alpha}{\alpha}} + \frac{1}{2}e^{-\frac{\tau^\alpha}{\alpha}} + 2. \end{aligned}$$

The series solution for $z(\xi,\tau)$ and $w(\xi,\tau)$ is

$$\begin{aligned}
z(\xi,\tau) &= z_0(\xi,\tau)+z_1(\xi,\tau)+z_2(\xi,\tau)+\cdots\\
&= \frac{3}{2}e^{\xi-\frac{\tau^\alpha}{\alpha}}+\frac{1}{2}e^{\xi+\frac{\tau^\alpha}{\alpha}}+\frac{\tau^{2\alpha}}{2\alpha^2}e^{\xi-\frac{\tau^\alpha}{\alpha}}-e^{\xi}\frac{\tau^\alpha}{\alpha}+\frac{\tau^\alpha}{\alpha}e^{-\frac{\tau^\alpha}{\alpha}}+\frac{3}{2}e^{-\frac{\tau^\alpha}{\alpha}}+\frac{1}{2}e^{\frac{\tau^\alpha}{\alpha}}\\
&+ \frac{\tau^\alpha}{\alpha}e^{\xi-\frac{\tau^\alpha}{\alpha}}-e^{\xi}-2+\cdots,\\
w(\xi,\tau) &= w_0(\xi,\tau)+w_1(\xi,\tau)+w_2(\xi,\tau)+\cdots\\
&= \frac{3}{2}e^{-\xi+\frac{\tau^\alpha}{\alpha}}+\frac{1}{2}e^{-\xi-\frac{\tau^\alpha}{\alpha}}+\frac{\tau^{2\alpha}}{2\alpha^2}e^{-\xi+\frac{\tau^\alpha}{\alpha}}+e^{-\xi}\frac{\tau^\alpha}{\alpha}+\frac{\tau^\alpha}{\alpha}e^{\frac{\tau^\alpha}{\alpha}}-\frac{3}{2}e^{\frac{\tau^\alpha}{\alpha}}-\frac{1}{2}e^{-\frac{\tau^\alpha}{\alpha}}\\
&- \frac{\tau^\alpha}{\alpha}e^{-\xi+\frac{\tau^\alpha}{\alpha}}-e^{-\xi}+2+\cdots,
\end{aligned}$$

Example 4.2. *Consider the following system of coupled nonlinear fractional partial differential equations [11, 21].*

$$\begin{aligned}
\frac{\partial^\alpha z(\varrho,\tau)}{\partial\varrho^\alpha}-w(\varrho,\tau)\frac{\partial z(\varrho,\tau)}{\partial\tau}+z(\varrho,\tau)\frac{\partial w(\varrho,\tau)}{\partial\tau} &= -1+sin(\tau)e^{\varrho},\\
\frac{\partial^\alpha w(\varrho,\tau)}{\partial\varrho^\alpha}+\frac{\partial z(\varrho,\tau)}{\partial\varrho}\frac{\partial w(\varrho,\tau)}{\partial\tau}+\frac{\partial z(\varrho,\tau)}{\partial\tau}\frac{\partial w(\varrho,\tau)}{\partial\varrho} &= -1-cos(\tau)e^{-\varrho}.
\end{aligned} \tag{4.10}$$

Boundary Conditions

$$\begin{aligned}
z_0(0,\tau) &= sin(\tau),\\
w_0(0,\tau) &= cos(\tau).
\end{aligned} \tag{4.11}$$

Exact Solution for $\alpha=1$

$$\begin{aligned}
z(\varrho,\tau) &= sin(\tau)e^{\varrho},\\
w(\varrho,\tau) &= cos(\tau)e^{-\varrho}.
\end{aligned} \tag{4.12}$$

A applying the Laplace transform on both sides Eq (10.20), we obtain

$$\begin{aligned}
\mathcal{L}_\alpha[z(\varrho,\tau)] &= \frac{z_0(0,\tau)}{s}+\frac{1}{s}\mathcal{L}_\alpha\left[-1+sin(\tau)e^{\frac{\varrho^\alpha}{\alpha}}+w(\varrho,\tau)\frac{\partial z(\varrho,\tau)}{\partial\tau}-z(\varrho,\tau)\frac{\partial w(\sigma,\tau)}{\partial\tau}\right],\\
\mathcal{L}_\alpha[w(\varrho,\tau)] &= \frac{w_0(0,\tau)}{s}-\frac{1}{s}\mathcal{L}_\alpha\left[1+cos(\tau)e^{-\frac{\varrho^\alpha}{\alpha}}+\frac{\partial z(\varrho,\tau)}{\partial\varrho}\frac{\partial w(\varrho,\tau)}{\partial\tau}+\frac{\partial z(\varrho,\tau)}{\partial\tau}\frac{\partial w(\varrho,\tau)}{\partial\varrho}\right].
\end{aligned} \tag{4.13}$$

Taking the inverse Laplace transform on both sides of Eq (10.20), we get

$$\begin{aligned}
z(\varrho,\tau) &= sin(\tau)+\mathcal{L}_\alpha{}^{-1}\left[\frac{1}{s}\mathcal{L}_\alpha\left(-1+sin(\tau)e^{\frac{\varrho^\alpha}{\alpha}}+w(\varrho,\tau)\frac{\partial z(\varrho,\tau)}{\partial\tau}-z(\varrho,\tau)\frac{\partial w(\sigma,\tau)}{\partial\tau}\right)\right],\\
w(\varrho,\tau) &= \cos(\tau)-\mathcal{L}_\alpha{}^{-1}\left[\frac{1}{s}\mathcal{L}_\alpha\left(1+cos(\tau)e^{-\frac{\varrho^\alpha}{\alpha}}+\frac{\partial z(\varrho,\tau)}{\partial\varrho}\frac{\partial w(\varrho,\tau)}{\partial\tau}+\frac{\partial z(\varrho,\tau)}{\partial\tau}\frac{\partial w(\varrho,\tau)}{\partial\varrho}\right)\right].
\end{aligned} \tag{4.14}$$

