Mellin-SBA method for solving nonliear partial differential equations

Abstract :
In this paper, we present a new numerical method for solving nonlinear differential and partial differential equations. This method is the Mellin-SBA method. In the first part, we describe the SBA and Mellin-SBA methods. The second part is devoted to solving two nonlinear problems by the Mellin-SBA method.

Key words : Partial differential equations, Mellin transformation, SBA method. Mellin-SBA Method.

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1. Introduction

Solving nonlinear partial differential equations has always been a challenge for mathematicians. Several numerical methods have been presented. The method we present in this paper is a coupling between the Mellin transformation and the SBA method.

2.Mellin transformation and resolution methods

2.1 Mellin transformation :

The Mellin transform is closely related to the Fourier and Laplace transforms. It can be successfully applied to the resolution of a class of plane harmonic problems in an areal domain, to problems of elasticity theory..

2.2 Definition [6] [7]

Let f be a causal function. The Mellin transform of f is defined by :

$$F(s) = \mathcal{M}(f(t)) = \int_{t=0}^{\infty} f(t)t^{s-1}dt \tag{1}$$

the inverse Mellin transformation is given by the formula [7] :

$$f(t) = \mathcal{M}^{-1}F(s) = \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} F(s)t^s ds \tag{2}$$

we also note

$$\mathcal{M}^{-1}[f(t); s] \tag{3}$$

properties [7] [6]

Let f be a numerical function and $\mathcal{M}(f(t) = F(s)$ its Mellin transformation. We have :

a. Linearity :

$$\mathcal{M}[\alpha f(t) + \beta g(t)] = \alpha\mathcal{M}(f(t)) + \beta\mathcal{M}(g(t)) \tag{4}$$

$$\begin{aligned}\mathcal{M}[\alpha f(t) + \beta g(t)] &= \int_{t=0}^{\infty}(\alpha f(t) + \beta g(t))t^{s-1}dt \\ &= \int_{t=0}^{\infty}(\alpha f(t)dt + \int_{t=0}^{\infty}(\beta g(t)dt \\ &= \alpha\mathcal{M}f(t) + \beta\mathcal{M}g(t)\end{aligned}$$

b.Convolution Product :

Let f and g be two causal functions. We have :

$$\mathcal{M}(f(t) * g(t)) = \mathcal{M}(f(s))(\mathcal{M}(g(s)) \tag{5}$$

c. Mellin transformation of $f(at)$.
we have

$$\mathcal{M}(f(at)) = \frac{F(s)}{a^s} \tag{6}$$

d.Mellin transformation of $t^\alpha f(t)$

$$\mathcal{M}(t^\alpha f(t)) = F(a+s) \tag{7}$$

e. Mellin transformation of the first derivative

$$\begin{aligned}\mathcal{M}\{f'(t)\} &= \int_0^{+\infty} t^{s-1} f'(t)dt \\ &= [t^{s-1}f(t)]_0^\infty - \int_0^\infty (s-1)t^{s-2}f(t)dt \\ \mathcal{M}\{f'(t)\} &= [t^{s-1}f(t)]_0^\infty - (s-1)F(s-1)\end{aligned}$$

f.Mellin transformation of the n order derivative

$$\forall n \in \mathbb{N}; \mathcal{M}(f^{(n)}(t))(s) = (-1)^n \frac{\Gamma(s)}{\Gamma(s-n)} \mathcal{M}(f)(s-n) + \sum_{k=0}^{n-1} (-1)^k \frac{\Gamma(s)}{\Gamma(s-k)} [t^{s-k-1} f^{(n-k-1)}(t)]_0^\infty \tag{8}$$

3.Resolution methods
3.1 Introduction
The main purpose of this chapter is to recall and describe the principles of the SBA method and the Mellin-SBA method.
3.2 SBA Method [2] [5] [8] [9] [1] [3] [4]
Consider the following functional equation :

$$\begin{cases} u_t(x,t) = R(u(x,t)) + N(u(x,t)), 0 < t < T \\ u(x,0) = f \end{cases} \tag{9}$$

In a suitably chosen space V, with R a linear differential operator, N the nonlinear term, u the unknown function such that $u_t = \frac{\partial u}{\partial t}$.
let's ask $L_t = \frac{\partial}{\partial t}(\cdot)$ and $L_t^{-1}(\cdot) = \int_0^t (\cdot) ds$.
Applying L_t^{-1} to the equation, we obtain :

$$u(x,t) = u(x,0) + L_t^{-1} R(u(x,t)) + L_t^{-1} N(u(x,t)) \tag{10}$$

Using the idea of the method of successive approximations, for any non-zero natural integer k, we obtain :

$$u^k(x,t) = u^k(x,0) + L_t^{-1} R(u^k(x,t)) + L_t^{-1} N(u^{k-1}(x,t)) \tag{11}$$

We seek the solution $u^k(x,t)$ in the form :

$$u^k(x,t) = \sum_{k=0}^{+\infty} u_n^k(x,t) \tag{12}$$

By injecting 12 into 11 we obtain by identification the SBA algorithm :

$$\begin{cases} u_0^k(x,t) & = u^k(x,0) + L_t^{-1}N(u^{k-1}(x,t)), 0 < t < T \\ u_n^k(x,t) & = L_t^{-1}R(u_{n-1}^k(x,t)); k = 1,2... \end{cases} \tag{13}$$

