**Short Research Article**

**Brief Discussion on Maximal Edge-Ideal in Graph Theory**

**Abstract:** The exploration of width parameters within the fields of graph theory and algebra has garnered significant interest. Among these parameters, tree-cut decomposition stands out as a vital metric. The "Edge-Tangle" concept is intrinsically linked to the width parameter known as "tree-cut width" in graph theory. In this paper, we introduce a new definition termed Maximal Edge-Ideal for graphs and demonstrate their equivalence to Edge-Tangles.

**Keyword:** Ideal, tangle, edge-tangle, tree-cut-width, tree-cut-decomposition

1. **Introduction**

**1.1. Graph and Graph Parameters**

A **graph** consists of vertices and edges [63]. Graphs are studied for applications in networks, artificial intelligence, and various other fields [65, 66]. A **graph parameter** is a numerical invariant measuring structural properties. In recent years, research interest in width parameters within graph theory and algebra has significantly increased [1-33]. These width parameters are metrics derived from tree-like structures, commonly studied through graph decompositions. Well-known examples of width parameters include tree-width, branch-width [12], path-width [10], hypertree-width [47,48], superhypertree-width [52,53], cut-width[67,68], linear-width [5,14], modular-width [54,55], Boolean-width[56,57], Band-width[58,59], and clique-width[60].

A key topic of study in this field is the concept of obstructions, which play a crucial role in determining width parameters. Notable examples include tangles [17], brambles [75, 76], ultrafilters [17, 77], blockages [64], and linear tangles [8]. In particular, the concept of an edge-tangle is closely related to the width parameter known as tree-cut width in graph theory [1]. These concepts are frequently used in the analysis of efficient algorithms and the exploration of mathematical structures in graph theory.

**1.2. Ideal in set theory and Graph Theory**

In set theory, an ideal is a collection of subsets of a given set that is closed under taking subsets and finite unions while excluding the entire set. The complementary structure to an ideal is called a filter, and similar concepts have been explored in graph theory as well. The concepts of Ideals and Filters have been widely studied across various fields [69-74].

The notion of maximality is frequently discussed in the context of ideals. A Maximal Ideal is an ideal that cannot be extended further without becoming the entire set. Its complement is called an Ultrafilter. These structures are known to possess various intriguing mathematical properties.

**1.3. Our Contribution of this paper**

This paper explains its contributions. As stated in the introduction, research on Ideals and Graph Width Parameters is significant. However, the relationship between Ideals and Graph Width Parameters has not been extensively explored.

In this paper, we introduce a new concept called Maximal-Edge-Ideal for graphs and demonstrate its connection to Edge-Tangles. Additionally, we briefly examine the maximality of Edge-Ideals and their structural similarities to traditional Ideals. Although the novelty of this work may be limited, we hope that it will make a modest contribution to the ongoing research on graph width parameters.

2. **Definitions and Notations in this paper**

This section provides the mathematical definitions of each concept. First, we briefly explain the notations used in this paper. Readers who wish to review the fundamentals of set theory and graph theory may refer to references [61-63] as needed.

**Notation 1:** In this short paper, we use expressions like A ⊆ X to indicate that *A* is a subset of *X,* *A ∪ B* to represent the union of two subsets *A* and *B,* both of which are subsets of *X,* or *A = ∅*  to signify an empty set. Specifically, *A ∩ B* denotes the intersection of subsets *A* and *B*. A similar logic applies to *A \ B.* The powerset of a set *A*, denoted as *2A*, is the set of all possible subsets of *A*, including the empty set and *A* itself.  
  
 **Notation 2:** Let *G* be a finite and undirected graph. The notation *V(G)* represents the set of vertices (nodes) in *G*, and *E(G)*  represents the set of edges in *G* . The expression *G = (V, E)* signifies that *G* is a graph defined by a pair of sets: *V* for vertices and *E* for edges. In this paper, we focus on the properties of undirected, finite, and simple graphs.

**Notation 3:** A natural number is a positive integer used for counting and ordering.In this paper, we utilize the natural number *k*.

Next, we briefly explain the definition of an edge-cut. This definition will be used later in the discussion of edge-tangles and edge-ideals.

**Definition 4:** An edge-cut *[A, B]* of a finite and undirected *G* is an ordered pair of disjoint subsets of *V(G)* such that *A ∪ B = V (G)*. And the order of an edge-cut *[A, B]* of G is the number of edges of *G* with one end in *A* and one end in *B* (cf: [1]).

