On Certain Results On The Diophantine Equation: $\sum_{r=1}^{n} w_r^2 + \frac{n}{3}d^2 = 3\left(\frac{nd^2}{3} + \sum_{r=1}^{\frac{n}{3}} w_{3r-1}^2\right)$

ORIGINAL RESEARCH ARTICLE

Abstract

Consider a sequence w_r in arithmetic progression with a common difference d. he exploration of Diophantine equations, which are polynomial equations seeking integer solutions, has been a fascinating endeavor in number theory. These equations have historically intrigued mathematicians due to their inherent complexities and their importance in understanding the properties of integers. In this study, we investigate a Diophantine equation that relates the sum of squares of integers from specific sequences to a variable d. Specifically, we extend existing results on the Diophantine equation: $\sum_{r=1}^{n} w_r^2 + \frac{n}{3}d^2 = 3\left(\frac{nd^2}{3} + \sum_{r=1}^{\frac{n}{3}} w_{3r-1}^2\right)$. We aim to determine the conditions under which integer solutions for w_r and d exist within this equation. Our methodology involves decomposing and factoring polynomials and exploring the solution set of the given equation.

Keywords: Sequences, Diophantine equation, Integer, Polynomial, Factorization

1 Introduction

Diophantine equations, tracing their origins back to the era of the ancient Greek mathematician Diophantus, continue to pose captivating challenges within number theory. These equations, which seek integer solutions, hold significant importance due to their real-life applications. Despite the extensive exploration of various Diophantine equations, including renowned challenges like Fermat's Last Theorem, the Ramanujan-Nagell equation, and the Lebesgue-Nagell equation, as well as studies focusing on polynomials of degree less than five, specific examinations of the Diophantine equation $\sum_{r=1}^{n} w_r^2 + \frac{n}{3}d^2 = 3\left(\frac{nd^2}{3} + \sum_{r=1}^{\frac{n}{3}} w_{3r-1}^2\right)$ remain largely uncharted. Recent research has delved into the intricacies of polynomials with degrees less than five [5, 1, 3, 15, 9, 13]. For a comprehensive understanding of studies related to Fermat's Last Theorem and the Ramanujan-Nagell equations, readers are encouraged to explore [4, 7, 10, 12, 8, 14, 2, 11, 16]. Within the existing body of work, the literature concerning the Diophantine equation remains largely unexplored. This study aims to contribute to this knowledge gap by extending existing results on the Diophantine equation $\sum_{r=1}^{n} w_r^2 + \frac{n}{3}d^2 = 3\left(\frac{nd^2}{3} + \sum_{r=1}^{\frac{n}{3}} w_{3r-1}^2\right)$ first introduced by Lao [12, 14], thus seeking to enhance our comprehension of this specific Diophantine equation within the broader landscape of mathematical exploration.

2 Main Results:

Theorem 1.1: Consider equation 1 satisfying condition $(n, w_1, w_2, ..., w_{15}, 9d) = (27, w_1, w_2, ..., w_{27}, 9d)$

Then, the diophantine equation:

$$w_1^2 + w_2^2 + w_3^2 + w_4^2 + w_5^2 + w_6^2 + w_7^2 + w_8^2 + w_9^2 + w_{10}^2 + w_{11}^2 + w_{12}^2 + w_{13}^2 + w_{14}^2 + w_{15}^2 + w_{16}^2 + w_{17}^2 + w_{18}^2 + w_{19}^2 + w_{20}^2 + w_{21}^2 + w_{22}^2 + w_{23}^2 + w_{24}^2 + w_{25}^2 + w_{26}^2 + w_{27}^2 + 9d^2 = 3(9d^2 + w_2^2 + w_5^2 + w_8^2 + w_{11}^2 + w_{14}^2 + w_{17}^2 + w_{20}^2 + w_{23}^2 + w_{26}^2)$$

