

Original Research Article

Complete monotonicity of functions defined by λ generalized psi function and its derivatives

Abstract

In this work, we firstly give integral representations of λ -psi (or λ -digamma) and λ -zeta functions and then obtain λ -generalization of Binet's first formula for the logarithms of λ -gamma function $\ln \Gamma_\lambda(x)$. As immediate consequences, we get some completely monotonicity properties on functions related to λ -psi function and its derivatives. At last, we obtain some mean inequalities on λ -psi function.

Keywords: λ -polygamma function, λ -psi function; inequality; Binet's first formula for $\ln \Gamma_\lambda(x)$

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1 Introduction

The gamma function, which is introduced by Euler, is defined by

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt$$

for $x > 0$. The logarithmic derivative of gamma function is called digamma (or psi) function and its integral representation is given by

$$\psi(x) = \int_0^\infty \left(\frac{e^{-t}}{t} - \frac{e^{-tx}}{1-e^{-t}} \right) dt \quad (1.1)$$

for $x > 0$.

Special functions are used in many mathematical subjects such as fractional calculus, inequalities etc. Generalizations of these special functions are also interested by many researchers. For instance; in order to define tempered fractional integrals, authors give λ -generalized incomplete gamma function as follows:

Definition 1.1. (Fu and Du, 2021; Mohammed and Baleanu, 2020) For the real number $x > 0$ and $a, \lambda \geq 0$, λ -incomplete gamma function can be defined by

$$\Gamma_{\lambda}(x, a) = \int_a^{\infty} t^{x-1} e^{-\lambda t} dt.$$

In 2022, Nantomah and Ege define the λ -analogue of the gamma function

Definition 1.2. (Nantomah and Ege, 2022) λ -gamma function can be defined by

$$\Gamma_{\lambda}(x) = \int_0^{\infty} t^{x-1} e^{-\lambda t} dt \quad (1.2)$$

$$= \lim_{k \rightarrow \infty} \frac{\lambda^{-x} k! k^x}{x(x+1)(x+2) \dots (x+k)} \quad (1.3)$$

for $x > 0$ and $\lambda > 0$.

In the same paper, authors also give some properties on the λ -gamma function.

Lemma 1.1. (Nantomah and Ege, 2022)

$$\Gamma_{\lambda}(x) = \lambda^{-x} \Gamma(x), \quad (1.4)$$

$$\Gamma_{\lambda}(1) = \frac{1}{\lambda}, \quad (1.5)$$

$$\Gamma_{\lambda}(x+1) = \frac{x}{\lambda} \Gamma_{\lambda}(x), \quad x > 0, \quad (1.6)$$

$$\Gamma_{\lambda}(k+1) = \frac{k!}{\lambda^{k+1}}, \quad k \in \mathbb{N}_0, \quad (1.7)$$

$$\Gamma_{\lambda}(x) \Gamma_{\lambda}(1-x) = \frac{\pi}{\lambda \sin(\pi x)}, \quad x \in (0, 1), \quad (1.8)$$

$$\Gamma_{\lambda}(1+x) \Gamma_{\lambda}(1-x) = \frac{\pi x}{\lambda^2 \sin(\pi x)}, \quad x \in (0, 1), \quad (1.9)$$

$$\Gamma_{\lambda}(x) \Gamma_{\lambda}\left(x + \frac{1}{2}\right) = 2^{1-2x} \sqrt{\pi} \frac{\Gamma_{\lambda}(2x)}{\lambda}, \quad x > 0, \quad (1.10)$$

$$\frac{\Gamma_{\lambda}(x+k)}{\Gamma_{\lambda}(x)} = \frac{(x)_k}{\lambda^k}, \quad x > 0, k \in \mathbb{N}_0, \quad (1.11)$$

$$\Gamma_{\lambda}\left(k + \frac{1}{2}\right) = \frac{(2k-1)!!}{2^k \lambda^k} \sqrt{\frac{\pi}{\lambda}}, \quad k \in \mathbb{N}_0. \quad (1.12)$$

where $(x)_k = x(x+1)(x+2) \dots (x+k-1)$ is Pochhammer symbol and $m!!$ is double factorial of m .

In an usual sense, authors define λ -analogues of beta and psi functions as follows:

Definition 1.3. (Nantomah and Ege, 2022) λ -beta function can be given by

$$\beta_{\lambda}(x, y) = \frac{\Gamma_{\lambda}(x) \Gamma_{\lambda}(y)}{\Gamma_{\lambda}(x+y)},$$

for $x > 0$ and $y > 0$. The function collide with classical beta function $\beta(x, y)$ since

$$\beta_{\lambda}(x, y) = \frac{\Gamma_{\lambda}(x) \Gamma_{\lambda}(y)}{\Gamma_{\lambda}(x+y)} = \frac{\lambda^{-x} \Gamma(x) \lambda^{-y} \Gamma(y)}{\lambda^{-(x+y)} \Gamma(x+y)} = \frac{\Gamma(x) \Gamma(y)}{\Gamma(x+y)} = \beta(x, y).$$

