**Quasi-Ultrafilters and Weak-Ultrafilters: Their Properties and Maximality**

**Abstract:** The concept of a Quasi-Ultrafilter and provides an axiomatic analysis of incomplete social judgments. The weak filter (or weak ultrafilter) was introduced by K. Schlechta in the 1990s as a relaxed version of an ultrafilter. A pair *(X, f)* of a finite set *X* and a symmetric submodular function *f* is called a connectivity system. This concept is commonly used in discussions of graph width parameters, such as branch-width and tree-width, to analyze graph structures. In this paper, we discussed about Quasi-Ultrafilter and Maximal Quasi-filter on Connectivity System.
**Keyword:** Quasi-Ultrafilter, Quasi-Filter, Ultrafilter, Connectivity System, Weak-Filter

1. **Introduction**
	1. **Set theory and Filter**

An ultrafilter is a maximal filter on a set, essential in set theory and topology for rigorously handling limits, convergence, and compactness. Its unique properties make it crucial in non-standard analysis, model theory, and first-order logic, providing powerful tools for both mathematical and logical applications [43-49].

Due to their versatility, various related concepts to ultrafilters have been proposed. One such concept is the Quasi-Ultrafilter, which offers an axiomatic analysis of incomplete social judgments [1-8]. The weak filter (or weak ultrafilter) was introduced by K. Schlechta in the 1990s as a relaxed version of an ultrafilter. It serves as a powerful tool for interpreting defaults using a generalized "most" quantifier in first-order logic [29-33].

* 1. **Graph Theory**

Graph theory, a fundamental branch of mathematics, studies networks of nodes and edges, focusing on their paths, structures, and properties [88]. A key metric in this field is the "graph width parameter," which measures the maximum width across all cuts or layers in a hierarchical decomposition of the graph, essential for analyzing a graph's complexity and structure.

Branch width is a significant graph width parameter that involves a branch decomposition where the leaves correspond to the graph's edges [22, 23]. Each edge is assigned a value from a symmetric submodular function, measuring connectivity. Branch width generalizes the width of symmetric submodular functions on graphs, making it crucial in graph theory. The study of branch width, along with other parameters like path-width, tree-width, and cut-width, has significantly impacted the field (e.g., [9-23, 34-42]).

A pair *(X, f)* of a finite set *X* and a symmetric submodular function *f* is called a connectivity system. This concept is commonly used in discussions of graph width parameters, such as branch-width and tree-width, to analyze graph structures. In graph width parameters, duality is often discussed to understand the relationship between different decompositions and measures of a graph's complexity. It is known that Quasi-Ultrafilters and Weak-Ultrafilters on Connectivity Systems have a dual relationship with branch-width [22, 23].

* 1. **Our Contribution**

In this paper, we discussed Quasi-Ultrafilters and Weak-Ultrafilters on Connectivity Systems. Here, we will explore the maximality of these concepts. Overall, this exploration contributes to a deeper understanding of these concepts and their relevance in various mathematical and logical contexts.

2. **Definitions and Notations in this paper**

This section provides mathematical definitions for each concept. Before delving into specific definitions, let's outline the basic mathematical concepts used in this text.

In set theory, a set is a collection of distinct elements or objects, considered as an entity and often denoted with curly braces. A subset is a set where all elements are also contained within another set. Boolean algebra *(X, ∪, ∩)* is a mathematical structure with a set *X*, union *(∪*), and intersection (*∩*), satisfying specific axioms for operations.

In this paper, we use expressions like *A ⊆ X* to indicate that *A* is a subset of *X, A ∪ B* to represent the union of two subsets *A* and *B* (both of which are subsets of *X*), and *A = ∅* to signify an empty set. Specifically, *A ∩ B* denotes the intersection of subsets *A* and *B*. Similarly, *A \ B* represents the difference between subsets *A* and *B*. The powerset of a set *A*, denoted as *2A*, is the set of all possible subsets of *A*, including the empty set and *A* itself.