using the MADM technique, the infinite series solution for $z_n(\varrho,\tau)$ and $w_n(\varrho,\tau)$ is

$$z(\xi,\tau)=\sum_{n=0}^{\infty} z_n(\varrho,\tau)=sin(\tau)+\mathcal{L}_\alpha{}^{-1}\left[\frac{1}{s}\mathcal{L}_\alpha\left(-1+sin(\tau)e^{\frac{\varrho^\alpha}{\alpha}}+\sum_{n=0}^{\infty}\vartheta_n\right)\right],$$
$$w(\xi,\tau)=\sum_{n=0}^{\infty} w_n(\varrho,\tau)=\cos(\tau)-\mathcal{L}_\alpha{}^{-1}\left[\frac{1}{s}\mathcal{L}_\alpha\left(1+cos(\tau)e^{-\frac{\varrho^\alpha}{\alpha}}+\sum_{n=0}^{\infty}\vartheta_n\right)\right]. \tag{4.15}$$

where θ_n and ϑ_n are Adomian polynomials the nonlinear terms.
The first few components of the θ_n and ϑ_n polynomials are given by

$$\theta_0=w_0z_{0_\tau}-z_0w_{0_\tau},$$
$$\theta_1=w_1z_{0_\tau}+w_0z_{1_\tau}-z_1w_{0_\tau}-z_0w_{1_\tau}+w_1z_{1_\tau}-z_1w_{1_\tau}$$
$$\vdots$$

$$\vartheta_0=z_{0_\tau}w_{0_\varrho}+z_{0_\varrho}w_{0_\tau},$$
$$\vartheta_1=w_{1_\varrho}z_{0_\tau}+w_{0_\varrho}z_{1_\tau}+z_{1_\varrho}w_{0_\tau}+z_{0_\varrho}w_{1_\tau}+w_{1_\varrho}z_{1_\tau}+z_{1_\varrho}w_{1_\tau}$$

and so on.
The subsequences of solutions given as

$$z_1(\varrho,\tau)=\mathcal{L}_\alpha{}^{-1}\left[\frac{1}{s}\mathcal{L}_\alpha\left(-1+sin(\tau)e^{\frac{\varrho^\alpha}{\alpha}}+\theta_0\right)\right],$$
$$w_1(\varrho,\tau)=-\mathcal{L}_\alpha{}^{-1}\left[\frac{1}{s}\mathcal{L}_\alpha\left(1+cos(\tau)e^{-\frac{\varrho^\alpha}{\alpha}}+\vartheta_0\right)\right],$$
$$\vdots$$
$$z_{n+1}(\varrho,\tau)=\mathcal{L}_\alpha{}^{-1}\left[\frac{1}{s}\mathcal{L}_\alpha\left(\theta_n\right)\right],\quad n\geq 1,$$
$$w_{n+1}(\varrho,\tau)=-\mathcal{L}_\alpha{}^{-1}\left[\frac{1}{s}\mathcal{L}_\alpha\left(\vartheta_n\right)\right],\quad n\geq 1.$$

The first three approximate terms of $z(\xi,\tau)$ and $w(\xi,\tau)$ are given as follows

$$\begin{aligned} z_0(\varrho,\tau) &= sin(\tau),\\ w_0(\varrho,\tau) &= cos(\tau), \end{aligned}$$

$$z_1(\varrho,\tau)=sin(\tau)\left(-1+e^{\frac{\varrho^\alpha}{\alpha}}\right),$$
$$w_1(\varrho,\tau)=-\frac{\varrho^\alpha}{\alpha}-cos(\tau)\left(1-e^{\frac{\varrho^{-\alpha}}{\alpha}}\right),$$

$$z_2(\varrho,\tau) \;=\; 0$$

$$w_2(\varrho,\tau) = \frac{\alpha^{1-\frac{1}{\alpha}}\varrho^{2-\frac{1}{\alpha}}}{2-\frac{1}{\alpha}} + \alpha^{1-\frac{1}{\alpha}}\Gamma\left(2-\frac{1}{\alpha}\right)cos(\tau)\mathcal{L}_{\alpha}{}^{-1}\left(\frac{(s-1)^{-2+\frac{1}{\alpha}}}{s}\right),$$

$$\vdots$$

The series solution for $z(\varrho,\tau)$ and $w(\varrho,\tau)$ is

$$\begin{aligned}
z(\varrho,\tau) &= z_0(\varrho,\tau)+z_1(\varrho,\tau)+z_2(\varrho,\tau)+\cdots\\
&= sin(\tau)+sin(\tau)\left(-1+e^{\frac{\varrho^{\alpha}}{\alpha}}\right) = sin(\tau)\left(e^{\frac{\varrho^{\alpha}}{\alpha}}\right)+\cdots,\\
w(\varrho,\tau) &= w_0(\xi,\tau)+w_1(\xi,\tau)+w_2(\xi,\tau)+\cdots\\
&= \cos(\tau)-\frac{\varrho^{\alpha}}{\alpha}-cos(\tau)\left(1-e^{\frac{\varrho^{-\alpha}}{\alpha}}\right)\\
&+ \frac{\alpha^{1-\frac{1}{\alpha}}\varrho^{2-\frac{1}{\alpha}}}{2-\frac{1}{\alpha}} + \alpha^{1-\frac{1}{\alpha}}\Gamma\left(2-\frac{1}{\alpha}\right)cos(\tau)\mathcal{L}_{\alpha}{}^{-1}\left(\frac{(s-1)^{-2+\frac{1}{\alpha}}}{s}\right)+\cdots
\end{aligned}$$