Definition 1.4. (Nantomah and Ege, 2022) λ -digamma (or λ -psi) function can be given by

$$\psi_{\lambda}(x) = \frac{d}{dx} \ln \Gamma_{\lambda}(x) \quad (1.13)$$

for $x > 0$. Some of the integral representations are

$$\begin{aligned} \psi_{\lambda}(x) &= \frac{\Gamma'_{\lambda}(x)}{\Gamma_{\lambda}(x)} = -\ln \lambda + \psi(x) \\ &= -(\ln \lambda + \gamma) + \int_0^1 \frac{1-t^{x-1}}{1-t} dt \\ &= -(\ln \lambda + \gamma) + \int_0^{\infty} \frac{e^{-t} - e^{-xt}}{1-e^{-t}} dt \\ &= -(\ln \lambda + \gamma) + \sum_{k=0}^{\infty} \frac{x-1}{(k+1)(k+x)} \end{aligned}$$

where $\gamma = \lim_{n \rightarrow \infty} (\sum_{r=1}^n \frac{1}{r} - \ln n) = 0.5772\dots$ is Euler-Mascheroni constant and $\psi(x) = \frac{d}{dx} \ln \Gamma(x)$ classical digamma (psi) function.

By using the equation (1.6), authors obtain the recurrence formula for λ -digamma function as

$$\psi_{\lambda}(x+1) = \frac{1}{x} + \psi_{\lambda}(x). \quad (1.14)$$

In the paper, authors obtain some well-known theorems, formulas, limit properties and inequalities on these functions, such as in the next theorem, they give arithmetic and geometric mean inequalities on λ -gamma function for x and $1/x$.

Theorem 1.2. For $x > 0$, the inequalities

$$\Gamma_{\lambda}(x)\Gamma_{\lambda}\left(\frac{1}{x}\right) \geq \lambda^{-(x+\frac{1}{x})}, \quad (1.15)$$

$$\Gamma_{\lambda}(x) + \Gamma_{\lambda}\left(\frac{1}{x}\right) \geq 2\lambda^{-\frac{1}{2}(x+\frac{1}{x})} \quad (1.16)$$

are satisfied. With equality when $x = 1$.

The interested readers can find more information about properties, inequalities and generalizations of special functions in (Gautschi, 1974; Qi, 2010; Whittaker and Watson, 1996; Qi and Guo, 2010; Batir, 2011; Chen and Batir, 2012; Guo and Qi, 2013; Qi, 2013; Guo and Luo, 2015; Guo et al., 2015; Qi and Guo, 2017; Batir, 2018; Qi et al., 2005; Alzer and Jameson, 2017; Kim et al., 2018; Diaz and Pariguan, 2007) and references therein.

Motivated by previous works, we introduce the definition of λ -Hurwitz zeta function and Binet's first formula for logarithms of λ -gamma function $\ln \Gamma_{\lambda}(x)$. Then we give other integral representation on λ -psi function and some complete monotonicity properties on the function related to λ -psi function and its derivatives. As applications, we lastly obtain arithmetic, geometric and harmonic mean inequalities on λ -psi function between x and $1/x$ for all positive values of x and λ .

2 Useful Lemmas and λ -Analogue of Hurwitz Zeta Function

In this section, we give some properties that help us to prove our main results

Lemma 2.1. (Qi et al., 2005) For $x > 0$ and any non-negative integer n , the integral

$$\frac{1}{x^{n+1}} = \frac{1}{n!} \int_0^\infty t^n e^{-xt} dt$$

holds true.

Lemma 2.2. (Spiegel and Ribero, 1970, pg.98, eq. 15.71)

$$\ln \frac{a}{b} = \int_0^\infty \frac{e^{-at} - e^{-bt}}{t} dt.$$

In the next result, we obtain new integral representation of λ -psi function:

Lemma 2.3. The λ -psi function can be also given by

$$\psi_\lambda(x) = \int_0^\infty \left[e^{-u} - \frac{\lambda^x}{(u+\lambda)^x} \right] \frac{du}{u} \quad (2.1)$$

$$= \int_0^\infty \left[e^{-u} - \frac{1}{\left(\frac{u}{\lambda} + 1\right)^x} \right] \frac{du}{u} \quad (2.2)$$

$$= \int_0^\infty \left[e^{-\lambda u} - \frac{1}{(t+1)^x} \right] \frac{dt}{t} \quad (2.3)$$

for all positive real values of x and λ .

Proof. By differentiating the λ -gamma function with respect to x and using Lemma 2.2, we get

$$\begin{aligned} \Gamma'_\lambda(x) &= \int_0^\infty t^{x-1} e^{-\lambda t} \ln t dt \\ &= \int_0^\infty t^{x-1} e^{-\lambda t} \int_0^\infty \frac{e^{-u} - e^{-ut}}{u} du dt \\ &= \int_0^\infty \int_0^\infty \frac{t^{x-1} e^{-\lambda t} e^{-u} - t^{x-1} e^{-\lambda t} e^{-ut}}{u} du dt \\ &= \int_0^\infty \left(e^{-u} \int_0^\infty t^{x-1} e^{-\lambda t} dt - \int_0^\infty t^{x-1} e^{-t(u+\lambda)} dt \right) \frac{du}{u}. \end{aligned}$$