Additionally, we use the following notation from graph theory: *V* represents the set of vertices in a graph, *E* represents the set of edges in a graph, and *G = (V, E)* denotes a graph *G* defined by its vertices *V* and edges *E.*

**2.1 Quasi-Ultrafilter on Boolean Algebras**First, we explain the concept of a Quasi-Ultrafilter on Boolean Algebras. Reference [1] introduces the Quasi-Ultrafilter and provides an axiomatic analysis of incomplete social judgments. Incomplete social judgments occur when people's preferences or opinions are only partially known or not fully specified. This can happen in contexts like voting, decision-making, and social choice theory. When individuals or groups do not provide a complete ranking of all options, it creates gaps in the collective assessment. Managing these incomplete judgments is essential for fair and representative decision-making and may involve techniques to infer or address the missing information. The Quasi-Ultrafilter is crucial to the proofs presented in this work.

This concept is illustrated using a Boolean algebra *(X, ∪, ∩)*. Although a Quasi-Ultrafilter shares many properties with an ultrafilter, it differs in property (QB1). The importance of the Quasi-Ultrafilter is underscored by its frequent mention in various related studies (e.g., [1-8]). The definition of a Quasi-Ultrafilter on Boolean Algebras is provided below.

**Definition 1:** In a Boolean algebra *(X,∪,∩)*, a set family *Q ⊆ 2X* satisfying the following conditions is called a Quasi-filter on the carrier set *X*.

(QB1) A*⊆ X*, B*⊆ X , A∉ Q , B ∉ Q* ⇒ *A ∪ B ∉ Q,*

(QB2) *A ∈ Q, A ⊆ B ⊆ X* ⇒ *B ∈ Q,*

(QB3) *∅* is not belong to *Q*.

(QB4) *∀A ⊆ X, either A ∈ Q or X / A ∈ Q.*

**2.2 Symmetric Submodular Function and Connectivity System**
The definition of a symmetric submodular function is presented below. This concept is extensively used and discussed in numerous scholarly articles.

**Definition 2:** Let *X* be a finite set. A function *f: X → ℕ* is called symmetric submodular if it satisfies the following conditions:
*· ∀A⊆X, f(A) = f(X\A).
· ∀A, B⊆X, f(A) + f(B) ≥ f(A∩B) + f(A∪B).*

A symmetric submodular function is a set function that is both submodular and symmetric, meaning it exhibits diminishing returns and remains invariant under the interchange of elements. While symmetric submodular functions are generally defined over real numbers, this paper specifically considers those restricted to natural numbers.In this short paper, a pair *(X, f)* of a finite set *X* and a symmetric submodular function *f* is called a connectivity system. This concept is commonly used in discussions of graph width parameters, such as branch-width and tree-width, to analyze graph structures (e.g., [9-21]).
The following is an example illustrating the concept of a symmetric submodular function.