Example 4.3. *Nonlinear thermoelastic system with coupled Fractional partial differential equations [26].*

$$\begin{aligned}
\frac{\partial^{\alpha+1}z(\xi,\tau)}{\partial\tau^{\alpha+1}} - \frac{\partial}{\partial\xi}\left(w(\xi,\tau)\frac{\partial}{\partial\xi}z(\xi,\tau)\right) + \frac{\partial w(\xi,\tau)}{\partial\xi} - 2\xi+6\xi^2+2\tau^2+2 = 0, \quad 0<\alpha\leq 1, \tau>0,\\
\frac{\partial^{\alpha}z(\xi,\tau)}{\partial\tau^{\alpha}} - \frac{\partial}{\partial\xi}\left(z(\xi,\tau)\frac{\partial}{\partial\xi}w(\xi,\tau)\right) + \frac{\partial^2 z(\xi,\tau)}{\partial\xi\partial\tau} + 6\xi^2-2\tau^2-2\tau = 0, \quad 0<\alpha\leq 1, \tau>0.
\end{aligned} \tag{4.16}$$

Initial Conditions

$$\begin{aligned} z(\xi,0) &= \xi^2,\\ w(\xi,0) &= \xi^2. \end{aligned} \tag{4.17}$$

Exact Solution for $\alpha = 1$

$$\begin{aligned} z(\xi,\tau) &= \xi^2-\tau^2,\\ w(\xi,\tau) &= \xi^2+\tau^2. \end{aligned} \tag{4.18}$$

By applying the Laplace transform and then its inverse on both sides Eq. (4.16), along with the initial conditions, we obtain

$$\begin{aligned}
z(\xi,\tau) &= \xi^2-\mathcal{L}_{\alpha}{}^{-1}\left[\frac{1}{s^2}\mathcal{L}_{\alpha}\left[-\frac{\partial}{\partial\xi}\left(w(\xi,\tau)\frac{\partial}{\partial\xi}z(\xi,\tau)\right)+\frac{\partial w(\xi,\tau)}{\partial\xi}-2\xi+6\xi^2+\frac{\tau^{2(\alpha+1)}}{(\alpha+1)^2}+2\right]\right],\\
w(\xi,\tau) &= \xi^2-\mathcal{L}_{\alpha}{}^{-1}\left[\frac{1}{s}\mathcal{L}_{\alpha}\left[-\frac{\partial}{\partial\xi}\left(z(\xi,\tau)\frac{\partial}{\partial\xi}w(\xi,\tau)\right)+\frac{\partial^2 z(\xi,\tau)}{\partial\xi\partial\tau}+6\xi^2-2\frac{\tau^{2\alpha}}{\alpha^2}-2\frac{\tau^{\alpha}}{\alpha}\right]\right].
\end{aligned} \tag{4.19}$$

By employing the MADM algorithm on system generated a recursive

$$z_0(\xi,\tau) = \xi^2,$$
$$w_0(\xi,\tau) = \xi^2,$$

$$z_1(\xi,\tau) = -\mathcal{L}_\alpha{}^{-1}\left[\frac{1}{s^2}\mathcal{L}_\alpha\left[-\theta_{0_\xi} + w_{0_\xi}(\xi,\tau) - 2\xi + 6\xi^2 + \frac{\tau^{2(\alpha+1)}}{(\alpha+1)^2} + 2\right]\right],$$
$$w_1(\xi,\tau) = -\mathcal{L}_\alpha{}^{-1}\left[\frac{1}{s}\mathcal{L}_\alpha\left[-\vartheta_{0_\xi} + z_{0_{\xi\tau}}(\xi,\tau) + 6\xi^2 - 2\frac{\tau^{2\alpha}}{\alpha^2} - 2\frac{\tau^\alpha}{\alpha}\right]\right],$$
$$\vdots$$
$$z_{n+1}(\xi,\tau) = \mathcal{L}_\alpha{}^{-1}\left[\frac{1}{s^2}\mathcal{L}_\alpha\left[\theta_{n_\xi} - w_{n_\xi}(\xi,\tau)\right]\right], \quad n \geq 1,$$
$$w_{n+1}(\xi,\tau) = \mathcal{L}_\alpha{}^{-1}\left[\frac{1}{s}\mathcal{L}_\alpha\left[\vartheta_{n_\xi} - z_{n_{\xi\tau}}(\xi,\tau)\right]\right], \quad n \geq 1.$$

The first tow approximate terms of $z(\xi,\tau)$ and $w(\xi,\tau)$ are given as follows

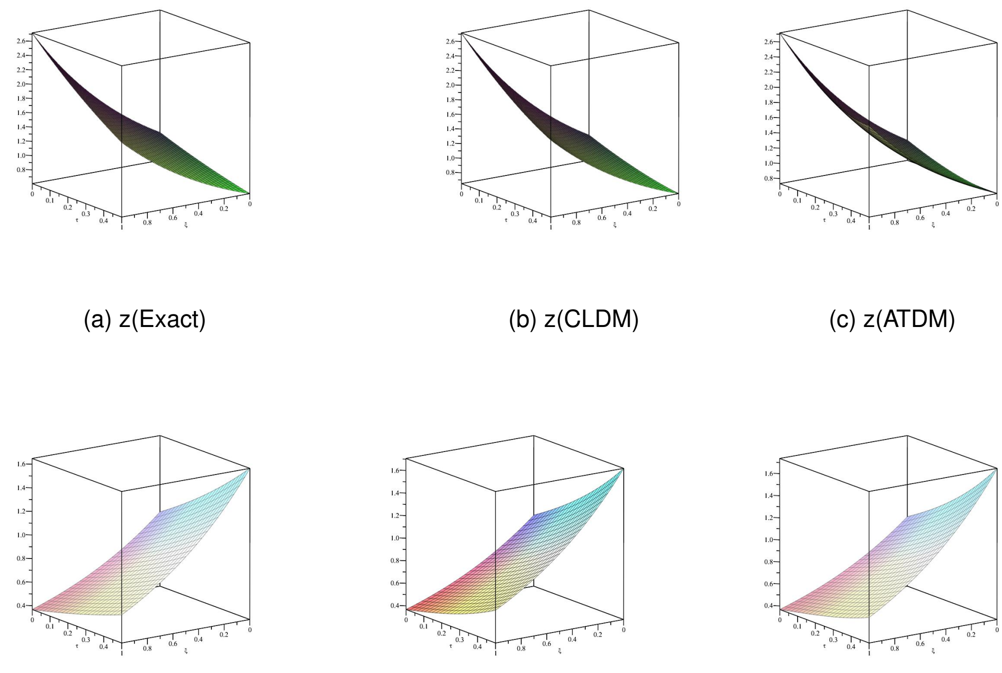
$$\begin{aligned} z_0(\xi,\tau) &= \xi^2, \\ w_0(\xi,\tau) &= \xi^2, \end{aligned}$$