Now, let us denote the second integral in the parentheses by I . Substituting $t(u+\lambda) = \lambda v$ yields that

$$\begin{aligned} I &= \int_0^\infty t^{x-1} e^{-t(u+\lambda)} dt = \int_0^\infty \left(\frac{\lambda v}{u+\lambda} \right)^{x-1} e^{-\lambda v} \frac{\lambda dv}{u+\lambda} = \frac{\lambda^x}{(u+\lambda)^x} \int_0^\infty v^{x-1} e^{-\lambda v} dv \\ &= \frac{\lambda^x}{(u+\lambda)^x} \Gamma_\lambda(x). \end{aligned}$$

Hence we get

$$\Gamma'_\lambda(x) = \int_0^\infty \left(e^{-u} \int_0^\infty t^{x-1} e^{-\lambda t} dt - \frac{\lambda^x}{(u+\lambda)^x} \Gamma_\lambda(x) \right) \frac{du}{u}.$$

Since the integral in the parentheses is equal to $\Gamma_\lambda(x)$, we get

$$\begin{aligned} \Gamma'_\lambda(x) &= \int_0^\infty \left(e^{-u} \Gamma_\lambda(x) - \frac{\lambda^x}{(u+\lambda)^x} \Gamma_\lambda(x) \right) \frac{du}{u} \\ \frac{\Gamma'_\lambda(x)}{\Gamma_\lambda(x)} &= \int_0^\infty \left[e^{-u} - \frac{\lambda^x}{(u+\lambda)^x} \right] \frac{du}{u} \end{aligned}$$

as desired. □

Next, we define the λ -analogue of Hurwitz zeta function:

Definition 2.1. The λ -Hurwitz zeta can be defined by

$$\zeta_\lambda(x, k) = \sum_{n=0}^{\infty} \frac{1}{(x+n)^k} \quad (2.4)$$

for $k > 1$ and all positive real values of x and λ .

We want to remark that taking $x = 1$ leads us to λ -Riemann zeta function (that is special case of λ -Hurwitz zeta function) as

$$\zeta_\lambda(1, k) = \sum_{n=0}^{\infty} \frac{1}{(n+1)^k} = \sum_{n=1}^{\infty} \frac{1}{n^k} = \zeta_\lambda(k).$$

Furthermore, the λ -psi function can be given by

$$\psi_\lambda^{(n)}(x) = \sum_{k=0}^{\infty} \frac{(-1)^k n!}{(x+k)^{n+1}} = (-1)^{n+1} n! \zeta_\lambda(x, n+1).$$

Hence $\psi_\lambda^{(n)}(1) = \zeta_\lambda(n)$.

Lemma 2.4. The integral representation of λ -zeta function can be given as

$$\zeta_\lambda(x) = \frac{1}{\Gamma_\lambda(x)} \int_0^\infty \frac{t^{x-1} dt}{e^{\lambda t} - 1}. \quad (2.5)$$

Proof. Substituting $u = nt$ in the integral representation of λ -gamma function (1.2) leads us that

$$\begin{aligned} \Gamma_\lambda(x) &= \int_0^\infty (nt)^{x-1} e^{-\lambda nt} n dt = \int_0^\infty n^{x-1} t^{x-1} e^{-\lambda nt} n dt \\ &= n^x \int_0^\infty t^{x-1} e^{-\lambda nt} dt \\ \Gamma_\lambda(x) \sum_{n=1}^{\infty} \frac{1}{n^x} &= \sum_{n=1}^{\infty} \int_0^\infty t^{x-1} e^{-\lambda nt} dt = \int_0^\infty t^{x-1} \sum_{n=1}^{\infty} e^{-\lambda nt} dt \end{aligned}$$

Since $\sum_{n=1}^{\infty} e^{-\lambda nt}$ is a geometric series and converges by the fact $|r| = \frac{1}{e^{\lambda t}} > 0$, the sum is equal to $\frac{1}{e^{\lambda t} - 1}$. Hence the desired result concludes from the last equation. \square

Taking derivatives of λ -psi function by n -times with respect to x yields that

$$\begin{aligned} \psi_\lambda^{(n)} &= (-1)^{n+1} n! \sum_{k=1}^{\infty} \frac{1}{(x+k-1)^{n+1}} = (-1)^{n+1} n! \sum_{k=0}^{\infty} \frac{1}{(x+k)^{n+1}} \\ &= (-1)^{n+1} n! \zeta_\lambda(x, n+1). \end{aligned}$$

Taking $x = 1$ in the last equation leads that $\psi_\lambda^{(n)}(1) = (-1)^{n+1} n! \zeta_\lambda(n+1)$. Hence from Taylor series expansion of λ -psi function at $x = 1$, we get

$$\begin{aligned} \psi_\lambda(x) &= \sum_{n=0}^{\infty} \frac{\psi_\lambda^{(n)}(1)(x-1)^n}{n!} = \psi_\lambda(1) + \sum_{n=1}^{\infty} \frac{\psi_\lambda^{(n)}(1)(x-1)^n}{n!} \\ &= -\ln \lambda - \gamma + \sum_{n=1}^{\infty} (-1)^{n+1} \zeta_\lambda(n+1)(x-1)^n. \end{aligned}$$

At last, by replacing $x + 1$ instead of x in the last equation, we obtain the power series of the λ -psi function as

$$\psi_\lambda(x+1) = -\ln \lambda - \gamma + \sum_{k=2}^{\infty} (-1)^k \zeta_\lambda(k) x^{k-1} \quad (2.6)$$

for $|x| < 1$.