has the solution in integers if $w_{27} - w_{26} = w_{26} - w_{25} = w_{25} - w_{24} = w_{24} - w_{23} = w_{23} - w_{22} = w_{22} - w_{21} = w_{21} - w_{20} = w_{20} - w_{19} = w_{19} - w_{18} = w_{18} - w_{17} = w_{17} - w_{16} = w_{16} - w_{15} = w_{15} - w_{14} = w_{14} - w_{13} = w_{13} - w_{12} = w_{12} - w_{11} = w_{11} - w_{10} = w_{10} - w_9 = w_9 - w_8 = w_8 - w_7 = w_7 - w_6 = w_6 - w_5 = w_5 - w_4 = w_4 - w_3 = w_3 - w_2 = w_2 - w_1 = d$

Proof: Consider the equation

$$w_1^2 + w_2^2 + w_3^2 + w_4^2 + w_5^2 + w_6^2 + w_7^2 + w_8^2 + w_9^2 + w_{10}^2 + w_{11}^2 + w_{12}^2 + w_{13}^2 + w_{14}^2 + w_{15}^2 + w_{16}^2 + w_{17}^2 + w_{18}^2 + w_{19}^2 + w_{20}^2 + w_{21}^2 + w_{22}^2 + w_{23}^2 + w_{24}^2 + w_{25}^2 + w_{26}^2 + w_{27}^2 + 9d^2 = 3(9d^2 + w_2^2 + w_5^2 + w_8^2 + w_{11}^2 + w_{14}^2 + w_{17}^2 + w_{20}^2 + w_{23}^2 + w_{26}^2)$$

And suppose that $w_2 = w_1 + d, w_3 = w_1 + 2d, w_4 = w_1 + 3d, w_5 = w_1 + 4d, w_6 = w_1 + 5d, w_7 = w_1 + 6d, w_8 = w_1 + 7d, w_9 = w_1 + 8d, w_{10} = w_1 + 9d, w_{11} = w_1 + 10d, w_{12} = w_1 + 11d, w_{13} = w_1 + 12d, w_{14} = w_1 + 13d, w_{15} = w_1 + 14d, w_{16} = w_1 + 15d, w_{17} = w_1 + 16d, w_{18} = w_1 + 17d, w_{19} = w_1 + 18d, w_{20} = w_1 + 19d, w_{21} = w_1 + 20dw_{22} = w_1 + 21dw_{23} = w_1 + 22d, w_{24} = w_1 + 23d, w_{25} = w_1 + 24d, w_{26} = w_1 + 25d, w_{27} = w_1 + 26d,$

The left hand side expressed as:

And suppose that $w_1^2 + (w_1 + d)^2 + (w_1 + 2d)^2 + (w_1 + 3d)^2 + (w_1 + 4d)^2 + (w_1 + 5d)^2 + (w_1 + 6d)^2 + (w_1 + 7d)^2 + (w_1 + 8d)^2 + (w_1 + 9d)^2 + (w_1 + 10d)^2 + (w_1 + 11d)^2 + (w_1 + 12d)^2 + (w_1 + 13d)^2 +$

$(w_1+14d)^2+(w_1+15d)^2+(w_1+16d)^2+(w_1+17d)^2+(w_1+18d)^2+(w_1+19d)^2+(w_1+20d)^2+(w_1+21d)^2+(w_1+22d)^2+(w_1+23d)^2+(w_1+24d)^2+(w_1+25d)^2+(w_1+26d)^2+9d^2$

Simplifies to

$27w_1^2+702w_1d+6210d^2=3(9w_1^2+234w_1d+2070d^2)$ ...(1.1)

Splitting equation (2.2) into thrice sums of squares we obtain:

3(9d$^2+(w_1^2+2w_1d+d^2)+(w_1^2+8w_1d+16d^2)+(w_1^2+14w_1d+49d^2)+(w_1^2+20w_1d+100d^2)+(w_1^2+26w_1d+169d^2)+(w_1^2+32w_1d+256d^2)+(w_1^2+38w_1d+361d^2)+(w_1^2+44w_1d+484d^2)+(w_1^2+50w_1d+625d^2))$