$$\begin{aligned} z_1(\xi,\tau) &= -\frac{\tau^{4(\alpha+1)}}{6(\alpha+1)^4} - \frac{\tau^{2(\alpha+1)}}{(\alpha+1)^2}, \\ w_1(\xi,\tau) &= \frac{\tau^{2\alpha}}{\alpha^2} + \frac{2\tau^{3\alpha}}{3\alpha^3}, \end{aligned}$$
$$\vdots$$

The series solution for $z(\xi,\tau)$ and $w(\xi,\tau)$ is

$$\begin{aligned} z(\xi,\tau) &= z_0(\xi,\tau) + z_1(\xi,\tau) + \cdots \\ &= \xi^2 - \frac{\tau^{4(\alpha+1)}}{6(\alpha+1)^4} - \frac{\tau^{2(\alpha+1)}}{(\alpha+1)^2} + \cdots, \\ w(\xi,\tau) &= w_0(\xi,\tau) + w_1(\xi,\tau) + \cdots \\ &= \xi^2 + \frac{\tau^{2\alpha}}{\alpha^2} + \frac{2\tau^{3\alpha}}{3\alpha^3} + \cdots \end{aligned}$$

The numerical results of the above examples are illustrated in the graphs below, respectively.



(a) z(Exact) (b) z(CLDM) (c) z(ATDM)

(d) w(Exact) (e) w(CLDM) (f) w(ATDM)

Figure 1: A comparison of the exact and approximate 3D solutions for $z(\xi,\tau)$ and $w(\xi,\tau)$ in the CLDM at $\alpha = 1$ (Example 1)

(a) z(CLDM) (b) w(CLDM)

Figure 2: Exact and CLDM Solutions for the first three terms of $z(\xi,\tau)$ and $w(\xi,\tau)$, at $\alpha = 1, \xi = 0.5$ for Example1

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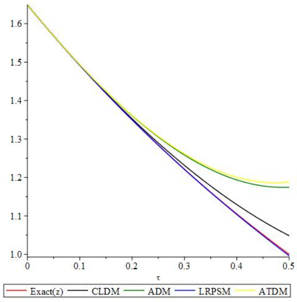
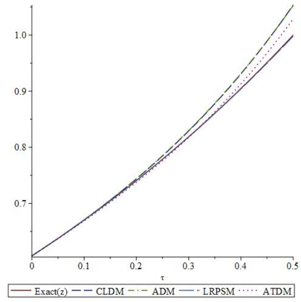
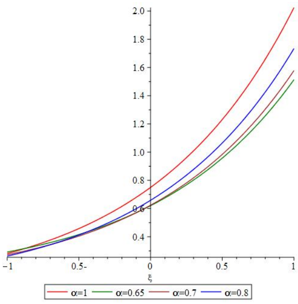


(a)

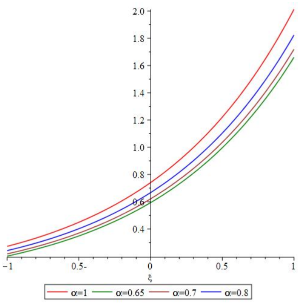


(b)

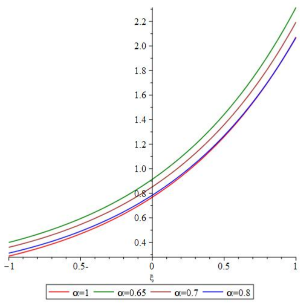
Figure 3: 2D plots of comparision of CLDM, ADM, LRPSM, ATDM and Exact solution at $\alpha = 1$, $\xi = 0.5$ for Example1



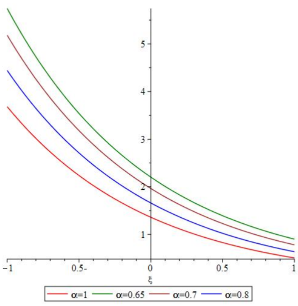
(a) z(CLDM)



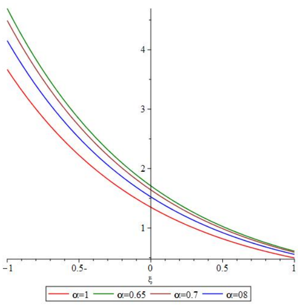
(b) z(LRPSM)



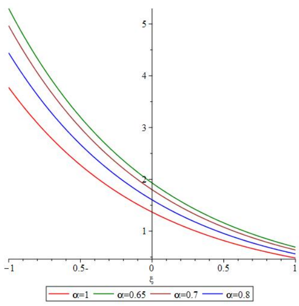
(c) z(ATDM)



(d) w(CLDM)



(e) w(LRPSM)



(f) w(ATDM)

Figure 4: 2D plots of CLDM, RPSM, ATDM solution of $z(\xi, \tau)$, $w(\xi, \tau)$ for different values of α at $\tau = 0.3$ of Example 1

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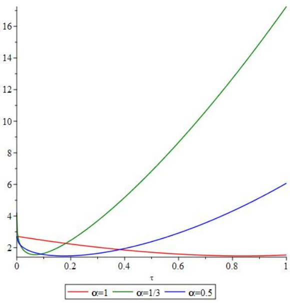
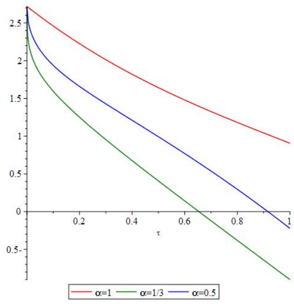
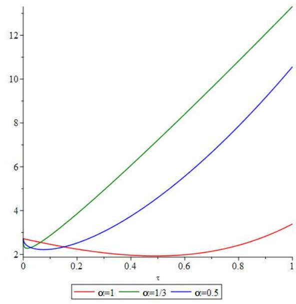