The resolution of the scheme (13) by the method of successive approximations, consists in determining at each iteration (k=1,2,...) approximate solutions $u^1(x,t)$, $u^2(x,t)$,...
The solution at each step is :

$$u^k(x,t) = \sum_{n=0}^{+\infty} u_n^k(x,t) \tag{14}$$

For the calculation of u^1, the choice of $u^0(x,t)$ is very important. According to the Picard principle, we choose $u^0(x,t)$ such that $N(u^0(x,t)) = 0$. The sought solution $u(x,t)$ of the problem (9) is obtained by :

$$u(x,t) = \lim_{k\to+\infty} u^k(x,t) \tag{15}$$

3.3 Convergence of the SBA method
The problems of the theoretical model are considered :

$$(P) \begin{cases} u_t = Lu(t) + N(u(t)),\ t_0 < t < t_1 \\ u(t_0) = u_0 \end{cases} \tag{16}$$

and the approximate model :

$$(P^k) \begin{cases} u_t^k = Lu^k + N(u^{k-1},t),\ t_0 < t < t_1 \\ u^k(t_0) = u_0,\ k = 1,2,3,... \end{cases} \tag{17}$$

under the following hypotheses :
(H1) : N is a Lypschitzian of constant k on a ball $B(u_0, r)$ with center u_0 and radius r.
(H2) : L continus on $C(t_0, t_1)$
Proposition : [9] [5]
if $u^0 \epsilon B(u_0, r)$ (that's to say : $||u^0 - u_0||_{c(t_0,t_1)} \max\limits_{t_0<t<t_1}$) then there exists δ >0 such that $t_0+\delta < t_1$ and the algorithm (P^k) admits a unique solution u^k such that $\max\limits_{t_0<t<t_1} |u^k(t) - u_0| \leq r$ for all k.
Preuve : for proof see [9] [5]
3.4 Mellin-SBA method
The Mellin-SBA method is a method that consists of combining the SBA method with the Mellin transformation.
We consider the problem :

$$\begin{cases} u_t(x,t) = R(u(x,t)) + N(u(x,t)), 0 < t < T \\ u(x,0) = f \end{cases}$$

let us remember

$$\mathcal{M}\{u_t(x,t)\} = [t^{s-1}u(x,t)]_0^T - (s-1)\mathcal{M}[u(x,t); s-1]$$

We apply the Mellin transformation to (9) and we obtain :

$$[t^{s-1}u(x,t)]_0^T - (s-1)\mathcal{M}[u(x,t); s-1] = \mathcal{M}[R(u(x,t)); s] + \mathcal{M}[N(u(x,t)); s] \tag{18}$$

we obtain :

$$(s-1)\mathcal{M}[u(x,t);s-1]=T^{s-1}u(x,T)-\mathcal{M}[R(u(x,t));s]-\mathcal{M}[N(u(x,t));s] \quad (19)$$

we obtain

$$\mathcal{M}[u(x,t);s-1]=\frac{T^{s-1}}{(s-1)}u(x,T)-\frac{1}{(s-1)}\mathcal{M}[R(u(x,t));s]-\frac{1}{(s-1)}\mathcal{M}[N(u(x,t));s] \quad (20)$$

Applying the inverse Mellin transformation, we obtain :

$$t^{-1}u(x,t)=\mathcal{M}^{-1}[\frac{T^{s-1}}{(s-1)}u(x,T)]-\mathcal{M}^{-1}[\frac{1}{(s-1)}\mathcal{M}[R(u(x,t));s]]-\mathcal{M}^{-1}[\frac{1}{(s-1)}\mathcal{M}[N(u(x,t));s]]$$

We draw $u(x,t)$ and we obtain :

$$u(x,t)=t\mathcal{M}^{-1}[\frac{T^{s-1}}{(s-1)}u(x,T)]-t\mathcal{M}^{-1}[\frac{1}{(s-1)}\mathcal{M}[R(u(x,t));s]]-t\mathcal{M}^{-1}[\frac{1}{(s-1)}\mathcal{M}[N(u(x,t));s]]$$

we pose

$$\bar{N}(u(x,t))=-t\mathcal{M}^{-1}[\frac{1}{(s-1)}\mathcal{M}[N(u(x,t));s]] \quad (21)$$

we obtain

$$u(x,t)=t\mathcal{M}^{-1}[\frac{T^{s-1}}{(s-1)}u(x,T)]-t\mathcal{M}^{-1}[\frac{1}{(s-1)}\mathcal{M}[R(u(x,t));s]]+\bar{N}(u(x,t))$$

Using the principle of successive approximations for $k\geq 1$, we obtain :

$$u^k(x,t)=t\mathcal{M}^{-1}[\frac{T^{s-1}}{(s-1)}u(x,T)]-t\mathcal{M}^{-1}[\frac{1}{(s-1)}\mathcal{M}[R(u^k(x,t));s]]+\bar{N}(u^{k-1}(x,t)) \quad (22)$$