3 Main Results

Theorem 3.1. *Binet's first formula for logarithms of λ -gamma function $\ln \Gamma_\lambda(x)$ can be given by*

$$\ln \Gamma_\lambda(x) = \left(x - \frac{1}{2}\right) \ln x - x \ln \lambda + \frac{1}{2} \ln(2\pi) + \int_0^\infty \left[\frac{1}{2} - \frac{1}{t} + \frac{1}{e^t - 1}\right] \frac{e^{-tx}}{t} dt \quad (3.1)$$

and as an immediate consequence; the λ -psi function can also be given by

$$\psi_\lambda(x) = \ln x - \frac{1}{x} - \ln \lambda + \int_0^\infty \left(\frac{1}{t} - \frac{1}{e^t - 1}\right) e^{-xt} dt \quad (3.2)$$

for all positive real values of x and λ .

Proof. By taking into account the singularity of the third equation in the Lemma 2.3 at $t = 0$, we can rewrite the equation as

$$\psi_\lambda(x) = \lim_{\varepsilon \rightarrow 0} \left[\int_\varepsilon^\infty \frac{e^{-\lambda u}}{u} du - \int_\varepsilon^\infty \frac{du}{u(u+1)^x} \right].$$

Then substituting $u = e^t - 1$ in the second integral leads us to

$$\begin{aligned} \psi_\lambda(x) &= \lim_{\varepsilon \rightarrow 0} \left[\int_\varepsilon^\infty \frac{e^{-\lambda u}}{u} du - \int_{\ln(1+\varepsilon)}^\infty \frac{e^t dt}{e^{tx}(e^t - 1)} \right] \\ &= \lim_{\varepsilon \rightarrow 0} \left[\int_\varepsilon^\infty \frac{e^{-\lambda u}}{u} du - \int_{\ln(1+\varepsilon)}^\infty \frac{e^{-xt} dt}{1 - e^{-t}} \right]. \end{aligned}$$

Since $\int_\varepsilon^{\ln(1+\varepsilon)} \frac{e^{-t} dt}{t} \leq \int_{\ln(1+\varepsilon)}^\varepsilon \frac{dt}{t} \rightarrow 0$, we get

$$\begin{aligned} \psi_\lambda(x) &= \lim_{\varepsilon \rightarrow 0} \left[\int_\varepsilon^\infty \frac{e^{-\lambda u}}{u} du - \int_{\ln(1+\varepsilon)}^\infty \frac{e^{-xt} dt}{1 - e^{-t}} \right] \\ &= \lim_{\varepsilon \rightarrow 0} \left[\int_\varepsilon^{\ln(1+\varepsilon)} \frac{e^{-\lambda u}}{u} du + \int_\varepsilon^\infty \frac{e^{-\lambda u}}{u} du - \int_{\ln(1+\varepsilon)}^\infty \frac{e^{-xt} dt}{1 - e^{-t}} \right] \\ &= \int_0^\infty \left[\frac{e^{-\lambda t}}{t} - \frac{e^{-xt}}{1 - e^{-t}} \right] dt. \end{aligned}$$

Using the equation $\ln \lambda = \int_0^\infty \frac{e^{-t} - e^{-\lambda t}}{t} dt$ yields that

$$\begin{aligned} \psi_\lambda(x) &= \int_0^\infty \left[\frac{e^{-\lambda t} - e^{-t} + e^{-t}}{t} - \frac{e^{-xt}}{1 - e^{-t}} \right] dt \\ &= - \int_0^\infty \frac{e^{-t} - e^{-\lambda t}}{t} dt + \int_0^\infty \left[\frac{e^{-t}}{t} - \frac{e^{-xt}}{1 - e^{-t}} \right] dt \\ &= -\ln \lambda + \int_0^\infty \left[\frac{e^{-t}}{t} - \frac{e^{-xt}}{1 - e^{-t}} \right] dt. \end{aligned} \quad (3.3)$$

By using last equation (3.3), $\frac{1}{2x} = \frac{1}{2} \int_0^\infty e^{-xt} dt$ and $\ln x = \int_0^\infty \frac{e^{-t} - e^{-xt}}{t} dt$, we obtain

$$\begin{aligned}\psi_\lambda(x+1) &= -\ln \lambda + \frac{1}{2x} + \ln x - \int_0^\infty \left[\frac{e^{-xt}}{2} + \frac{e^{-t} - e^{-xt}}{t} - \frac{e^{-t}}{t} + \frac{e^{-xt}}{e^t - 1} \right] dt \\ &= \ln x - \ln \lambda + \frac{1}{2x} - \int_0^\infty \left[\frac{1}{2} - \frac{1}{t} + \frac{1}{e^t - 1} \right] e^{-xt} dt.\end{aligned}$$