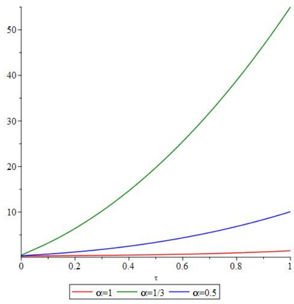
(a) z(CLDM)



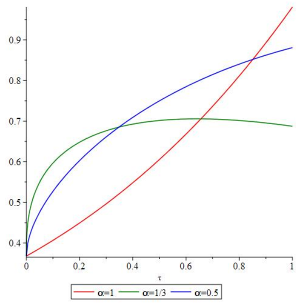
(b) z(LRPSM)



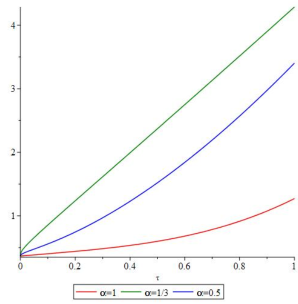
(c) z(ATDM)



(d) w(CLDM)



(e) w(LRPSM)



(f) w(ATDM)

Figure 5: 2D plots of CLDM, LRPSM, ATDM solution for $z(\xi, \tau)$ and $w(\xi, \tau)$ at different value $\alpha, \xi = 1$ (example.1)

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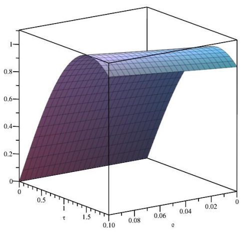
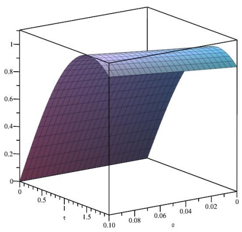
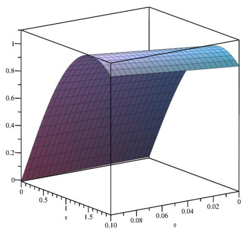


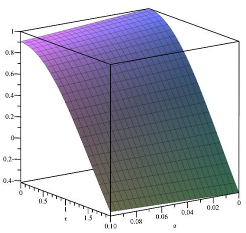
(a) z(Exact)



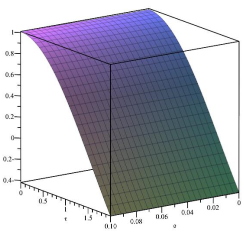
(b) z(CLDM)



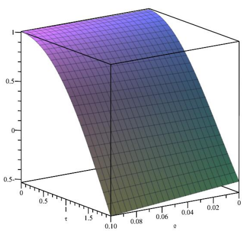
(c) z(LDM)



(d) w(Exact)



(e) w(CLDM)



(f) w(LDM)

Figure 6: CLDM and LDM Solutions for $z(\varrho, \tau)$ and $w(\varrho, \tau)$ at $\alpha = 1$ (Example 2)

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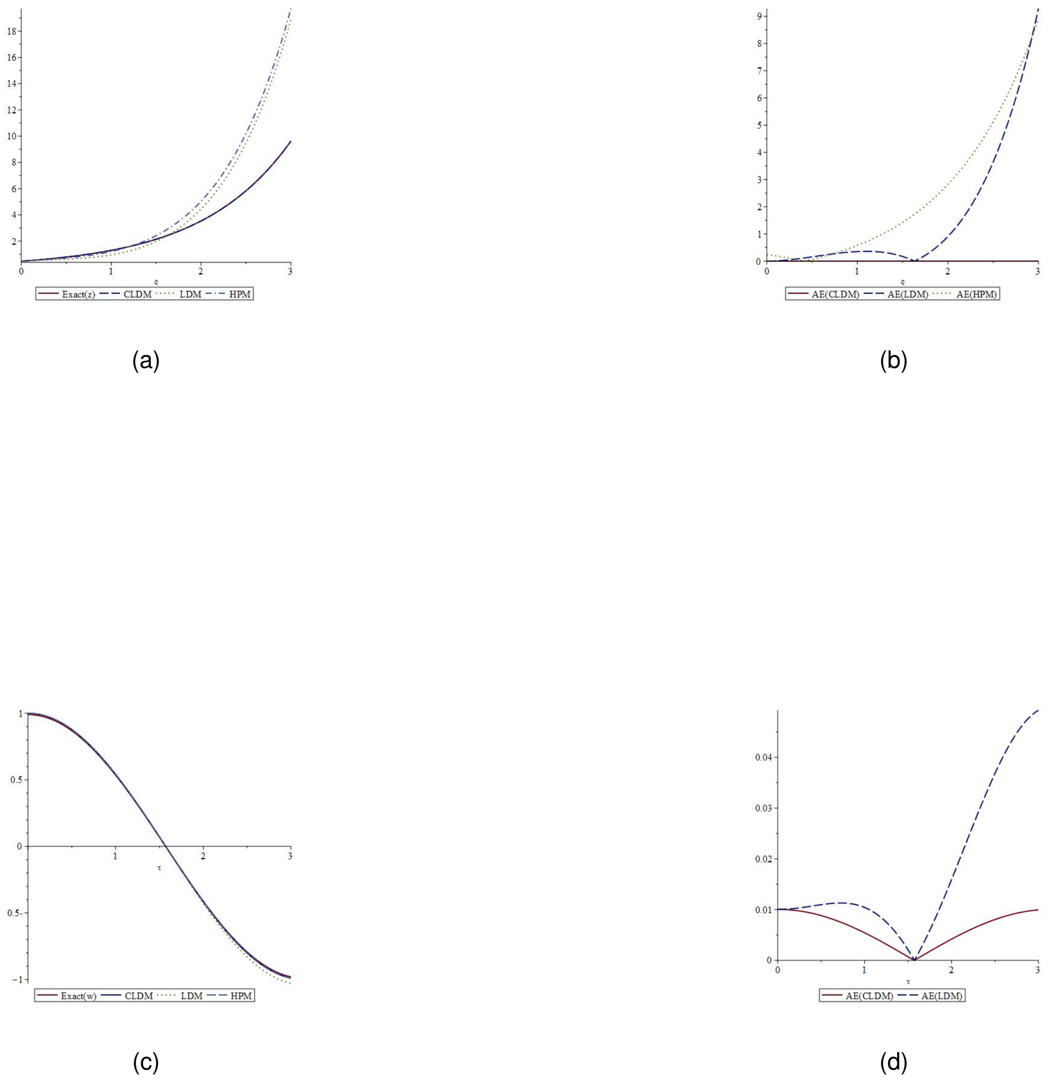
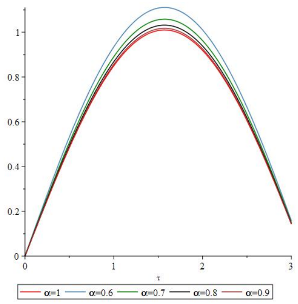
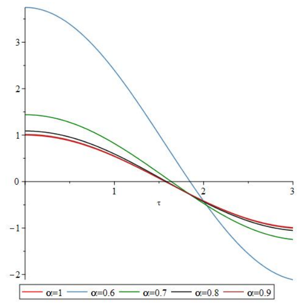