Applying the SBA algorithm (13) to (22), we obtain :

$$\begin{cases} u_0^k(x,t)=t\mathcal{M}^{-1}[\frac{T^{s-1}}{(s-1)}u(x,T)]+\bar{N}(u^{k-1}(x,t)) \\ u_n^k(x,t)=-t\mathcal{M}^{-1}[\frac{1}{(s-1)}\mathcal{M}[R(u_{n-1}^k(x,t));s]] \end{cases} \quad (23)$$

k^{th} **iteration** : Calculation of $u^k(x,t)$

$$\begin{cases} u_0^k(x,t)=t\mathcal{M}^{-1}[\frac{T^{s-1}}{(s-1)}u(x,T)]+\bar{N}(u^{k-1}(x,t)) \\ u_1^k(x,t)=-t\mathcal{M}^{-1}[\frac{1}{(s-1)}\mathcal{M}[R(u_0^k(x,t));s]] \\ \vdots \\ u_n^k(x,t)=-t\mathcal{M}^{-1}[\frac{1}{(s-1)}\mathcal{M}[R(u_{n-1}^k(x,t));s]] \end{cases} \quad (24)$$

The approximate solution of this kth iteration is :
$u^k(x,t)=\sum_{n=0}^{+\infty}u_n^k(x,t)$
The solution to the problem is given by :

$$u(x,t)=\lim_{k\to+\infty}u^k(x,t) \quad (25)$$

property
if

$$u^1(x,t) = u^2(x,t) \tag{26}$$

then

$$u(x,t) = u^1(x,t) = u^2(x,t) \tag{27}$$

4.Resolution of some partial differential equations by the Mellin-SBA method
The aim of this chapter is to apply the Mellin-SBA method to solving a partial differential equation.
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Example 1 : Nonlinear diffusion equation

$$\begin{cases} u_t(x,t) = \varepsilon u_{xx} + u^3 + (u_{xx})^3 \quad 0 \le t \le 1 \\ u(x,0) = sinx \\ u(x,1) = sinxe^{-\varepsilon} \end{cases} \tag{28}$$

we pose $L_t = \frac{\partial}{\partial}(\cdot)$ et $N(\cdot) = (\cdot)^3 + \frac{\partial^2}{\partial x^2}(\cdot)$.
The problem becomes :

$$\begin{cases} u_t(x,t) = \varepsilon u_{xx} + N(u) \\ u(x,0) = sinx \\ u(x,1) = sinxe^{-\varepsilon} \end{cases} \tag{29}$$

Applying the Mellin transformation, we obtain :

$$\mathcal{M}\{u_t(x,t))\} = [t^{s-1}u(x,t)]_0^{\infty} - (s-1)\mathcal{M}\{(u(x,t); s-1)\} \tag{30}$$

let

$$[t^{s-1}u(x,t)]_0^{\infty} - (s-1)[\mathcal{M}(u(x,t)); s-1] = \varepsilon \mathcal{M}(u_{xx}; s) + \mathcal{M}(N(u)) \tag{31}$$

Noting that $0 \le t \le 1$ we obtain :

$$[t^{s-1}u(x,t)]_0^{1} - (s-1)[\mathcal{M}(u(x,t)); s-1] = \varepsilon \mathcal{M}(u_{xx}; s) + \mathcal{M}(N(u)) \tag{32}$$

we pose $a(x) = [t^{s-1}u(x,t)]_0^1 = u(x,1) = sinxe^{-\varepsilon}$
we obtain :

$$\mathcal{M}\{(u(x,t); s-1)\} = \frac{-sinxe^{-\varepsilon}}{1-s} + \frac{1}{1-s}k\mathcal{M}(u_{xx}; s) + \frac{1}{1-s}\mathcal{M}(N(u)) \tag{33}$$

Let's apply the inverse transformation :

$$t^{-1}u(x,t) = sinxe^{-\varepsilon}t^{-1} + \varepsilon\mathcal{M}^{-1}[\frac{1}{1-s}\mathcal{M}(u_{xx}; s)] + \mathcal{M}^{-1}\frac{1}{1-s}\mathcal{M}(N(u)) \tag{34}$$

we obtain

$$u(x,t) = sinxe^{-\varepsilon} + t\varepsilon\mathcal{M}^{-1}[\frac{-1}{s-1}\mathcal{M}(u_{xx}; s)] + t\mathcal{M}^{-1}[\frac{1}{1-s}\mathcal{M}(N(u))] \tag{35}$$

Using the idea of the principle of successive approximations we obtain the canonical form SBA :

$$u^k(x,t) = sinxe^{-\varepsilon} + t\varepsilon\mathcal{M}^{-1}[\frac{-1}{s-1}\mathcal{M}(u^k_{xx}; s)] + t\mathcal{M}^{-1}[\frac{1}{1-s}\mathcal{M}(N(u^{k-1}))] \tag{36}$$

We obtain the SBA algorithm :

$$\begin{cases} u_0^k & = sinxe^{-\varepsilon} + t\mathcal{M}^{-1}[\frac{1}{1-s}\mathcal{M}(N(u^{k-1}))] \\ u_n^k(x,t) & = \varepsilon t\mathcal{M}^{-1}[\frac{-1}{s-1}\mathcal{M}[\frac{\partial^2 u_{n-1}^k}{\partial x^2}(x,t)]] \end{cases} \tag{37}$$