The integrand is continuous as $t \rightarrow 0$ and since $\frac{1}{2} - \frac{1}{t} + \frac{1}{e^t - 1}$ is bounded as $t \rightarrow \infty$, the integral is uniformly convergence for $x > 0$. By integrating from 1 to x , we obtain

$$\ln \Gamma_\lambda(x+1) - \ln \Gamma_\lambda = x \ln x - x + 1 - (x-1) \ln \lambda + \frac{1}{2} \ln x + \int_0^\infty \left[\frac{1}{2} - \frac{1}{t} + \frac{1}{e^t - 1} \right] \frac{e^{-xt} - e^{-t}}{t} dt.$$

Now using recurrence formula on λ -gamma function in (1.6) and the equation $\Gamma_\lambda(2) = \frac{1}{\lambda^2}$ leads us that

$$\begin{aligned}\ln \Gamma_\lambda + \ln x - \ln \lambda + 2 \ln \lambda &= x \ln x - x + 1 - (x-1) \ln \lambda + \frac{1}{2} \ln x \\ &\quad + \int_0^\infty \left[\frac{1}{2} - \frac{1}{t} + \frac{1}{e^t - 1} \right] \frac{e^{-xt} - e^{-t}}{t} dt \\ \ln \Gamma_\lambda(x) &= (x - \frac{1}{2}) \ln x - x - x \ln \lambda + 1 + \int_0^\infty \left[\frac{1}{2} - \frac{1}{t} + \frac{1}{e^t - 1} \right] \frac{e^{-xt}}{t} dt \\ &\quad - \int_0^\infty \left[\frac{1}{2} - \frac{1}{t} + \frac{1}{e^t - 1} \right] \frac{e^{-t}}{t} dt.\end{aligned}$$

For evaluating the second integral on the right side of the last equation, let us denote

$$I = \int_0^\infty \left[\frac{1}{2} - \frac{1}{t} + \frac{1}{e^t - 1} \right] \frac{e^{-t}}{t} dt \text{ and } J = \int_0^\infty \left[\frac{1}{2} - \frac{1}{t} + \frac{1}{e^t - 1} \right] \frac{e^{-t/2}}{t} dt. \text{ Taking } x = 1/2 \text{ in the last equation yields that } \ln \Gamma_\lambda(1/2) = \frac{1}{2} - \frac{\ln \lambda}{2} + J - I \text{ and then using } \Gamma_\lambda\left(\frac{1}{2}\right) = \frac{1}{2}(\ln \pi - \ln \lambda) \text{ leads us to } J - I = \frac{1}{2}(1 - \ln \pi). \text{ On the other hand, by substituting } u = \frac{t}{2} \text{ in the integral } I, \text{ we get } I = \int_0^\infty \left[\frac{1}{2} - \frac{2}{t} + \frac{1}{e^{t/2} - 1} \right] \frac{e^{-t/2}}{t} dt. \text{ Hence, we find}$$

$$\begin{aligned}J - I &= \int_0^\infty \left[\frac{1}{2} - \frac{1}{t} + \frac{1}{e^t - 1} - \frac{1}{2} + \frac{2}{t} - \frac{1}{e^{t/2} - 1} \right] \frac{e^{-t/2}}{t} dt \\ &= \int_0^\infty \left(\frac{1}{t} - \frac{e^{-t/2}}{e^t - 1} \right) \frac{e^{-t/2}}{t} dt = \int_0^\infty \left(\frac{e^{-t/2}}{2} - \frac{1}{e^t - 1} \right) \frac{dt}{t}\end{aligned}$$

By using the last equation and integral representation of I , we obtain

$$\begin{aligned}J &= \int_0^\infty \left(\frac{e^{-t/2}}{2} - \frac{1}{e^t - 1} + \frac{e^{-t/2}}{2} - \frac{e^{-t}}{t} + \frac{e^t}{e^t - 1} \right) \frac{dt}{t} \\ &= \int_0^\infty \left[\frac{e^{-t/2} - e^{-t}}{t} - \frac{e^{-t}}{2} \right] \frac{dt}{t} \\ &= \int_0^\infty \left[-\frac{d}{dt} \left(\frac{e^{-t/2} - e^{-t}}{t} \right) - \frac{\frac{e^{-t/2}}{2} - e^{-t}}{t} - \frac{e^{-t}}{2t} \right] dt \\ &= \left(\frac{e^{-t/2} - e^{-t}}{t} \right) \Big|_0^\infty + \frac{1}{2} \int_0^\infty \frac{e^{-t} - e^{-t/2}}{t} dt \\ &= \frac{1}{2} + \frac{1}{2} \ln \frac{1}{2}.\end{aligned}$$

Hence we get $I = 1 - \frac{1}{2} \ln(2\pi)$. So by writing the solution of integral I in the last equation, we find

$$\ln \Gamma_\lambda(x) = \left(x - \frac{1}{2}\right) \ln x - x - x \ln \lambda + \frac{1}{2} \ln(2\pi) + \int_0^\infty \left[\frac{1}{2} - \frac{1}{t} + \frac{1}{e^t - 1} \right] \frac{e^{-xt}}{t} dt$$

for all $x, \lambda > 0$. By differentiating the last equation, one can obtain the equation (3.2). \square

By using Theorem 3.1, we give the following completely monotonicity properties on the function involving λ -psi function and its first derivative:

Theorem 3.2. *The functions*

$$\begin{aligned} & \psi_\lambda(x) + \ln \lambda - \ln x + \frac{1}{2x} + \frac{1}{12x^2} \\ & \ln x - \frac{1}{2x} - \ln \lambda - \psi_\lambda(x) \\ & \psi'_\lambda(x) - \frac{1}{x} - \frac{1}{2x^2} - \frac{1}{6x^3} + \frac{1}{30x^5} \\ & \frac{1}{x} + \frac{1}{2x^2} + \frac{1}{6x^3} - \psi'_\lambda(x) \end{aligned}$$

are completely monotonic for all $x, \lambda > 0$.