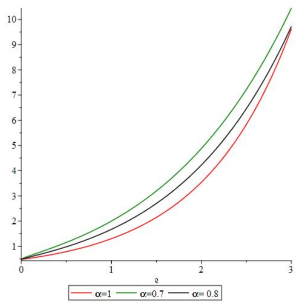
Figure 7: Compare of the approximate solutions and approximate error of the CLDM, LDM for $z(\varrho, \tau)$ at $\alpha = 1, \varrho = 0.01$ and CLDM, LDM, and HPM for $w(\varrho, \tau)$ at $\alpha = 1$, $\tau = 0.5$ in Example2



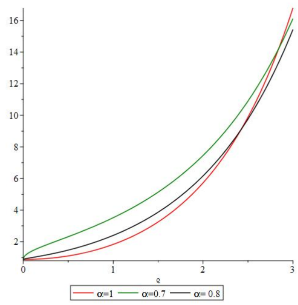
(a) (CLDM)



(b) (CLDM)

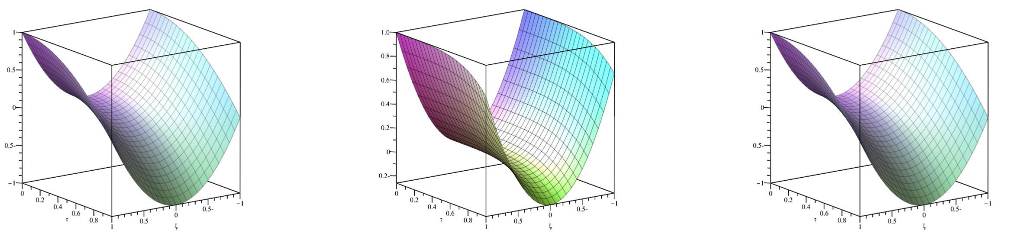


(c) (CLDM)

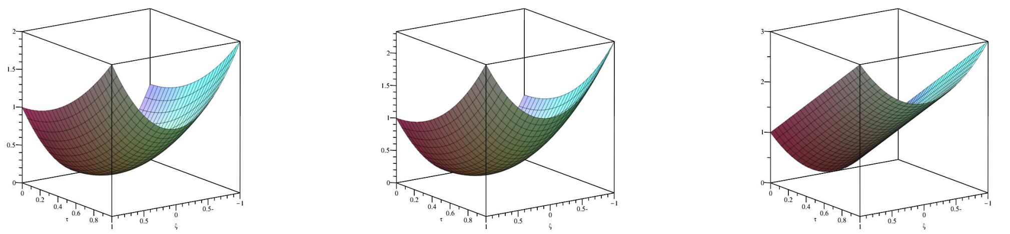


(d) (CLDM)

Figure 8: 2D plots of CLDM solution of $z(\varrho,\tau), w(\varrho,\tau)$ for different values of α at $\varrho = 0.01$ and $\tau = 0.5$ of Example 2



(a) z(Exact) (b) z(CLDM) (c) z(LRPSM)



(d) w(Exact) (e) w(CLDM) (f) w(LRPSM)

Figure 9: CLDM and LRPSM Solutions for $z(\varrho,\tau)$ and $w(\varrho,\tau)$ at $\alpha = 1$ (Example.3)

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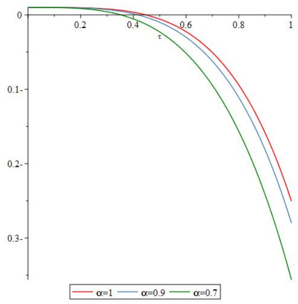
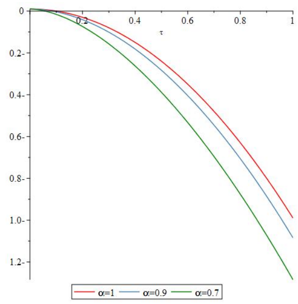
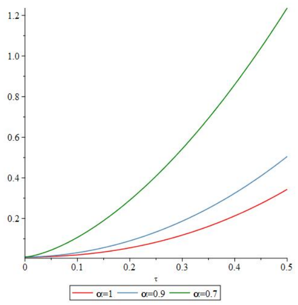


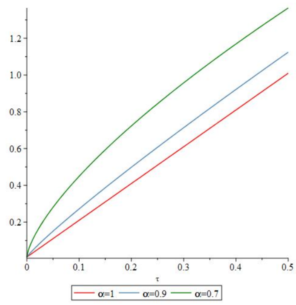
(a) (CLDM)