$=3(9d^2)+(w_1+d)^2+(w_1+4d)^2+(w_1+7d)^2+(w_1+10d)^2+(w_1+13d)^2+(w_1+16d)^2+(w_1+19d)^2+(w_1+22d)^2+(w_1+25d)^2$

$=3(9d^2+w_2^2+w_5^2+w_8^2+w_{11}^2+w_{14}^2+w_{17}^2+w_{20}^2+w_{23}^2+w_{26}^2)$

This completesthe proof □.

Theorem 1.2: Consider equation 1 satisfying condition $(n,w_1,w_2,...,w_{30},10d)=(30,w_1,w_2,...,w_{30},10d)$

Then, the diophantine equation:

$w_1^2+w_2^2+w_3^2+w_4^2+w_5^2+w_6^2+w_7^2+w_8^2+w_9^2+w_{10}^2+w_{11}^2+w_{12}^2+w_{13}^2+w_{14}^2+w_{15}^2+w_{16}^2+w_{17}^2+w_{18}^2+w_{19}^2+w_{20}^2+w_{21}^2+w_{22}^2+...+w_{28}^2+w_{29}^2+w_{30}^2+10d^2$
=3(10d$^2+w_2^2+w_5^2+w_8^2+w_{11}^2+w_{14}^2+w_{17}^2+w_{20}^2+w_{23}^2+w_{26}^2+w_{29}^2)$

has the solution in integers if $w_{30}-w_{29}=w_{29}-w_{28}=w_{28}-w_{27}=w_{27}-w_{26}=w_{26}-w_{25}=w_{25}-w_{24}=w_{24}-w_{23}=w_{23}-w_{22}=w_{22}-w_{21}=w_{20}-w_{20}=w_{19}-w_{18}=w_{18}-w_{17}=w_{17}-w_{16}=w_{16}-w_{15}=w_{15}-w_{14}=w_{14}-w_{13}=w_{13}-w_{12}=w_{12}-w_{11}=w_{11}-w_{10}=w_{10}-w_9=w_9-w_8=w_8-w_7=w_7-w_6=w_6-w_5=w_5-w_4=w_4-w_3=w_3-w_2=w_2-w_1=d$

Proof: Consider the equation

$w_1^2+w_2^2+w_3^2+w_4^2+w_5^2+w_6^2+w_7^2+w_8^2+w_9^2+w_{10}^2+w_{11}^2+w_{12}^2+w_{13}^2+w_{14}^2+w_{15}^2+w_{16}^2+w_{17}^2+w_{18}^2+w_{19}^2+w_{20}^2+w_{21}^2+w_{22}^2+...+w_{28}^2+w_{29}^2+w_{30}^2+10d^2$
=3(6d$^2+w_2^2+w_5^2+w_8^2+w_{11}^2+w_{14}^2+w_{17}^2+w_{20}^2+w_{23}^2+w_{26}^2+w_{29}^2)$

And suppose that $w_2=w_1+d,w_3=w_1+2d,w_4=w_1+3d,w_5=w_1+4d,w_6=w_1+5d,w_7=w_1+6d,w_8=w_1+7d,w_9=w_1+8d,w_{10}=w_1+9d,w_{11}=w_1+10d,w_{12}=w_1+11d,w_{13}=w_1+12d,w_{14}=w_1+13d,w_{15}=w_1+14d,w_{16}=w_1+15d,w_{17}=w_1+16d,w_{18}=w_1+17d,w_{19}=w_1+18d,w_{20}=w_1+19d,w_{21}=w_1+20d,w_{22}=w_1+21d,,w_{23}=w_1+22d,w_{24}=w_1+23d,w_{25}=w_1+24d,w_{26}=w_1+5d,w_{27}=w_1+26d,w_{28}=w_1+27d,w_{29}=w_1+28d,w_{30}=w_1+29d$