**Example 3:** Consider a simple undirected graph *G = (V, E)* where *V* is the set of vertices and *E* is the set of edges. Let *V = {1, 2, 3, 4}* and the edges *E = {(1, 2), (2, 3), (3, 4), (4, 1)}*, forming a cycle.
Define a function f: 2V → ℕ as follows:
*f(A) = |E(A, V \ A)|*
where *E(A, V \ A)* is the set of edges with one endpoint in *A* and the other endpoint in *V \ A.*Verification of Symmetric Submodularity:
**Symmetry:** For any subset *A ⊆ V: f(A) = |E(A, V \ A)|.* Since *E(A, V \ A) = E(V \ A, A),* we have: *f(A) = f(V \ A).* For example, if *A = {1, 2}*, then *V \ A = {3, 4}* and *E(A, V \ A) = {(2, 3), (4, 1)}. Thus, f(A) = 2 and f(V \ A) = 2.*S**ubmodularity:** For any subsets *A, B ⊆ V: f(A) + f(B) ≥ f(A ∩ B) + f(A ∪ B).* Let's take *A = {1, 2}* and *B = {2, 3}. A ∩ B = {2}.A ∪ B = {1, 2, 3}.f(A) = |E({1, 2}, {3, 4})| = 2.f(B) = |E({2, 3}, {1, 4})| = 2.f(A ∩ B) = |E({2}, {1, 3, 4})| = 1.f(A ∪ B) = |E({1, 2, 3}, {4})| = 1.* Thus: *f(A) + f(B) = 2 + 2 = 4 f(A ∩ B) + f(A ∪ B) = 1 + 1 = 2*Since *4 ≥ 2*, the submodularity condition is satisfied.
It is known that a symmetric submodular function *f* satisfies the following properties:
**Lemma 4:** Let *X* be a finite set. A symmetric submodular function *f* satisfies:
*1. ∀A⊆X, f(A) ≥ f(∅) = f(X) = 0.
2. ∀A, B⊆X, f(A) + f(B) ≥ f(A\B) + f(B\A).***Proof.** By the definition of a symmetric submodular function, we have: *f(A) = f(X \ A) for all A ⊆ X.*For the empty set, *f(∅) = f(X)* because *∅* is the complement of *X.*Given that *f* is a symmetric submodular function and assuming *f i*s non-negative (a common assumption in submodular functions): *f(∅) = 0* and since *f* is non-negative*, f(A) ≥ 0 for all A ⊆ X.*Therefore, *f(A) ≥ f(∅) = f(X) = 0 for all A ⊆ X.*By the submodularity property, *for all A, B ⊆ X, f(A) + f(B) ≥ f(A ∩ B) + f(A ∪ B).*Using the symmetric property of *f:
f(A ∩ B) = f(X \ (A ∩ B)) = f((X \ A) ∪ (X \ B)).
f(A ∪ B) = f(X \ (A ∪ B)) = f((X \ A) ∩ (X \ B)).*Recognizing that *A \ B* is the same as *A ∩ (X \ B)* and *B \ A* is the same as *B ∩ (X \ A), f(A \ B) = f(A ∩ (X \ B)) and f(B \ A) = f(B ∩ (X \ A)).* By applying submodularity to *A* and *X \ B,* and *B* and *X \ A,* we get:
*f(A) + f(B) ≥ f(A ∩ (X \ B)) + f((X \ A) ∩ B).* Simplifying the intersections, we get *f(A) + f(B) ≥ f(A \ B) + f(B \ A).* Therefore, we have shown that a symmetric submodular function *f* satisfies both properties stated in Lemma 4. ■In this short paper, we use the notation *f* for a symmetric submodular function, a finite set *X*, and a natural number *k*. A set A is *k*-efficient if *f(A)≤k*. Unless otherwise specified, in this paper, the underlying set *X* is assumed to be a non-empty finite set.

**2.4　Branch-decomposition of a connectivity system**
In graph theory, branch width is a crucial graph width parameter. It involves a branch decomposition where the leaves of the decomposition correspond to the graph's edges. Each edge is associated with a value from a symmetric submodular function, which measures connectivity between edges. Branch width significantly broadens the application of symmetric submodular functions to graphs. The definition of branch decomposition is provided below. Due to its importance, branch decomposition has been extensively researched [9-23].
Definition of branch decomposition is below. Note that bijection is a one-to-one correspondence between two sets and a ternary tree is a tree with each node having three children.

**Definition 5:** Let *(X, f)* be a connectivity system. The pair *(T, μ)* is a branch decomposition tree of *(X, f)* if *T* is a ternary tree such that *|L(T)| = |X|* and μ is a bijection from *L(T)* to *X*, where *L(T)* denotes the leaves in *T*. For each *e ∈ E(T)*, we define *bw(T, μ, e)* as *f(∪v∈L(T1) μ(v))*, where *T1* is a tree obtained by removing *e* from *T* (taking into account the symmetry property of f). The width of *(T, μ)* is defined as the maximum value among *bw(T, μ, e)* for all *e ∈ E(T)*. The branch-width of *X*, denoted by *bw(X)*, is defined as the minimum width among all possible branch decomposition trees of *X*.

**3.　 Quasi-Ultrafilter on Connectivity System**
We introduce the Quasi-Ultrafilter on the Connectivity System *(X,f)* as an extension of the Quasi-Ultrafilter on Boolean Algebras.First, we introduce the definition of a Quasi-filter.
 **Definition 6[22]:** Let *X* be a finite set and *f* be a symmetric submodular function. In a connectivity system, the set family *Q ⊆2X* is called a Quasi filter of order *k+1* if the following axioms hold true:(Q0) *∀A ∈ Q, f(A) ≤ k*(Q1) *A ⊆ X, B⊆ X , A∉ Q , B ∉ Q ⇒ A ∪ B ∉ Q*(Q2) *A ∈Q, A ⊆ B ⊆ X*, f(B) ≤ k ⇒ *B ∈ Q*(Q3) *∅* is not belong to *Q*.