(b) (LRPSM)

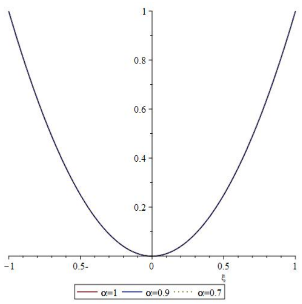


(c) (CLDM)

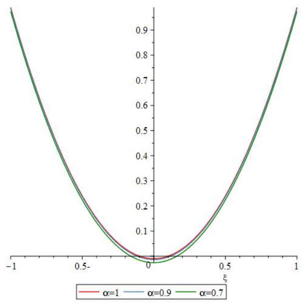


(d) (LRPSM)

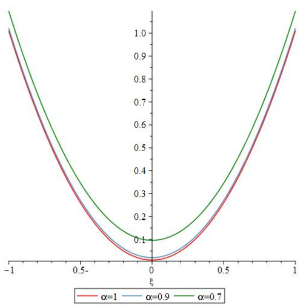
Figure 10: 2D plots of CLDM and LRPSM solutions of $z(\xi,\tau), w(\xi,\tau)$ for different values of α at $\xi = 0.1$ of Example 3



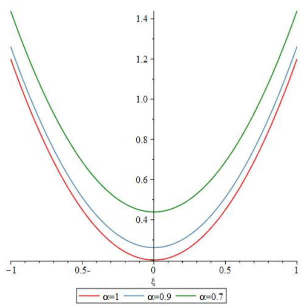
(a) (CLDM)



(b) (LRPSM)



(c) (CLDM)



(d) (LRPSM)

Figure 11: 2D plots of CLDM and LRPSM solutions of $z(\xi,\tau), w(\xi,\tau)$ for different values of α at $\tau = 0.1$ of Example 3

Table 1: Comparison of CLDM and ATDM at $\alpha = 1$.

Method	Variable	Exact Value	Numerical Value	Relative Error (%)	Accuracy (%)
CLDM	$z(0.5, 0.4)$	1.1052	1.1050	0.0181	99.9819
ATDM	$z(0.5, 0.4)$	1.1052	1.1045	0.0633	99.9367
CLDM	$w(0.5, 0.4)$	0.9048	0.9045	0.0332	99.9668
ATDM	$w(0.5, 0.4)$	0.9048	0.9040	0.0884	99.9116

Table 2: Comparison of CLDM and ATDM at $\alpha = 0.8$.

Method	Variable	Exact Value	Numerical Value	Relative Error (%)	Accuracy (%)
CLDM	$z(0.5, 0.4)$	1.1052	1.1050	0.0181	99.9819
ATDM	$z(0.5, 0.4)$	1.1052	1.1040	0.1085	99.8915
CLDM	$w(0.5, 0.4)$	0.9048	0.9045	0.0332	99.9668
ATDM	$w(0.5, 0.4)$	0.9048	0.9040	0.0884	99.9116

Table 3: Comparison of CLDM and ATDM at $\alpha = 0.5$.

Method	Variable	Exact Value	Numerical Value	Relative Error (%)	Accuracy (%)
CLDM	$z(0.5, 0.4)$	1.1052	1.1048	0.0362	99.9638
ATDM	$z(0.5, 0.4)$	1.1052	1.1035	0.1538	99.8462
CLDM	$w(0.5, 0.4)$	0.9048	0.9042	0.0663	99.9337
ATDM	$w(0.5, 0.4)$	0.9048	0.9035	0.1437	99.8563

Table 4: Comparison of CLDM and LDM at $\alpha = 0.8$.

Method	Point	Exact Value	Numerical Value	Relative Error (%)	Accuracy (%)
CLDM	$z(2, 0.5)$	3.541	3.540	0.028	99.972
LDM	$z(2, 0.5)$	3.541	3.535	0.169	99.831
CLDM	$w(0.5, 2)$	-0.252	-0.251	0.397	99.603
LDM	$w(0.5, 2)$	-0.252	-0.250	0.794	99.206

UNDER PEER REVIEW

5. Analysis and Discussion of Results

Based on the graphs, tables, and literature review, the following conclusions are drawn:

1. **Convergence of Numerical Solutions:** The CLDM method yields numerical solutions that converge with the exact solution with high accuracy, outperforming traditional methods like finite difference and finite element methods, as noted in [4, 5].

2. **Superiority in Accuracy:** The CLDM method is more accurate than ATDM, HPM, RPSM, and LDM for various values of α, aligning with findings in [8, 21].

3. **Stability:** The CLDM method shows stability across different α values, crucial for applications like fluid dynamics and heat transfer [24, 25].

4. **Effect of α on Accuracy:** As α approaches 1, the accuracy of CLDM improves, while decreasing α reduces accuracy, as discussed in [7, 20].

5. **Handling Complex Systems:** The CLDM method performs better as system complexity increases, making it suitable for complex fractional systems in fields like materials science and biology [22, 26].

6. **Suitability for Complex Fractional Systems:** The CLDM method excels in solving fractional-order differential equations in fluid dynamics, heat transfer, and wave propagation, as well as in modeling biological and financial systems with memory effects [1, 10, 24, 27].

6. Conclusion

The Conformable Laplace Decomposition Method (CLDM) effectively solves nonlinear fractional partial differential equations (FPDEs), outperforming methods like ATDM, HPM, RPSM, and LDM in accuracy, stability, and efficiency. Its precision in handling complex systems, robustness across varying α, and performance in modeling memory-dependent systems make it ideal for applications in fluid dynamics, heat transfer, materials science, and biology. Building on the work of Podlubny [25] and Li and Chen [21], CLDM advances fractional calculus, offering a reliable tool for complex systems. Future research could explore its use in quantum mechanics and climate modeling.“

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References

[1] T. Abdeljawad; On conformable fractional calculus, *Journal of Computational and Applied Mathematics*, 279 (2015), 5766.

[2] G. Adomian; System of nonlinear partial differential equations, *Journal of Mathematical Analysis and Applications*, 115(1) (1986), 235238.