The left hand side expressed as:

And suppose that $w_1^2+(w_1+d)^2+(w_1+2d)^2+(w_1+3d)^2+(w_1+4d)^2+(w_1+5d)^2+(w_1+6d)^2+(w_1+7d)^2+(w_1+8d)^2+(w_1+9d)^2+(w_1+10d)^2+(w_1+11d)^2+(w_1+12d)^2+(w_1+13d)^2+(w_1+14d)^2+(w_1+15d)^2+(w_1+16d)^2+(w_1+17d)^2+(w_1+18d)^2+(w_1+19d)^2+(w_1+20d)^2+(w_1+21d)^2+(w_1+22d)^2+$

$(w_1+23d)^2+(w_1+24d)^2+(w_1+25d)^2+(w_1+26d)^2+(w_1+27d)^2+(w_1+28d)^2+(w_1+29d)^2+10d^2$

Simplifies to

$30w_1^2+870w_1d+8565d^2 = 3(10w_1^2+290w_1d+2855d^2)$ ...(1.2)

Splitting equation (1.2) into thrice sums of squares we obtain:

3(10d$^2+(w_1^2+2w_1d+d^2)+(w_1^2+8w_1d+16d^2)+(w_1^2+14w_1d+49d^2)+(w_1^2+20w_1d+100d^2)+(w_1^2+26w_1d+169d^2+(w_1^2+32w_1d+256d^2)+(w_1^2+38w_1d+361d^2)+(w_1^2+44w_1d+484d^2)+(w_1^2+50w_1d+625d^2)+(w_1^2+56w_1d+784d^2))$

$= 3(10d^2)+(w_1+d)^2+(w_1+4d)^2+(w_1+7d)^2+(w_1+10d)^2+(w_1+13d)^2+(w_1+16d)^2+(w_1+19d)^2+(w_1+22d)^2+(w_1+25d)^2+(w_1+28d)^2$

$= 3(6d^2+w_2^2+w_5^2+w_8^2+w_{11}^2+w_{14}^2+w_{17}^2)$

This completesthe proof □.

Theorem 1.3: Consider equation 1 satisfying condition $(n, w_1, w_2, ..., w_{33}, 11d) = (33, w_1, w_2, ..., w_{33}, 11d)$

Then, the diophantine equation:

$w_1^2+w_2^2+w_3^2+w_4^2+w_5^2+w_6^2+w_7^2+w_8^2+w_9^2+w_{10}^2+w_{11}^2+w_{12}^2+w_{13}^2+w_{14}^2+w_{15}^2+w_{16}^2+w_{17}^2+w_{18}^2+w_{19}^2+w_{20}^2+w_{21}^2+w_{22}^2+w_{23}^2+w_{24}^2+w_{25}^2+w_{26}^2+w_{27}^2+w_{28}^2+w_{29}^2+w_{30}^2+w_{31}^2+w_{32}^2+w_{33}^2+11d^2$
=3(11d$^2+w_2^2+w_5^2+w_8^2+w_{11}^2+w_{14}^2+w_{17}^2+w_{20}^2+w_{23}^2+w_{26}^2+w_{29}^2+w_{32}^2)$

has the solution in integers if $w_{33}-w_{32}=w_{32}-w_{31}=w_{31}-w_{30}=w_{30}-w_{29}=w_{29}-w_{28}=w_{28}-w_{27}=w_{27}-w_{26}=w_{26}-w_{25}=w_{25}-w_{24}=w_{24}-w_{23}=w_{23}-w_{22}=w_{22}-w_{21}=w_{21}-w_{20}=w_{20}-w_{19}=w_{19}-w_{18}=w_{18}-w_{17}=w_{17}-w_{16}=w_{16}-w_{15}=w_{15}-w_{14}=w_{14}-w_{13}=w_{13}-w_{12}=w_{12}-w_{11}=w_{11}-w_{10}=w_{10}-w_9=w_9-w_8=w_8-w_7=w_7-w_6=w_6-w_5=w_5-w_4=w_4-w_3=w_3-w_2=w_2-w_1=d$