Proof. Let us denote $f_1(x) = \psi_\lambda(x) + \ln \lambda - \ln x + \frac{1}{2x} + \frac{1}{12x^2}$. By using the definition of the λ -psi function (3.2) and Lemma 2.1, we get

$$\begin{aligned} f_1(x) &= \psi_\lambda(x) + \ln \lambda - \ln x + \frac{1}{2x} + \frac{1}{12x^2} \\ &= \int_0^\infty \left[\frac{1}{t} - \frac{1}{e^t - 1} - \frac{1}{2} + \frac{t}{12} \right] e^{-xt} dt \\ &= \int_0^\infty \left[\frac{(12 - 6t + t^2)e^t - (12 + 6t + t^2)}{12t(e^t - 1)} \right] e^{-xt} dt. \end{aligned}$$

Since the nominator $d_1(t) = (12 - 6t + t^2)e^t - (12 + 6t + t^2) > 0$ and $d_1(0) = 0$, we get that $(-1)^n f_1^{(n)}(x) > 0$ as desired.

Now, let us define the function f_2 by $f_2(x) = \psi_\lambda(x) + \ln \lambda - \ln x + \frac{1}{2x}$. Then we obtain

$$\begin{aligned} f_2(x) &= \psi_\lambda(x) + \ln \lambda - \ln x + \frac{1}{2x} \\ &= \int_0^\infty \left(\frac{1}{t} - \frac{1}{e^t - 1} - \frac{1}{2} \right) e^{-xt} dt \\ &= \int_0^\infty \frac{(2 - t)e^t - (t + 2)}{2t(e^t - 1)} e^{-xt} dt. \end{aligned}$$

Since $d_2(t) = (2 - t)e^t - (t + 2) < 0$ and $d_2(0) = 0$ for all $t > 0$, we obtain $(-1)^{n+1} f_2^{(n)}(x) > 0$ as desired.

By differentiating the definition of the λ -psi function (3.2), we get

$$\psi'_\lambda(x) = \frac{1}{x} + \frac{1}{x^2} - \int_0^\infty \left(1 - \frac{t}{e^t - 1} \right) e^{-xt} dt$$

for all $x, \lambda > 0$. Then using the last equation and Lemma 2.1 yields that

$$\begin{aligned} f_3(x) &= \psi'_\lambda(x) - \frac{1}{x} - \frac{1}{2x^2} - \frac{1}{6x^3} + \frac{1}{30x^5} \\ &= \int_0^\infty \left(\frac{t}{e^t - 1} - 1 + \frac{t}{2} - \frac{t^2}{12} + \frac{t^4}{720} \right) e^{-xt} dt \\ &= \int_0^\infty \left[\frac{(t^4 - 60t^2 + 360t - 720)e^t - (t^4 - 60t^2 - 360t - 720)}{720(e^t - 1)} \right] e^{-xt} dt \end{aligned}$$

Since the nominator of the function $d_3(t) = (t^4 - 60t^2 + 360t - 720)e^t - (t^4 - 60t^2 - 360t - 720)$ is positive for all $t \geq 0$, it concludes that $(-1)^n f_3^{(n)}(x) > 0$ for all positive real value of x and any nonnegative integer n .

At last, let us define the function f_4 by $f_4(x) = \frac{1}{x} + \frac{1}{2x^2} + \frac{1}{6x^3} - \psi'_\lambda(x)$. Then by using the derivative of the λ -psi function (3.2) and Lemma 2.1, we obtain

$$f_4(x) = \int_0^\infty \left[\frac{(12 - 6t + t^2)e^t - (t^2 + 6t + 12)}{12(e^t - 1)} \right] e^{-xt} dt.$$

Since the nominator of the function is the same as the function f_1 , we get that the function f_4 is also completely monotonic for all $x > 0$ and $n \in \mathbb{Z}^+ \cup \{0\}$. \square

As an immediate consequence of the previous theorem, we get the following double sided inequalities on λ -psi function and its first derivative.

Corollary 3.3. *The following double sided inequalities*

$$\ln x \ln \lambda - \frac{1}{2x} - \frac{1}{12x^2} < \psi_\lambda(x) < \ln x - \ln \lambda - \frac{1}{2x}, \quad (3.4)$$

$$\frac{1}{x} + \frac{1}{2x} + \frac{1}{6x^3} - \frac{1}{30x^5} < \psi'_\lambda(x) < \frac{1}{x} + \frac{1}{2x^2} + \frac{1}{6x^3} \quad (3.5)$$

and

$$-\frac{1}{kx^2} - \frac{1}{x^3} - \frac{1}{2x^4} < \psi''_\lambda(x) < -\frac{1}{x^2} - \frac{1}{x^3} \quad (3.6)$$

hold true for $x > 0$.