A Quasi-Ultrafilter is a quasi-filter that satisfies the following condition (Q4). The Quasi-Ultrafilter on a Connectivity System has a dual relationship with branch-width, a graph width parameter [22].
(Q4) *∀A ⊆ X, f(A)* ≤ k ⇒ *either A ∈ Q or X / A ∈ Q.*
A quasi-filter is maximal if *Q* cannot be extended by including any additional subsets while still satisfying these axioms. In mathematical research, the maximality of filters is often a topic of discussion.And Quasi-filter is non-principal if the Quasi-filter satisfies following axiom:(Q5) *A ∉Q* for all *A ⊆ X* with *|A| = 1.*
Non-principal refers to a filter or ideal that does not contain any singletons (i.e., sets with exactly one element). It is not generated by any finite set.

The following duality theorem is known for branch-width and quasi ultrafilter.

**Theorem 7:** Let *X* be a finite set and *f* be a symmetric submodular function. Branch-width of the connectivity system *(X, f)* is at most *k* if and only if no (non-principal) Quasi Ultrafilter of order *k+1* exists.

The main theorem of this paper is presented as follows.
**Theorem 8:** Let *X* be a finite set and *f* be a symmetric submodular function. In a connectivity system, maximal quasi-filter *Q* of order *k+1* satisfies axiom (Q4).(Q4) *∀A ⊆ X, f(A)* ≤ k ⇒ *either A ∈ Q or X / A ∈ Q.***Proof.** To prove Theorem 6 by contradiction, we start by assuming that there exists a set *A ⊆ X* such that *f(A) ≤ k* and both *A ∉ Q* and *X \ A ∉ Q.*Given that *Q* is a maximal quasi-filter of order *k+1*, it satisfies the following axioms:(Q0) *∀A ∈ Q, f(A) ≤ k*

(Q1) *A ⊆ X, B ⊆ X, A ∉ Q, B ∉ Q ⇒ A ∪ B ∉ Q*

(Q2) *A ∈ Q, A ⊆ B ⊆ X, f(B) ≤ k ⇒ B ∈ Q*

(Q3) *∅ ∉ Q*

Assume *A* and *X \ A* are both not in *Q*, violating the statement to be proven.

Since *A ∉ Q* and *X \ A ∉ Q,* by axiom (Q1), we have:

*A ∪ (X \ A) = X ∉ Q*

But this directly contradicts axiom (Q2) and axiom (Q0), because if *X ∉ Q,* it would imply *f(X) > k.* However, since *X* is the whole set and by the property of symmetric submodular functions, we know that *f(X) = f(∅)*. From Lemma 4, *f(∅) = f(X).* Thus, by axiom (Q0), *f(∅) ≤ k.*

The contradiction implies that our initial assumption is false. Therefore, for any *A ⊆ X* with *f(A) ≤ k*, either *A ∈ Q* or *X \ A ∈ Q.*

Thus, we have shown that a maximal quasi-filter *Q* of order *k+1* satisfies the axiom (Q4). Hence, Theorem 8 is proved. ■

**4. Quasi-ideal on Connectivity System**

We discussed about a quasi–ideal. A quasi-ideal can be defined as the complement of a quasi-filter. This concept is analogous to the relationship between ideals and filters in order theory.
Let's define the quasi-ideal with the following properties, which mirror the axioms of a quasi-filter but with the roles of inclusion and complementation interchanged.