Proof: Consider the equation

$w_1^2+w_2^2+w_3^2+w_4^2+w_5^2+w_6^2+w_7^2+w_8^2+w_9^2+w_{10}^2+w_{11}^2+w_{12}^2+w_{13}^2+w_{14}^2+w_{15}^2+w_{16}^2+w_{17}^2+w_{18}^2+w_{19}^2+w_{20}^2+w_{21}^2+w_{22}^2+w_{23}^2+w_{24}^2+w_{25}^2+w_{26}^2+w_{27}^2+w_{28}^2+w_{29}^2+w_{30}^2+w_{31}^2+w_{32}^2+w_{33}^2+11d^2$
=3(11d$^2+w_2^2+w_5^2+w_8^2+w_{11}^2+w_{14}^2+w_{17}^2+w_{20}^2+w_{23}^2+w_{26}^2+w_{29}^2+w_{32}^2)$

And suppose that $w_2=w_1+d, w_3=w_1+2d, w_4=w_1+3d, w_5=w_1+4d, w_6=w_1+5d, w_7=w_1+6d, w_8=w_1+7d, w_9=w_1+8d, w_{10}=w_1+9d, w_{11}=w_1+10d, w_{12}=w_1+11d, w_{13}=w_1+12d, w_{14}=w_1+13d, w_{15}=w_1+14d, w_{16}=w_1+15d, w_{17}=w_1+16d, w_{18}=w_1+17d, w_{19}=w_1+18d, w_{20}=w_1+19d, w_{21}=w_1+20d, w_{22}=w_1+21d, w_{23}=w_1+22d, w_{24}=w_1+23d, w_{25}=w_1+24d, w_{26}=w_1+25d, w_{27}=w_1+26d, w_{28}=w_1+27d, w_{29}=w_1+28d, w_{30}=w_1+29d, w_{31}=w_1+30d, w_{32}=w_1+31d, w_{33}=w_1+32d$

The left hand side expressed as:

And suppose that $w_1^2+(w_1+d)^2+(w_1+2d)^2+(w_1+3d)^2+(w_1+4d)^2+(w_1+5d)^2+(w_1+6d)^2+(w_1+7d)^2+(w_1+8d)^2+(w_1+9d)^2+(w_1+10d)^2+(w_1+11d)^2+(w_1+12d)^2+(w_1+13d)^2+(w_1+14d)^2+(w_1+15d)^2+(w_1+16d)^2+(w_1+17d)^2+(w_1+18d)^2+(w_1+19d)^2+(w_1+20d)^2+$

$(w_1+21d)^2+(w_1+22d)^2+(w_1+23d)^2+(w_1+24d)^2+(w_1+25d)^2+(w_1+26d)^2+(w_1+27d)^2+(w_1+28d)^2+(w_1+29d)^2+(w_1+30d)^2+(w_1+31d)^2+(w_1+32d)^2+11d^2$

Simplifies to

$33w_1^2+1056w_1d+11451d^2=3(11w_1^2+352w_1d+3817d^2)$ ...(1.3)

Splitting equation (1.3) into thrice sums of squares we obtain:

3(11d$^2+(w_1^2+2w_1d+d^2)+(w_1^2+8w_1d+16d^2)+(w_1^2+14w_1d+49d^2)+(w_1^2+20w_1d+100d^2)+(w_1^2+26w_1d+169d^2+(w_1^2+32w_1d+256d^2)+(w_1^2+38w_1d+361d^2)+(w_1^2+44w_1d+484d^2)+(w_1^2+50w_1d+625d^2)+(w_1^2+56w_1d+784d^2)+(w_1^2+62w_1d+961d^2))$

$=3(11d^2)+(w_1+d)^2+(w_1+4d)^2+(w_1+7d)^2+(w_1+10d)^2+(w_1+13d)^2+(w_1+16d)^2+(w_1+19d)^2+(w_1+22d)^2+(w_1+25d)^2+(w_1+28d)^2+(w_1+31d)^2$

$=3(11d^2+w_2^2+w_5^2+w_8^2+w_{11}^2+w_{14}^2+w_{17}^2+w_{20}^2+w_{23}^2+w_{26}^2+w_{29}^2+w_{32}^2)$

This completesthe proof □.