To simplify the writing, we designate by $T(\cdot)$, the operator defined by :
$T(u) = \mathcal{M}^{-1}[\frac{-1}{s-1}\mathcal{M}[\frac{\partial^2 u_{n-1}}{\partial x^2}(x,t)]]$
the algorithm becomes

$$\begin{cases} u_0^k & = sinxe^{-\varepsilon} + t\mathcal{M}^{-1}[\frac{1}{1-s}\mathcal{M}(N(u^{k-1}))] \\ u_n^k(x,t) & = \varepsilon tT(u^k) \end{cases} \tag{38}$$

first iteration :$k = 1$

$$\begin{cases} u_0^1 & = sinxe^{-\varepsilon} + t\mathcal{M}^{-1}[\frac{1}{1-s}\mathcal{M}(N(u^0))] \\ u_n^1(x,t) & = \varepsilon tT(u^1) \end{cases} \tag{39}$$

By the Picard principle, there exists u^0 such that $N(u^0) = 0$.
we obtain : $u_0^1 = sinxe^{-\varepsilon}$
We note that for : $u_0 = u_0^1 = sinxe^{-\varepsilon}$, on a :
$\frac{\partial^2 u_0^1}{\partial x^2}(x,t) = -u_0^1$ and that :

$$P(t^n u_0) = \mathcal{M}^{-1}[\frac{-1}{s-1}\mathcal{M}[t^n\frac{\partial^2 u_0}{\partial x^2}(x,t)]] \tag{40}$$

$$= \mathcal{M}^{-1}[\frac{1}{s-1}\mathcal{M}[t^n u_0 \tag{41}$$

$$= u_0\mathcal{M}^{-1}[\frac{1}{(s-1)(s+n)}] \tag{42}$$

$$= u_0\mathcal{M}^{-1}[\frac{\frac{-1}{n+1}}{s+n} + \frac{\frac{1}{n+1}}{s-1}] \tag{43}$$

$$P(t^n u_0) = u_0[-\frac{t^n}{n+1} + \frac{t^{-1}}{n+1}] \tag{44}$$

$$\tag{45}$$

we obtain that : :

$$tP(t^n u_0) = u_0[-\frac{t^{n+1}}{n+1} + \frac{1}{n+1}] \tag{46}$$

calculation of u_1^1 we obtain

$$u_1^1(x,t) = \varepsilon tp(u_0) = \varepsilon tp(t^0 u_0] = \varepsilon u_0(-t+1)$$
$$= -\varepsilon tu_0 + \varepsilon u_0$$
$$u_1^1(x,t) = -\varepsilon u_0(t-1)$$

Calculation of u_2^1

$$\begin{aligned}
u_2^1(x,t) &= \varepsilon t T(u_1^1) = -\varepsilon^2 t T(tu_0) + \varepsilon^2 t T(u_0) \\
&= \varepsilon^2 u_0(\frac{t^2}{2} - \frac{1}{2}) + \varepsilon^2 u_0(-t+1) \\
&= \frac{\varepsilon^2}{2} t^2 u_0 - t u_0 + \frac{1}{2} u_0 \\
u_2^1(x,t) &= u_0 \frac{\varepsilon^2 (t-1)^2}{2}
\end{aligned}$$

Calculation of u_3

$$\begin{aligned}
u_3^1(x,t) &= \varepsilon t T(u_2^1) = \frac{1}{2}\varepsilon^3 t T(t^2 u_0) - \varepsilon^3 t T(tu_0) + \frac{1}{2}\varepsilon^3 t T(u_0) \\
&= u_0 \varepsilon^3 \frac{1}{2}(-\frac{t^3}{3} + \frac{1}{3}) - u_0 \varepsilon^3 (-\frac{t^2}{2} + \frac{1}{2}) + \frac{1}{2} u_0 \varepsilon^3 (-t+1) \\
&= -u_0 \varepsilon^3 (\frac{t^3}{6} - \frac{1}{6} - \frac{t^2}{2} + \frac{1}{2} + \frac{t}{2} - \frac{1}{2}) \\
&= \frac{-\varepsilon^3}{6} t^3 u_0 + \frac{\varepsilon^3}{2} t^2 + \frac{k^3}{2} t u_0 - \frac{\varepsilon^3}{2} t u_0)) \\
&= -\frac{u_0 \varepsilon^3}{6}(t^3 - 3t^2 + 3t - 1)) \\
u_3^1(x,t) &= -\frac{u_0 \varepsilon^3 (t-1)^3}{3!}
\end{aligned}$$

In recurring ways, we obtain :

$$u_n^1 = u_0 \frac{[-\varepsilon(t-1)]^n}{n!} \tag{47}$$

indeed

$$u_n^1 = u_0 \frac{(\varepsilon - \varepsilon t)^n}{n!} = \frac{u_0 \varepsilon^n}{n!} \sum_{i=0}^{n} C_n^i (-t)^i \tag{48}$$

At order $(n+1)$ we have :