Theorem 3.4. *For all positive real values of x and λ , the function*

$$P_\lambda(x) = \psi_\lambda(x) + \psi_\lambda\left(\frac{1}{x}\right) + 2 \ln \lambda \quad (3.7)$$

is concave.

Proof. Taking derivatives of the function P_λ yields that

$$\begin{aligned} P'_\lambda(x) &= \psi_\lambda(x) - \frac{1}{x^2} \psi'_\lambda\left(\frac{1}{x}\right) \\ P''_\lambda(x) &= \psi''_\lambda(x) + \frac{2}{x^3} \psi'_\lambda\left(\frac{1}{x}\right) + \frac{1}{x^4} \psi''_\lambda\left(\frac{1}{x}\right) \\ x^4 P''_\lambda(x) &= \psi''_\lambda \psi''_\lambda(x) + \frac{2}{x^3} \psi'_\lambda\left(\frac{1}{x}\right) + \frac{1}{x^4} \psi''_\lambda\left(\frac{1}{x}\right) + 2x \psi'_\lambda\left(\frac{1}{x}\right) + x^4 \psi''_\lambda(x). \end{aligned}$$

By using the recurrence formulas $\psi'_\lambda(x+1) = \psi'_\lambda(x) - \frac{1}{x^2}$ and $\psi''_\lambda(x+1) = \psi''_\lambda(x) + \frac{2}{x^3}$ and inequalities (3.5) and (3.6), we obtain

$$\begin{aligned} x^4 P''_\lambda(x) &= -2x^3 + \psi''_\lambda\left(\frac{1}{x}\right) + 2x \left[x^2 + \psi'_\lambda\left(\frac{1}{x} + 1\right) \right] + x^4 \psi''_\lambda(x) \\ &< -2x^3 + \left(-\frac{1}{(1/x+1)^2} - \frac{1}{(1+1/x)^3} \right) + 2x \left(x^2 + \frac{1}{1+1/x} + \frac{1}{2(1+1/x)^2} + \frac{1}{6(1+1/x)^3} \right) \\ &\quad + x^4 \left(-\frac{1}{x^2} - \frac{1}{x^3} \right) \\ &= -\frac{x}{3(1+x)^3} [3x^4 + 2x^3 + 9x^2 + 9x + 3]. \end{aligned}$$

Hence it is easy to see that since $x^4 P''_\lambda(x) < 0 \Rightarrow P''_\lambda(x) < 0$, the function P_λ is concave for $\forall x, \lambda > 0$. \square

Corollary 3.5. For $x \in \mathbb{R}^+ - \{1\}$, the inequality

$$\psi_\lambda(x) + \psi_\lambda\left(\frac{1}{x}\right) + 2\ln\lambda < 2(\psi_\lambda(1) + \ln\lambda) \quad (3.8)$$

is valid.

Proof. Using the concavity of the function P_λ in Theorem 3.4 leads us that the function P'_λ is decreasing for $x > 0$. Since $P'_\lambda(x) = \psi_\lambda(x) - \frac{1}{x^2}\psi'_\lambda\left(\frac{1}{x}\right) \Rightarrow P'_\lambda(1) = 0$, we get that from $P'_\lambda(x) > P'_\lambda(1) = 0$ for $0 < x < 1$, the function P_λ increases and also from $P'_\lambda(x) < P'_\lambda(1) = 0$ for $1 < x$, the function P_λ increases. Hence for $x \neq 1$ and $x > 0$, we find $P_\lambda(x) < P_\lambda(1) = 2(\psi_\lambda(1) + \ln\lambda)$ as desired. \square

We want to note that since the classical psi function $\psi(x)$ at $x = 1$ is equal to $-\gamma$, by using the relation between classical and λ -psi function $\psi_\lambda(x) = -\ln\lambda + \psi(x)$, we clearly see that for all positive value of λ , $P_\lambda(1) = 2(\psi_\lambda(1) + \ln\lambda) = 2\psi(1) = -2\gamma < 0$.

Theorem 3.6. The inequality

$$[\psi_\lambda(1+y) + \ln\lambda][\psi_\lambda(1-y) + \ln\lambda] < [\psi_\lambda(1) + \ln\lambda]^2 \quad (3.9)$$

holds true for $\forall y \in (0, 1)$.