**Definition 9:** Let *X* be a finite set and *f* be a symmetric submodular function. In a connectivity system, the set family *I ⊆ 2X* is called a quasi-ideal of order k+1 if the following axioms hold true:
(QI0) *∀A ∈ I, f(A) ≤ k.*(QI1) *A ⊆ X, B ⊆ X, A ∉ I, B ∉ I ⇒ A ∩ B ∉ I.*(QI2) *A ∈ I, B ⊆ A ⊆ X, f(B) ≤ k ⇒ B ∈ I.*
(QI3) *X ∈ I.*

A quasi-ideal *I* is maximal if *I* cannot be extended by including any additional subsets while still satisfying these axioms. **Theorem 10:** Let *X* be a finite set and *f* be a symmetric submodular function. In a connectivity system, maximal quasi-ideal *I* of order *k+1* satisfies axiom (QI4).(QI4) *∀A ⊆ X, f(A)* ≤ k ⇒ *either A ∈ I or X / A ∈I.***Proof.** The proof can be done in the same manner as Theorem 8. ■

**5. Weak ultrafilter** **on a connectivity system**
We will discuss the concept of a Weak Ultrafilter. The definition of a Weak Ultrafilter on a connectivity system is provided below. Also we will discuss about maximal weak ideal on a connectivity system.

**Definition 11[23]:** Let *X* be a finite set and *f* be a symmetric submodular function. In a connectivity system, the set family *W ⊆2X* is called a weak filter of order *k+1* if the following axioms hold true:
(FB) For every *A ∈W, f(A) ≤ k.*
(FH) If *A, B ⊆ X*, *f(B) ≤ k,* A is a proper subset of *B* and *A* belongs to *W*, then *B* belongs to *W*.
(WIS) If *A* belongs to *W,* B belongs to *W and f(A ∩ B) ≤ k*, then *A ∩ B ≠ ∅*.
(FW) *∅* does not belong to *W.*A Weak Ultrafilter is a weak filter that satisfies the following condition (FE). The Weak Ultrafilter on a Connectivity System has a dual relationship with branch-width, a graph width parameter [23].

(FE) If *∀A ⊆ X* and  *f(A) ≤ k, then* either *A ∈ W* or *X / A ∈ W*.
A Weak filter is maximal if *Q* cannot be extended by including any additional subsets while still satisfying these axioms. And Weak filter is non-principal if the Weak filter satisfies following axiom:(FS) *A ∉Q* for all *A ⊆ X* with *|A| = 1.*

We will verify whether the following can be proven.
**Theorem 12:** Let X be a finite set and f be a symmetric submodular function. In a connectivity system (X, f), every maximal weak-filter W of order k+1 satisfies the axiom (FE):

(FE) For all A subset of X with f(A) *≤ k*, either A is in W or (X \ A) is in W.

**Proof :**

We prove this by contradiction. Suppose there exists a set A subset of X such that f(A) <= k

but neither A nor X \ A is in the weak-filter W. Recall that W is maximal, so we cannot add

A or X \ A without violating at least one of the weak-filter axioms:

(1) (FB) For every S in W, f(S) *≤ k*.

(2) (FH) If S, T subset of X, f(T) *≤ k*, S is a proper subset of T, and S is in W, then T is in W.

(3) (WIS) If S and T are in W and f(S ∩ T) *≤ k*, then S ∩ T is not empty.

(4) (FW) The empty set is not in W.

Since f(A) *≤ k*, adding A to W does not violate (FB) or (FH). The only potential issue is (WIS):

if we add A to W, there must exist some B in W such that f(A ∩ B) *≤ k* but A ∩ B = ∅, which

would contradict (WIS). Consequently, there must be a set B in W disjoint from A.

Similarly, we cannot add X \ A to W for the same reason: there must be a set B' in W disjoint

from X \ A. Thus, B subset of (X \ A) and B' subset of A. This implies B and B' are disjoint,

so B ∩ B' = ∅. By symmetry and non-negativity, we have f(B ∩ B') = f(∅) = 0 *≤ k*.

But both B and B' are in W, which contradicts (WIS), since (WIS) requires B ∩ B' to be non-empty.

This contradiction shows our assumption was false. Hence, for every A with f(A) *≤ k*,

either A is in W or (X \ A) is in W, completing the proof of Theorem 12.

**6. Conclusion and Future tasks of this paper**

This section presents the conclusions of this paper and outlines future perspectives.