$$
\begin{aligned}
u_{n+1}^1 = \varepsilon t T(u_n^1) &= \varepsilon t \frac{u_0 \varepsilon^n}{n!} \sum_{i=0}^{n} C_n^i (-1)^i T(t^i u_0) \\
&= \varepsilon \frac{u_0 \varepsilon^n}{n!} \sum_{i=0}^{n} C_n^i (\varepsilon^{n-i})(-1)^i (\frac{t^{i+1}}{i+1} + \frac{1}{i+1}) \\
&= \varepsilon^{n+1} \frac{u_0}{n!} \sum_{i=0}^{n} \frac{C_n^i}{(i+1)} (-1)^i (1 - t^{i+1}) \\
&= \varepsilon^{n+1} \frac{u_0}{n!} \sum_{i=0}^{n} \frac{n!}{(n-i)!i!(i+1)} (-1)^{i+1} (t^{i+1} - 1) \\
&= \varepsilon^{n+1} \frac{u_0}{(n+1)!} \sum_{i=0}^{n} \frac{(n+1)!}{(n-i)!i!(i+1)} (-1)^{i+1} (t^{i+1} - 1) \\
&= \varepsilon^{n+1} \frac{u_0}{(n+1)!} \sum_{i=0}^{n} \frac{(i+1)! C_{n+1}^{i+1}}{i!(i+1)} (-1)^{i+1} (t^{i+1} - 1) \\
&= \varepsilon^{n+1} \frac{u_0}{(n+1)!} \sum_{i=0}^{n} C_{n+1}^{i+1} (-1)^{i+1} (t^{i+1} - 1) \\
&= \varepsilon^{n+1} \frac{u_0}{(n+1)!} \sum_{i=1}^{n+1} C_{n+1}^{i} (-1)^{i} (t^{i} - 1) \\
&= \varepsilon^{n+1} \frac{u_0}{(n+1)!} [\sum_{i=1}^{n+1} C_{n+1}^{i} (-1)^{i} t^{i} - \sum_{i=1}^{n+1} C_{n+1}^{i} (-1)^{i}] \\
&= \varepsilon^{n+1} \frac{u_0}{(n+1)!} [\sum_{i=1}^{n+1} C_{n+1}^{i} (-1)^{i} t^{i} + 1] \\
&= \varepsilon^{n+1} \frac{u_0}{(n+1)!} [\sum_{i=0}^{n+1} C_{n+1}^{i} (-1)^{i} t^{i}] \\
&= u_0 \frac{(\varepsilon - \varepsilon t)^{n+1}}{(n+1)!}
\end{aligned}
$$

Conclusion :

$$
u_n^1 = u_0 \frac{(\varepsilon - \varepsilon t)^n}{n!} \tag{49}
$$

$$
u^1(x,t) = \sum_{n=0}^{+\infty} u_n^1 = sinx e^{-\varepsilon t} \tag{50}
$$

algorithm at step $k = 2$:

$$
\begin{cases}
u_0^2 & = sinx e^{-\varepsilon} + t \mathcal{M}^{-1} [\frac{1}{1-s} \mathcal{M}(N(u^1))] \\
u_n^2(x,t) & = \varepsilon t T(u_n^2)
\end{cases} \tag{51}
$$

$$
\begin{aligned}
N(u^1) &= (u^1)^3 + [\frac{\partial^2}{\partial x^2}(u^1)]^3 \\
&= sin^3 x e^{-3\varepsilon t} - sin^3 x e^{-3\varepsilon t} = 0
\end{aligned}
$$

we obtain $u_0^2 = sinxe^{-\varepsilon}$
we calculate the u_i^2 for $i \in \{1, 2, ...\}$

$$\begin{cases} u_0^2 & = sinxe^{-\varepsilon} \\ u_1^2(x,t) & = -\varepsilon u_0(t-1) \\ u_2^2(x,t) & = u_0 \frac{\varepsilon^2(t-1)^2}{2} \\ u_3^2(x,t) & = -\frac{u_0\varepsilon^3(t-1)^3}{3!} \\ \vdots \\ u_n^2 & = u_0 \frac{(\varepsilon-\varepsilon t)^n}{n!} \frac{u_0\varepsilon^n}{n!} \sum_{i=0}^n C_n^i(-t)^i \end{cases} \tag{52}$$

$$u^2(x,t) = \sum_{n=0}^{+\infty} u_n^2 = sinxe^{-\varepsilon t} \tag{53}$$

$$u^1(x,t) = u^2(x,t) \tag{54}$$

We deduce the general solution to the problem :

$$u(x,t) = sinxe^{-\varepsilon t} \tag{55}$$

Example 2 : nonlinear evolution equation

$$\begin{cases} \frac{\partial u(t,x,y)}{\partial t} = \beta u(t,x,y) + (\Delta u(t,x,y))^{2m} - (\lambda\omega^2 e^{\beta t} - \omega^2 u(t,x,y))^{2m} \\ u(0,x,y) = \lambda + cos(\omega x) + sin(\omega y); \omega > 0 \\ u(1,x,y) = (\lambda + cos(\omega x) + sin(\omega y))e^{\beta}; \omega > 0 \end{cases} \tag{56}$$

with $\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$
we pose $L_t = \frac{\partial}{\partial_t}(\cdot)$ et $N(\cdot) = [\Delta(\cdot)]^{2m} - [\lambda\omega^2 e^{\beta t} - \omega^2(\cdot)]^{2m}$.
the equation becomes

$$L_t u(t,x,y) = \beta u(t,x,y) + N(u(t,x,y)) \tag{57}$$

Let us apply the Mellin transformation to the equation. We obtain :

$$[t^{s-1}u(t,x,y)]_0^\infty - (s-1)\mathcal{M}\{(u(t,x,y); s-1)\} = \beta\mathcal{M}(u(t,x,y)) + \mathcal{M}(N(u(t,x,y))) \tag{58}$$

we obtain :

$$\mathcal{M}\{(u(t,x,y); s-1)\} = -\frac{1}{1-s}[t^{s-1}u(t,x,y)]_0^\infty + \frac{\beta}{1-s}\mathcal{M}(u(t,x,y)) + \frac{1}{1-s}\mathcal{M}(N(u(t,x,y))) \tag{59}$$