Proof. Since for all $\lambda > 0$, the λ -psi function is completely monotonic on $(0, \infty)$, there is a $x_0 \in (0, \infty)$ such that $\psi_\lambda(x) + \ln\lambda = 0$ and on the other hand, from the first inequality (3.4), we get that $\psi_\lambda(x) + \ln\lambda > \ln x - \frac{1}{2x} - \frac{1}{12x^2} \Rightarrow x_0 \gtrsim 1,46321$. So if $y \in [x_0 - 1, 1)$, then $\psi_\lambda(1-y) + \ln\lambda < 0 \leq \psi_\lambda(1+y) + \ln\lambda$. Hence we get desired result. Now if $y \in (0, x_0 - 1)$, then $\psi_\lambda(1-y) + \ln\lambda < 0$ and $\psi_\lambda(1+y) + \ln\lambda < 0$. By using the power series on the λ -psi function in the equation (2.6), we get the following inequalities

$$0 < -(\psi_\lambda(1+y) + \ln\lambda) \leq \gamma - \zeta_\lambda(2)y + \zeta_\lambda(3)y^2$$

and

$$\begin{aligned} 0 < -(\psi_\lambda(1-y) + \ln\lambda) &\leq \gamma + \zeta_\lambda(2)y + \zeta_\lambda(3)\sum_{k=2}^{\infty} y^k \\ &= \gamma + \zeta_\lambda(2)y + \zeta_\lambda(3)\frac{y^3}{1-y} \\ &\leq \gamma + \zeta_\lambda(2)y + \zeta_\lambda(3)y^2. \end{aligned}$$

Thus these inequalities leads us to the following inequality

$$(\psi_\lambda(1+y) + \ln\lambda)(\psi_\lambda(1-y) + \ln\lambda) \leq \gamma^2 - [\zeta_\lambda(2)^2 - \zeta_\lambda(3)]y^2 - \zeta_\lambda(2)\zeta_\lambda(3)y^3 + 2\zeta_\lambda(3)^2y^4.$$

Since $\zeta_\lambda(2)^2 - \zeta_\lambda(3) > 0$, $\zeta_\lambda(2)\zeta_\lambda(3) > 0$ and $2\zeta_\lambda(3)^2 > 0$, the inequality $\zeta_\lambda(2)\zeta_\lambda(3) > 2\zeta_\lambda(3)^2$ is valid. Hence we obtain that the inequality $\zeta_\lambda(2)\zeta_\lambda(3)y^3 > 2\zeta_\lambda(3)^2y^4$. Thus the proof is completed. \square

Theorem 3.7. For $x \in \mathbb{R} - \{1\}$, the inequality

$$[\psi_\lambda(x) + \ln\lambda] \left[\psi_\lambda\left(\frac{1}{x}\right) + \ln\lambda \right] < [\psi_\lambda(1) + \ln\lambda]^2 \quad (3.10)$$

holds true.

Proof. If $x \geq x_0$, then the inequality is valid since $\left[\psi_\lambda\left(\frac{1}{x}\right) + \ln \lambda\right] < 0 \leq [\psi_\lambda(x) + \ln \lambda]$.

Now, let $1 + y = x \in (1, x_0)$. By using the inequality $\frac{1}{x} > 1 - y$ and monotonicity property of the λ -psi function $(0, \infty)$, we find that the inequality

$$[\psi_\lambda(1 - y) + \ln \lambda] < \left[\psi_\lambda\left(\frac{1}{x}\right) + \ln \lambda\right] < 0$$

holds true. Hence from the last inequality and Theorem 3.6, we get

$$\begin{aligned} [\psi_\lambda(x) + \ln \lambda] \left[\psi_\lambda\left(\frac{1}{x}\right) + \ln \lambda\right] &= [\psi_\lambda(1 + y) + \ln \lambda] \left[\psi_\lambda\left(\frac{1}{x}\right) + \ln \lambda\right] \\ &< [\psi_\lambda(1 + y) + \ln \lambda][\psi_\lambda(1 - y) + \ln \lambda] \\ &< [\psi_\lambda(1) + \ln \lambda]^2 \end{aligned}$$

as desired.

If $x \in (0, 1)$, then we take $z = 1/x$ and use the method above for the result. □

Theorem 3.8. *The inequality*

$$\psi_\lambda(1) + \ln \lambda \leq \frac{2 [\psi_\lambda(x) + \ln \lambda] \left[\psi_\lambda\left(\frac{1}{x}\right) + \ln \lambda\right]}{\psi_\lambda(x) + \psi_\lambda\left(\frac{1}{x}\right) + 2 \ln \lambda} \quad (3.11)$$

holds true for all positive real values of x .

Proof. By using the inequalities (3.8) and (3.10) and taking into account that the inequality (3.8) is negative, we get

$$\begin{aligned} 2 [\psi_\lambda(x) + \ln \lambda] \left[\psi_\lambda\left(\frac{1}{x}\right) + \ln \lambda\right] &< 2 [\psi_\lambda(1) + \ln \lambda]^2 \\ \frac{2 [\psi_\lambda(x) + \ln \lambda] \left[\psi_\lambda\left(\frac{1}{x}\right) + \ln \lambda\right]}{\psi_\lambda(x) + \psi_\lambda\left(\frac{1}{x}\right) + 2 \ln \lambda} &> \frac{2 [\psi_\lambda(1) + \ln \lambda]^2}{\psi_\lambda(x) + \psi_\lambda\left(\frac{1}{x}\right) + 2 \ln \lambda} \\ &> \frac{2 [\psi_\lambda(1) + \ln \lambda]^2}{[\psi_\lambda(1) + \ln \lambda]^2} \\ &= [\psi_\lambda(1) + \ln \lambda]^2 \end{aligned}$$

for all positive real values of x except for 1. The sign of the equality holds if and only if $x = 1$. □

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