Theorem 1.4: Consider equation 1 satisfying condition $(n,w_1,w_2,...,w_{36},12d)=(36,w_1,w_2,...,w_{36},12d)$

Then, the diophantine equation:

$w_1^2+w_2^2+w_3^2+w_4^2+w_5^2+w_6^2+w_7^2+w_8^2+w_9^2+w_{10}^2+w_{11}^2+w_{12}^2+w_{13}^2+w_{14}^2+w_{15}^2+w_{16}^2+w_{17}^2+w_{18}^2+w_{19}^2+w_{20}^2+w_{21}^2+w_{22}^2+w_{23}^2+w_{24}^2+w_{25}^2+w_{26}^2+w_{27}^2+w_{28}^2+w_{29}^2+w_{30}^2+w_{31}^2+w_{32}^2+w_{33}^2+w_{34}^2+w_{35}^2+w_{36}^2+12d^2$
=3(12d$^2+w_2^2+w_5^2+w_8^2+w_{11}^2+w_{14}^2+w_{17}^2+w_{20}^2+w_{23}^2+w_{26}^2+w_{29}^2+w_{32}^2+w_{35}^2)$

has the solution in integers if $w_{36}-w_{35}=w_{35}-w_{34}=w_{34}-w_{33}=w_{33}-w_{32}=w_{32}-w_{31}=w_{31}-w_{30}=w_{30}-w_{29}=w_{29}-w_{28}=w_{28}-w_{27}=w_{27}-w_{26}=w_{26}-w_{25}=w_{25}-w_{24}=w_{24}-w_{23}=w_{23}-w_{22}=w_{22}-w_{21}=w_{21}-w_{20}=w_{20}-w_{19}=w_{19}-w_{18}=w_{18}-w_{17}=w_{17}-w_{16}=w_{16}-w_{15}=w_{15}-w_{14}=w_{14}-w_{13}=w_{13}-w_{12}=w_{12}-w_{11}=w_{11}-w_{10}=w_{10}-w_9=w_9-w_8=w_8-w_7=w_7-w_6=w_6-w_5=w_5-w_4=w_4-w_3=w_3-w_2=w_2-w_1=d$

Proof: Consider the equation

$w_1^2+w_2^2+w_3^2+w_4^2+w_5^2+w_6^2+w_7^2+w_8^2+w_9^2+w_{10}^2+w_{11}^2+w_{12}^2+w_{13}^2+w_{14}^2+w_{15}^2+w_{16}^2+w_{17}^2+w_{18}^2+w_{19}^2+w_{20}^2+w_{21}^2+w_{22}^2+w_{23}^2+w_{24}^2+w_{25}^2+w_{26}^2+w_{27}^2+w_{28}^2+w_{29}^2+w_{30}^2+w_{31}^2+w_{32}^2+w_{33}^2+w_{34}^2+w_{35}^2+w_{36}^2+12d^2$
=3(12d$^2+w_2^2+w_5^2+w_8^2+w_{11}^2+w_{14}^2+w_{17}^2+w_{20}^2+w_{23}^2+w_{26}^2+w_{29}^2+w_{32}^2+w_{35}^2)$