Noting that $0 < t < 1$, we find

$$\mathcal{M}\{(u(t,x,y); s-1)\} = -\frac{1}{1-s}[t^{s-1}u(t,x,y)]_0^1 + \frac{\beta}{1-s}\mathcal{M}(u(t,x,y)) + \frac{1}{1-s}\mathcal{M}(N(u(t,x,y))) \tag{60}$$

We finally obtain :

$$\mathcal{M}\{(u(t,x,y); s-1)\} = -\frac{1}{1-s}u(1,x,y)] + \frac{\beta}{1-s}\mathcal{M}(u(t,x,y)) + \frac{1}{1-s}\mathcal{M}(N(u(t,x,y))) \tag{61}$$

Let us apply the inverse transformation to the equation :

$$t^{-1}u(t,x,y) = t^{-1}u(1,x,t) + \mathcal{M}^{-1}[\frac{\beta}{1-s}\mathcal{M}(u(t,x,y))] + \mathcal{M}^{-1}[\frac{1}{1-s}\mathcal{M}(N(u(t,x,y)))] \tag{62}$$

we obtain :

$$u(t,x,y) = u(1,x,t) + t\mathcal{M}^{-1}[\frac{\beta}{1-s}\mathcal{M}(u(t,x,y))] + t\mathcal{M}^{-1}[\frac{1}{1-s}\mathcal{M}(N(u(t,x,y)))] \tag{63}$$

let's ask

$$N(\bar{u}(t,x,y)) = t\mathcal{M}^{-1}[\frac{1}{1-s}\mathcal{M}(N(u(t,x,y)))] \tag{64}$$

we obtain :

$$u(t,x,y) = u(1,x,t) + t\mathcal{M}^{-1}[\frac{\beta}{1-s}\mathcal{M}(u(t,x,y))] + N(\bar{u}(t,x,y)) \tag{65}$$

We use the idea of the principle of successive approximations and we obtain for any non-zero natural integer k :

$$u^k(t,x,y) = u(1,x,t) + t\mathcal{M}^{-1}[\frac{\beta}{1-s}\mathcal{M}(u^k(t,x,y))] + N(\bar{u}^k(t,x,y)) \tag{66}$$

We deduce the canonical form SBA below :

$$\begin{cases} u_0^k(t,x,y) = u(1,x,t) + N(\bar{u}^{k-1}(t,x,y)) \\ u_n^k(t,x,y) = t\mathcal{M}^{-1}[\frac{\beta}{1-s}\mathcal{M}(u_{n-1}^k(t,x,y))] \end{cases} \tag{67}$$

firts iteration : $k=1$

$$\begin{cases} u_0^1(t,x,y) = u(1,x,t) + N(\bar{u}^0(t,x,y)) \\ u_n^1(t,x,y) = t\mathcal{M}^{-1}[\frac{\beta}{1-s}\mathcal{M}(u_{n-1}^1(t,x,y))] \end{cases} \tag{68}$$

According to Picard's principle, there exists u^0 such that $Nu^0 = 0$. We obtain :

$$\begin{cases} u_0^1(t,x,y) = (\lambda + cos(\omega x) + sin(\omega y))e^{\beta} \\ u_n^1(t,x,y) = t\mathcal{M}^{-1}[\frac{\beta}{1-s}\mathcal{M}(u_{n-1}^1(t,x,y))] \end{cases} \tag{69}$$

calculation of u_n^1

$$\begin{aligned} u_1^1(t,x,y) &= t\mathcal{M}^{-1}[\frac{\beta}{1-s}\mathcal{M}(u_0^1(t,x,y))] \\ &= t(\lambda + cos(\omega x) + sin(\omega y))e^{\beta}\mathcal{M}^{-1}[\frac{\beta}{s(1-s)} \\ &= t\beta(\lambda + cos(\omega x) + sin(\omega y))e^{\beta}\mathcal{M}^{-1}[\frac{1}{s} + \frac{1}{1-s}] \\ &= t\beta(\lambda + cos(\omega x) + sin(\omega y))e^{\beta}[1 - \frac{1}{t} \\ u_1^1(t,x,y) &= (\lambda + cos(\omega x) + sin(\omega y))e^{\beta}\beta(t-1) \end{aligned}$$

$$
\begin{aligned}
u_2^1(t,x,y) &= t\mathcal{M}^{-1}[\frac{\beta}{1-s}\mathcal{M}(u_1^1(t,x,y))] \\
&= t\beta^2(\lambda + cos(\omega x) + sin(\omega y))e^{\beta}\mathcal{M}^{-1}[\frac{1}{1-s}\mathcal{M}(t-1)) \\
&= t\beta^2(\lambda + cos(\omega x) + sin(\omega y))e^{\beta}\mathcal{M}^{-1}[-\frac{1}{s(1-s)} + \frac{1}{(s+1)(1-s)} \\
&= t\beta^2(\lambda + cos(\omega x) + sin(\omega y))e^{\beta}\mathcal{M}^{-1}[-\frac{1}{s} - \frac{1}{1-s} + \frac{\frac{1}{2}}{s+1} + \frac{\frac{1}{2}}{1-s} \\
&= t\beta^2(\lambda + cos(\omega x) + sin(\omega y))e^{\beta}\mathcal{M}^{-1}[-\frac{1}{s} + \frac{\frac{1}{2}}{s+1} + \frac{\frac{1}{2}}{s-1} \\
&= \beta^2(\lambda + cos(\omega x) + sin(\omega y))e^{\beta}[-t + \frac{t^2}{2} + \frac{1}{2}] \\
u_2^1(t,x,y) &= (\lambda + cos(\omega x) + sin(\omega y))e^{\beta}\frac{(\beta(t-1))^2}{2}
\end{aligned}
$$