**Example 5:** Let finite and undirected graph *G = (V, E)* be defined as follows:

*V = {a, b, c, d}, E = {{a, b}, {b, c}, {c, d}, {d, a}, {a, c}}*

Define the subsets: *A = {a, b} B = {c, d}.*

These subsets satisfy:

*A ∩ B = ∅* (they are disjoint), *A ∪ B = V* (their union covers all vertices).

According to Definition 4, the edge-cut *[A, B]* is the ordered pair *(A, B)* and its order is the number of edges with one endpoint in *A* and the other in *B.*

Now, inspect each edge:

* Edge *{a, b}*: Both endpoints are in *A* — not part of the cut.
* Edge *{b, c}: b* *∈* A and c *∈* *B* — included in the cut.
* Edge *{c, d}:* Both endpoints are in *B* — not part of the cut.
* Edge *{d, a}: d ∈ B* and *a ∈ A* — included in the cut.
* Edge *{a, c}: a ∈ A* and *c ∈ B* — included in the cut.

Thus, the edge-cut *[A, B*] contains the edges: *{b, c}, {d, a}, {a, c},* and its order is 3 (since there are 3 edges crossing from *A* to B).

**2.1 Ideals on Boolean Algebras**

We provide an explanation of Ideals in Boolean Algebras.   
 **Definition 6:** In a Boolean algebra *(X,∪,∩)*, a set family *I ⊆ 2X* satisfying the following conditions is called an ideal on the carrier set *X*.

(IB1) *A, B ∈ I*  ⇒ *A ∪ B ∈ I (*Closure under Union*),*

(IB2) *B ∈ I, A ⊆ B ⊆ X* ⇒ *A∈ I (*Closure under Superset*),*

(IB3) *X i*s not belong to *I (*Exclusion of the Universal Set*)*.

In a Boolean algebras *(X,∪,∩)*, A maximal ideal satisfies the following axiom (IB4):

(IB4) *∀A ⊆ X, either A ∈ I or X / A ∈ I*

**2.2　Edge*-*Tangle on the graph**  
We describe the concept of an edge-tangle in graphs. Below, we provide the definition of a tangle within graph theory. Edge-Tangles are renowned for their profound association with tree-cut decompositions [1]. These decompositions have been extensively studied by various researchers [24-31].  
 **Definition 7 [1, 23]:** Let *G* be a finite and undirected graph. A *edge-tang*le *E* of order *k* is a set of edge-cuts of G such that the following hold.   
(E1) For all edge-cut *[A, B]* of *G* of order less than *k*, either *[A, B] ∈ E* or *[B, A] ∈ E* .  
(E2) If *[A1, B1],[A2, B2],[A3, B3] ∈ E* then *A1 ∪ A2 ∪ A3 ≠ G.*  
(E3) If *[A, B] ∈ E*, then G has at least k edges incident with vertices in *B*.

**3. Some Properties of Maximal Ideals of Edge-Cuts: Their Relation to Tree-Cut Decomposition**  
The definition of a Maximal Edge-ideal on the graph is given below. We naturally extend the definition from Boolean algebras to a set of edge-cuts.

**Definition 8:** Let *G* be a finite and undirected graph. An Edge-ideal *I* of order *k* is a set of edge-cuts of G such that the following hold.

(I1) *[A2, B2] ∈ I, A1 ⊆ A2 , [A1, B1]* of order less than *k*⇒ *[A1, B1] ∈ I*,

(I2) *[A1, B1]* *∈ I, [A2, B2] ∈ I*, *[A1∪A2, B1∩ B2 ]* of order less than *k*

⇒ *[A1 ∪ A2, B1∩ B2 ] ∈ I*,

(I3)If *[A, B] ∈ I,* then *G* has at least *k* edges incident with vertices in *B.*(I4) *If V(A) ＝ V(G), then [A, B] ∈I.*

A Maximal Edge-ideal satisfies the following additional axiom (I5):  
For all edge-cut *[A, B]*  of *G* of order less than *k*, either *[A, B] ∈ I* or *[B, A] ∈ I*

Proving the Main Theorem of this paper, which establishes the　equivalence between Maximal Edge-ideal and Edge-Tangles.  
**Theorem 9:** Let *G* be a finite and undirected graph. *E* is anedge-Tangle of separations of order *k* in graph if and only if *E* is a Maximal Edge-ideal of separations of order *k* in graph.  
**Proof.** (⇒) Assume that *E* is an edge-tangle of separations of order *k* in *G.*

We show that *E* satisfies the axioms (I1)–(I4) for an edge-ideal and, moreover, the maximality condition (I5).