And suppose that $w_2=w_1+d,w_3=w_1+2d,w_4=w_1+3d,w_5=w_1+4d,w_6=w_1+5d,w_7=w_1+6d,w_8=w_1+7d,w_9=w_1+8d,w_{10}=w_1+9d,w_{11}=w_1+10d,w_{12}=w_1+11d,w_{13}=w_1+12d,w_{14}=w_1+13d,w_{15}=w_1+14d,w_{16}=w_1+15d,w_{17}=w_1+16d,w_{18}=w_1+17d,w_{19}=w_1+18d,w_{20}=w_1+19d,w_{21}=w_1+20d,w_{22}=w_1+21d,w_{23}=w_1+22d,w_{24}=w_1+23d,w_{25}=w_1+24d,w_{26}=w_1+25d,w_{27}=w_1+26d,w_{28}=w_1+27d,w_{29}=w_1+28d,w_{30}=w_1+29d,w_{31}=w_1+30d,w_{32}=w_1+31d,w_{33}=w_1+32d,w_{34}=w_1+23d,w_{35}=w_1+34d,w_{36}=w_1+35d$

The left hand side expressed as:

And suppose that $w_1^2+(w_1+d)^2+(w_1+2d)^2+(w_1+3d)^2+(w_1+4d)^2+(w_1+5d)^2+(w_1+6d)^2+(w_1+7d)^2+(w_1+8d)^2+(w_1+9d)^2+(w_1+10d)^2+(w_1+11d)^2+(w_1+12d)^2+(w_1+13d)^2+(w_1+14d)^2+$

$(w_1+15d)^2+(w_1+16d)^2+(w_1+17d)^2+(w_1+18d)^2+(w_1+19d)^2+(w_1+20d)^2+(w_1+21d)^2+(w_1+22d)^2+(w_1+23d)^2+(w_1+24d)^2+(w_1+25d)^2+(w_1+26d)^2+(w_1+27d)^2+(w_1+28d)^2+(w_1+29d)^2+(w_1+30d)^2+(w_1+31d)^2+(w_1+32d)^2+(w_1+33d)^2+(w_1+34d)^2+(w_1+35d)^2+12d^2$

Simplifies to

$36w_1^2+1260w_1d+14922d^2=3(12w_1^2+420w_1d+4974d^2)$ ...(1.4)

Splitting equation (1.3) into thrice sums of squares we obtain:

3(12d$^2+(w_1^2+2w_1d+d^2)+(w_1^2+8w_1d+16d^2)+(w_1^2+14w_1d+49d^2)+(w_1^2+20w_1d+100d^2)+(w_1^2+26w_1d+169d^2+(w_1^2+32w_1d+256d^2)+(w_1^2+38w_1d+361d^2)+(w_1^2+44w_1d+484d^2)+(w_1^2+50w_1d+625d^2)+(w_1^2+56w_1d+784d^2)+(w_1^2+62w_1d+961d^2)+(w_1^2+68w_1d+1156d^2))$

$=3(12d^2+w_2^2+w_5^2+w_8^2+w_{11}^2+w_{14}^2+w_{17}^2+w_{20}^2+w_{23}^2+w_{26}^2+w_{29}^2+w_{32}^2+w_{35}^2)$

This completesthe proof □.

3 Conclusion

In summary, the solution of the diophantine equation $\sum_{r=1}^{n} w_r^2+\frac{n}{3}d^2=3(\frac{nd^2}{3}+\sum_{n=1}^{\frac{n}{3}} w_{3r-1}^2)$, under the specified conditions of a common difference d between consecutive terms $w_n, w_{n1}, , w_2, w_1$ where $w_nw_{n1}=w_{n1}w_{n2}= \ =w_2w_1=d$ has been achieved for some cases. This solution provides valuable insights into the relation among the sequence terms, enhancing our understanding of the inherent patterns and structures within the equation. For future investigations, it is recommended to explore extensions of this diophantine equation by proving conjecture (1).

Disclaimer (Artificial Intelligence)

Author(s) hereby declare that NO generative AI technologies such as Large Language Models (ChatGPT, COPILOT, etc) and text-to-image generators have been used during writing or editing of manuscripts.

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Competing Interests

Authors have declared that no competing interests exist.

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