$$
\begin{aligned}
u_3^1(t,x,y) &= t\mathcal{M}^{-1}[\frac{\beta}{1-s}\mathcal{M}(u_2^1(t,x,y))] \\
&= t\beta^3(\lambda + cos(\omega x) + sin(\omega y))e^{\beta}\mathcal{M}^{-1}[\frac{1}{1-s}\mathcal{M}[-t + \frac{t^2}{2} + \frac{1}{2}]) \\
&= t\beta^3(\lambda + cos(\omega x) + sin(\omega y))e^{\beta}\mathcal{M}^{-1}[\frac{1}{1-s}[-\frac{1}{s+1} + \frac{1}{2(s+2)} + \frac{1}{2s}]) \\
&= t\beta^3(\lambda + cos(\omega x) + sin(\omega y))e^{\beta}\mathcal{M}^{-1}[\frac{1}{(s+1)(s-1)} - \frac{1}{2(s+2)(s-1)} - \frac{1}{2s(s-1)}
\end{aligned}
$$

we obtain :
$u_3^1(t,x,y) = t\beta^3(\lambda + cos(\omega x) + sin(\omega y))e^{\beta}\mathcal{M}^{-1}[\frac{\frac{1}{2}}{s-1} + \frac{-\frac{1}{2}}{s+1} - \frac{1}{2}(\frac{-\frac{1}{3}}{s+2} + \frac{\frac{1}{3}}{s-1}) - \frac{1}{2}(\frac{-1}{s} + \frac{1}{s-1})$.

either finally

$$
\begin{aligned}
u_3^1(t,x,y) &= t\beta^3(\lambda + cos(\omega x) + sin(\omega y))e^{\beta}(\frac{1}{2}t^{-1} - \frac{1}{2}t + \frac{1}{6}(t^2 - t^{-1}) + \frac{1}{2}(1 - t^{-1})) \\
&= \beta^3(\lambda + cos(\omega x) + sin(\omega y))e^{\beta}(\frac{3 - 3t^2 + t^3 - 1 + 3t - 3}{6}) \\
&= \beta^3(\lambda + cos(\omega x) + sin(\omega y))e^{\beta}(\frac{t^3 - 3t^2 + 3t - 1}{3!}) \\
u_3^1(t,x,y) &= (\lambda + cos(\omega x) + sin(\omega y))e^{\beta}\frac{[\beta(t-1)]^3}{3!}
\end{aligned}
$$

It has been shown repeatedly that

$$
u_n^1(x,t) = (\lambda + cos(\omega x) + sin(\omega y))e^{\beta}\frac{[\beta(t-1)]^n}{n!} \quad (70)
$$

$$
u^1(x,t) = \sum_{n=0}^{+\infty} u_n^1 = (\lambda + cos(\omega x) + sin(\omega y))e^{\beta}e^{(\beta(t-1))} = (\lambda + cos(\omega x) + sin(\omega y))e^{\beta t} \quad (71)
$$

Calculation of $u^2(t,x,y)$

$$\begin{cases} u_0^2(t,x,y) = u(1,x,t) + N(\bar{u}^1(t,x,y)) \\ u_n^2(t,x,y) = t\mathcal{M}^{-1}[\frac{\beta}{1-s}\mathcal{M}(u_{n-1}^2(t,x,y))] \end{cases} \tag{72}$$

$$N(\bar{u}^1(t,x,y)) = t\mathcal{M}^{-1}[\frac{1}{1-s}\mathcal{M}(N(u(t,x,y)))]$$

with

$$\begin{aligned} N(u^1(t,x,y))) &= [\Delta u^1(t,x,y))]^{2m} - [\lambda\omega^2 e^{\beta t} - \omega^2 u^1(t,x,y))]^{2m} \\ &= [(-\omega^2 cos(\omega x))e^{\beta t} - \omega^2 sin(\omega x))e^{\beta t}]^{2m} - [\lambda\omega^2 e^{\beta t} - \omega^2(\lambda + cos(\omega x) + sin(\omega y))e^{\beta t}]^{2m} \\ &= [(cos(\omega x) + sin(\omega y))e^{\beta t}]^{2m} - [(cos(\omega x) + sin(\omega y))e^{\beta t}]^{2m} \\ &= 0 \end{aligned}$$