**6.1. Conclusion of this paper**

In this paper, we discussed Quasi-Ultrafilters and Weak-Ultrafilters on Connectivity Systems.

 **6.2. Future tasks: Superfilter on a connectivity system**

We will consider about superfilter on a connectivity system. Superfilters are generalizations of ultrafilters and play a crucial role in Ramsey-theoretic theorems such as van der Waerden’s Theorem. They were first identified in Berge’s 1959 monograph under the name "grill" and have also been referred to as "coideal" in other works. Additionally, superfilters are related to Banakh and Zdomskyy’s concepts of semifilters and unsplit semifilters [24-27].

The definition of a superfilter is as follows.

**Definition 13 [27]:** Let *n* integer. For a set *S,*  *[S]n = {F ⊆ S: |F| = n}*, and *[S]∞* is the family of infinite subsets of *S.* A nonempty family *S ⊆ [N] ∞* is a superfilter if for all *A, B ⊆ ℕ:*

(S0) If *A ∈ S* and *B ⊇ A,* then *B ∈ S.*

(S1) If *A ∪ B ∈ S*, then *A ∈ S* or *B ∈ S.*

The definition of a superfilter, incorporating the condition of symmetric submodularity within a finite connectivity system, is as follows. In the future, we intend to investigate how this definition relates to graph width parameters.

**Definition 14:** Let *X* be a finite set and *f* be a symmetric submodular function. In a connectivity system,

the set family *S ⊆ 2X* is called a superfilter of order *k+1* if the following axioms hold true:

(SF0) For every *A ∈S, f(A) ≤ k.*

(SF1) *A ∈ S*, *f(B) ≤ k,* and *B ⊇ A,* then *B ∈ S.*

(SF2) If *A ∪ B ∈ S, f(A)* *≤ k, and f(B) ≤ k,* then *A ∈ S* or *B ∈ S.*
We will also consider semifilters on a connectivity system [28]. In the future, we intend to investigate how semifilters relate to graph width parameters.

**6.3. Future tasks: Quasi-Ultrafilter on hyperstructures and hyperlattices**

We aim to delve into algebraic hyperstructures and hyperlattices [50–55]. Algebraic hyperstructures extend traditional algebraic frameworks by allowing the composition of two elements to produce a set rather than a single element. In a classical lattice, each pair of elements has a unique least upper bound (join) and a unique greatest lower bound (meet). A hyperlattice generalizes this idea by permitting joins and meets to consist of sets of elements instead of individual elements.

Building on this foundation, we intend to explore superhyperstructures [84-86] and superhyperlattices [87], focusing on properties such as Quasi-Ultrafilters and their mathematical characteristics when symmetric submodular conditions are applied. These extensions promise to reveal new insights into their structure and behavior.

Additionally, prior research has utilized concepts such as hyperideals and hyperfilters within these frameworks [56–58]. We aim to further investigate the connections between these notions and graph parameters, seeking to uncover deeper relationships and applications.

**6.4. Future tasks: Uncertain Connectivity System**Various set concepts have been developed to address uncertainty, including Fuzzy Sets[59-61], Intuitionistic Fuzzy Sets [62-64], Vague Sets [71-73], Neutrosophic Sets [65-70], Soft Sets [81-83], and Plithogenic Sets [74-80]. In the future, we aim to extend these concepts to frameworks such as Connectivity Systems, Quasi-Ultrafilters, and Weak-Ultrafilters, and to explore their properties in depth.

**Data Availability**

This research is purely theoretical, involving no data collection or analysis. We encourage future researchers to pursue empirical investigations to further develop and validate the concepts introduced here.

**Conflict of Interest Statement**

The author declares no conflicts of interest.

**Ethical Approval**

As this research is entirely theoretical in nature and does not involve human participants or animal subjects, no ethical approval is required.

**Disclaimer**

This work presents theoretical concepts that have not yet undergone practical testing or validation. Future researchers are encouraged to apply and assess these ideas in empirical contexts. While every effort has been made to ensure accuracy and appropriate referencing, unintentional errors or omissions may still exist. Readers are advised to verify referenced materials on their own. The views and conclusions expressed here are the authors’ own and do not necessarily reflect those of their affiliated organizations.

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