* (I1) Downward Closure: Let *[A2, B2] ∈ E* and suppose *A1 ⊆ A2* with *[A1, B1]* an edge-cut of order less than k. By the tangle property (E1), either *[A1, B1] ∈ E* or *[B1, A1] ∈ E.*
* (I2) Closure under 'Union': Let *[A1, B1] ∈ E* and *[A2, B2] ∈ E*. If *[A1 ∪ A2, B1 ∩ B2]* is an edge-cut of order less than *k,* then *[A1 ∪ A2, B1 ∩ B2] ∈ E.*
* (I3) Connectivity Condition: If *[A, B] ∈ E,* then *G* has at least *k* edges incident with vertices in B.
* (I4) Inclusion of Trivial Edge-Cuts: If *V(A) = V(G)*, then *[A, B]* is included in *E.*
* (I5) Maximality: For every edge-cut *[A, B]* of order less than *k,* either *[A, B] ∈ E* or *[B, A] ∈ E.*

Thus, if *E* is an edge-tangle, then *E* is a maximal edge-ideal.

(⇐) Conversely, assume that *E* is a maximal edge-ideal, that is, *E* satisfies axioms (I1)–(I4) and the maximality condition (I5). We show that *E* is an edge-tangle.

* Edge-Cut Orientation (E1): Axiom (I5) states that for every edge-cut *[A, B]* of order less than *k*, either *[A, B] ∈ E* or *[B, A] ∈ E.*
* Non-Covering Condition (E2): Suppose for contradiction that there exist edge-cuts *[A1, B1], [A2, B2], [A3, B3] ∈ E* such that *A1 ∪ A2 ∪ A3 = V(G).* This contradicts the connectivity condition (I3).
* Connectivity (E3): Axiom (I3) of the edge-ideal directly provides the connectivity requirement (E3).

Thus, *E* satisfies all the conditions for being an edge-tangle. This completes the proof of the Theorem.

**Theorem 10:**  Let *G* be a finite and undirected graph. The following holds.  
(i) An Edge-Ideal *I* of order *k* in a graph *G* possesses the structure of an ideal analogous to ideals in Boolean algebras.  
(ii) Moreover, a maximal Edge-Ideal satisfies the additional axiom (I5).  
  
**Proof:** Each can be proved as follows.

(i) Edge-Ideal as an Ideal Structure:  
Downward Closure (I1): If *[A2, B2] ∈ I* and *A1 ⊆ A2,* then *[A1, B1] ∈ I.*  
Closure under 'Union' (I2): If *[A1, B1]* and *[A2, B2]* are in *I,* then *[A1 ∪ A2, B1 ∩ B2] ∈ I.*Connectivity Condition (I3): If *[A, B] ∈ I,* then *G* has at least *k* edges incident with vertices in *B.*Trivial Edge-Cut Inclusion (I4): If *V(A) = V(G),* then *[A, B] ∈ I.*(ii) Maximal Edge-Ideals Satisfy Axiom (I5):  
By definition, a maximal Edge-Ideal is one that cannot be extended further without violating (I1)–(I4). For every edge-cut *[A, B]* of order less than *k*, if *[A, B]* is not in *I,* then its complement *[B, A]* must be in *I.*  
Thus, a maximal Edge-Ideal not only has the ideal structure but also satisfies the additional maximality condition (I5). This completes the proof of the Theorem.  
  
**4. Conclusion and Future Direction of This Research**

In this paper, we explored the characteristics of Edge-Ideals as an extension of traditional Ideals. As a future research direction, we aim to extend this concept to Hypergraphs[37-39], SuperHyperGraphs[34-36], Fuzzy Graphs[40-42], Neutrosophic Graphs[43-46], and Plithogenic Graphs[49-51], further investigating their properties and potential applications.

**Data Availability**

This research is purely theoretical, involving no data collection or analysis. We encourage future researchers to pursue empirical investigations to further develop and validate the concepts introduced here.

**Ethical Approval**

As this research is entirely theoretical in nature and does not involve human participants or animal subjects, no ethical approval is required.

**Disclaimer**

This work presents theoretical concepts that have not yet undergone practical testing or validation. Future researchers are encouraged to apply and assess these ideas in empirical contexts. While every effort has been made to ensure accuracy and appropriate referencing, unintentional errors or omissions may still exist. Readers are advised to verify referenced materials on their own. The views and conclusions expressed here are the authors’ own and do not necessarily reflect those of their affiliated organizations.

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