We deduce that $N(\bar{u}^1(t,x,y)) = 0$.
We obtain :

$$\begin{cases} u_0^2(t,x,y) = u(1,x,t) \\ u_n^2(t,x,y) = t\mathcal{M}^{-1}[\frac{\beta}{1-s}\mathcal{M}(u_{n-1}^2(t,x,y))] \end{cases} \tag{73}$$

calculation of u_n^2

$$\begin{aligned} u_1^2(t,x,y) &= t\mathcal{M}^{-1}[\frac{\beta}{1-s}\mathcal{M}(u_0^2(t,x,y))] \\ &= t(\lambda + cos(\omega x) + sin(\omega y))e^{\beta}\mathcal{M}^{-1}[\frac{\beta}{s(1-s)} \\ &= t\beta(\lambda + cos(\omega x) + sin(\omega y))e^{\beta}\mathcal{M}^{-1}[\frac{1}{s} + \frac{1}{1-s}] \\ &= t\beta(\lambda + cos(\omega x) + sin(\omega y))e^{\beta}[1 - \frac{1}{t} \\ u_1^2(t,x,y) &= (\lambda + cos(\omega x) + sin(\omega y))e^{\beta}\beta(t-1) \end{aligned}$$

$$\begin{aligned} u_2^2(t,x,y) &= t\mathcal{M}^{-1}[\frac{\beta}{1-s}\mathcal{M}(u_1^2(t,x,y))] \\ &= t\beta^2(\lambda + cos(\omega x) + sin(\omega y))e^{\beta}\mathcal{M}^{-1}[\frac{1}{1-s}\mathcal{M}(t-1)) \\ &= t\beta^2(\lambda + cos(\omega x) + sin(\omega y))e^{\beta}\mathcal{M}^{-1}[-\frac{1}{s(1-s)} + \frac{1}{(s+1)(1-s)} \\ &= t\beta^2(\lambda + cos(\omega x) + sin(\omega y))e^{\beta}\mathcal{M}^{-1}[-\frac{1}{s} - \frac{1}{1-s} + \frac{\frac{1}{2}}{s+1} + \frac{\frac{1}{2}}{1-s} \\ &= t\beta^2(\lambda + cos(\omega x) + sin(\omega y))e^{\beta}\mathcal{M}^{-1}[-\frac{1}{s} + \frac{\frac{1}{2}}{s+1} + \frac{\frac{1}{2}}{s-1} \\ &= \beta^2(\lambda + cos(\omega x) + sin(\omega y))e^{\beta}[-t + \frac{t^2}{2} + \frac{1}{2}] \\ u_2^1(t,x,y) &= (\lambda + cos(\omega x) + sin(\omega y))e^{\beta}\frac{(\beta(t-1))^2}{2} \end{aligned}$$

$$
\begin{aligned}
u_3^2(t,x,y) &= t\mathcal{M}^{-1}[\frac{\beta}{1-s}\mathcal{M}(u_2^2(t,x,y))] \\
&= t\beta^3(\lambda + cos(\omega x) + sin(\omega y))e^{\beta}\mathcal{M}^{-1}[\frac{1}{1-s}\mathcal{M}[-t+\frac{t^2}{2}+\frac{1}{2}]) \\
&= t\beta^3(\lambda + cos(\omega x) + sin(\omega y))e^{\beta}\mathcal{M}^{-1}[\frac{1}{1-s}[-\frac{1}{s+1}+\frac{1}{2(s+2)}+\frac{1}{2s}]) \\
&= t\beta^3(\lambda + cos(\omega x) + sin(\omega y))e^{\beta}\mathcal{M}^{-1}[\frac{1}{(s+1)(s-1)}-\frac{1}{2(s+2)(s-1)}-\frac{1}{2s(s-1)}
\end{aligned}
$$

we obtain :
$u_3^2(t,x,y) = t\beta^3(\lambda + cos(\omega x) + sin(\omega y))e^{\beta}\mathcal{M}^{-1}[\frac{\frac{1}{2}}{s-1}+\frac{-\frac{1}{2}}{s+1}-\frac{1}{2}(\frac{-\frac{1}{3}}{s+2}+\frac{\frac{1}{3}}{s-1})-\frac{1}{2}(\frac{-1}{s}+\frac{1}{s-1})$ we obtain

$$
\begin{aligned}
u_3^2(t,x,y) &= t\beta^3(\lambda + cos(\omega x) + sin(\omega y))e^{\beta}(\frac{1}{2}t^{-1}-\frac{1}{2}t+\frac{1}{6}(t^2-t^{-1})+\frac{1}{2}(1-t^{-1})) \\
&= \beta^3(\lambda + cos(\omega x) + sin(\omega y))e^{\beta}(\frac{3-3t^2+t^3-1+3t-3}{6}) \\
&= \beta^3(\lambda + cos(\omega x) + sin(\omega y))e^{\beta}(\frac{t^3-3t^2+3t-1}{3!}) \\
u_3^2(t,x,y) &= (\lambda + cos(\omega x) + sin(\omega y))e^{\beta}\frac{[\beta(t-1)]^3}{3!}
\end{aligned}
$$

It has been shown repeatedly that

$$
u_n^2(x,t) = (\lambda + cos(\omega x) + sin(\omega y))e^{\beta}\frac{[\beta(t-1)]^n}{n!} \tag{74}
$$

$$
u^2(t,x,y) = \sum_{n=0}^{+\infty} u_n^2 = (\lambda + cos(\omega x) + sin(\omega y))e^{\beta}e^{(\beta(t-1))} = (\lambda + cos(\omega x) + sin(\omega y))e^{\beta t} \tag{75}
$$

we obtain :

$$
u^1(t,x,y) = u^2(t,x,y) \tag{76}
$$

We deduce the general solution to the problem :

$$
u(t,x,y) = (\lambda + cos(\omega x) + sin(\omega y))e^{\beta t} \tag{77}
$$

5. Conclusion :

In this paper, we have successfully tested the solution of some nonlinear problems by the Mellin-SBA method. In the future, we plan to use the method to solve fractional order partial differential equations.

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