[3] A. Aghili, B. P. Moghaddam; Certain theorems on two-dimensional Laplace transform and non-homogeneous parabolic partial differential equations, *Surveys in Mathematics and its Applications*, 6 (2011), 165174.

[4] H. Aljarrah, M. Alaroud, A. Ishak, M. Darus; Approximate solution of nonlinear time-fractional PDEs by Laplace residual power series method, *Mathematics*, 10(12) (2022), 1980.

[5] S. A. Alomari, Y. Q. Hasan; Enhanced Adomian Decomposition Method for Accurate Numerical Solutions of PDE Systems, *Asian Research Journal of Mathematics*, 20(11) (2024), 2641.

[6] A. Atangana, D. Baleanu, A. Alsaedi; New properties of conformable derivative, *Open Mathematics*, 13 (2015), 110.

[7] M. Ayata, O. Ozkan; A new application of conformable Laplace decomposition method for fractional Newell-Whitehead-Segel equation, *AIMS Mathematics*, 5(6) (2020), 74027412.

[8] A. A. Bashir; The Adomian decomposition method for solving a system of nonlinear equations (Doctoral dissertation), 2021.

[9] M. E. Benatta, K. Belghaba; Shehu conformable fractional transform, theories and applications, *Cankaya University Journal of Science and Engineering*, 18(1) (2021), 2432.

[10] J. Biazar, M. Eslami; A new homotopy perturbation method for solving the fractional Jaulent-Miodek system via a modified Laplace decomposition method, *Waves in Random and Complex Media*, 11 (2011), 114.

[11] M. H. Cherif, D. Ziane; A new numerical technique for solving systems of nonlinear fractional partial differential equations, *International Journal of Analysis and Applications*, 15(2) (2017), 188197.

[12] N. M. Dabwan, Y. Q. Hasan; Solutions of fractional differential equations by modified Adomian decomposition method, *Advances in Mathematics: Scientific Journal*, 9(7) (2020), 47614767.

UNDER PEER REVIEW

[13] M. Dehghan, J. Manafian, A. Saadatmandi; Solving nonlinear fractional partial differential equations using the homotopy analysis method, *Numerical Methods for Partial Differential Equations*, 26(2) (2010), 448465.

[14] V. P. Dubey, R. Kumar, D. Kumar, I. Khan, J. Singh; An efficient computational scheme for nonlinear time fractional systems of partial differential equations arising in physical sciences, *Advances in Difference Equations*, 127 (2020).

[15] A. E. Hamza, A. K. Sedeeg, R. Saadeh, A. Qazza, R. Khalil; A new approach in solving regular and singular conformable fractional coupled Burgers equations, *arXiv preprint arXiv:2306.10030*, 2023.

[16] Y. Q. Hasan, L. M. Zhu; Modified Adomian decomposition method for singular initial value problems in the second-order ordinary differential equations, *Surveys in Mathematics and its Applications*, 3 (2008), 183193.

[17] K. Ilhem, M. Al Horani, R. R. Khalil; Solution of non-linear fractional Burgers type equations using the Laplace transform decomposition method, *Results in Nonlinear Analysis*, 5(2) (2022), 131150.

[18] O. S. Iyiola, E. R. Nwaeze; Some new results on the new conformable fractional calculus with application using DAlembert approach, *Progress in Fractional Differentiation and Applications*, 2(2) (2016), 115122.

[19] R. Khalil, M. Al Horani, A. Yousef, M. Sababheh; A new definition of fractional derivative, *Journal of Computational and Applied Mathematics*, 264 (2014), 6570.

[20] H. Khan, R. Shah, P. Kumam, D. Baleanu, M. Arif; Laplace decomposition for solving nonlinear system of fractional order partial differential equations, *Advances in Difference Equations*, 118 (2020).

[21] A. A. Kilbas, H. M. Srivastava, J. J. Trujillo; Theory and applications of fractional differential equations, Elsevier, 2006.

[22] P. Kittipoom; Efficient computation of Laplace residual power series with explicit coefficient formulas, *arXiv preprint arXiv:2407.04705*, 2024.

[23] S. Mahmood, R. Shah, H. Khan, M. Arif; Laplace Adomian decomposition method for multi-dimensional time fractional model of Navier-Stokes equation, *Symmetry*, 11(2) (2019), 149.

[24] K. S. Miller; An introduction to the fractional calculus and fractional differential equations, John Wiley and Sons, 1993.

[25] I. Podlubny; Fractional differential equations: An introduction to fractional derivatives, fractional differential equations, to methods of their solution and some of their applications, Elsevier, 1998.

[26] A. Rab, S. Khan, H. Khan, F. Tchier, S. Jebran, F. Tawfiq, M. Nadeem; The fractional analysis of thermo-elasticity coupled systems with non-linear and singular nature, *Scientific Reports*, 14(1) (2024), 9663.

[27] G. Chen, J. Zhou; Numerical Methods for Partial Differential Equations, Springer, 1998.

[28] A. Constantin, J. Escher; Wave Breaking for Nonlinear Nonlocal Shallow Water Equations, *Acta Mathematica*, 181(2) (1998), 229243.

[29] P. Drabek, J. Milota; Methods of Nonlinear Analysis: Applications to Differential Equations, Birkhuser, 2007.

[30] L. C. Evans; Partial Differential Equations, American Mathematical Society, 2010.

[31] Y. Awad; Exploring solutions for nonlinear fractional differential equations with multiple fractional derivatives and integral boundary conditions, *Jordan Journal of Mathematics and Statistics*, 2024.

[32] Z. A. Mohammed, M. Damak, F. S. Fadhel; Existence and Uniqueness Theorems of MultiDimensional Integro-Differential Equations with Conformable Fractional Differointegrations, *Jordan Journal of Mathematics and Statistics*, 2024.

[33] M. Derhab; On a class of Caputo modified fractional differential equations with advanced arguments, *Jordan Journal of Mathematics and Statistics*, 2024.