Original Research Article

Dynamical analysis of delayed autonomous and non-autonomous oscillators

Abstract: In this paper, the oscillatory behavior of the solutions for a five-dimensional system of coupled van der Pol-Hamiltonian-Duffing oscillator with delays is investigated. We extend the existing result in the literature from mathematical point of view. Some sufficient conditions to guarantee the oscillation of the solutions are provided and computer simulations are given to support the present criteria.

Keywords: nonlinear oscillator, delay, instability, oscillation

AMS Mathematical Subject Classification: 34K11

1 Introduction

It is known that various van der Pol oscillators, Duffing equations, Hamiltonian-Duffing oscillators which discribe many kinds of nonlinear oscillatory systems in various biological, physical and engineering systems. Recently, Ma and Zhang have investigated the following hybrid van der Pol-Duffing-Rayleigh system[1]:

$$x'' - x + \gamma x^3 - (\alpha - \beta_1 x^2 - \beta_2 x^4)x' + (k - \gamma x^2)x\cos(2\omega t) = (f + g\cos(n\omega t))\cos(\omega t), \quad (1)$$

where $\gamma, \alpha, \beta_1, \beta_2$, and k are system parameters. A bursting oscillation with two pulse-shaped explosions has been observed. By treating the cosine function $cos(\omega t)$ as a slowly varying variable δ, system (1) can be rewritten as a generalized autonomous system, expressed as

$$x'' - x + \gamma x^3 - (\alpha - \beta_1 x^2 - \beta_2 x^4)x' + (k - \gamma x^2)x(2\delta^2 - 1) = f\delta. \quad (2)$$

A bifurcation structure has been obsevered for the given parameter conditions $\alpha = 0.5, \beta_1 = \beta_2 = 0.2, \gamma = k = 1, f = 3$, and $\omega = 0.005$. A coupled system of simple oscillators may often exhibit many interesting phenomena different from their behavior in isolation. For example, Jiang et al. have studied a coupled four dimensional coupled Mathieu-van der Pol system[2]:

$$\begin{cases} x^{'} = y, \\ y^{'} = -(h+bu)x - (h+bu)x^3 - cy + (d+w)u, \\ u^{'} = v, \\ v^{'} = -cu + f(1-u^2)v + gx. \end{cases} \tag{3}$$

Using the bifurcation theory and fast-slow analysis, the bifurcation diagrams and an intriguing phenomenon were observed in model (2) as the parameters of the fast and slow systems change in the orbits. Savostianov et al. investigated the synchronized dynamics of two coupled van der Pol oscillators[3]. Liu and Zhang have discussed multiple Hopf bifurcations of four coupled van der Pol oscillators with delay as follows:

$$\begin{cases} x_1^{''}(t) = \alpha(p^2 - x_1^2)x_1^{'} - x_1 + ax_1^{'}(t-\tau) + bx_2^{'}(t-\tau) + cx_3^{'}(t-\tau) + x_4^{'}(t-\tau), \\ x_2^{''}(t) = \alpha(p^2 - x_2^2)x_2^{'} - x_2 + ax_2^{'}(t-\tau) + bx_3^{'}(t-\tau) + cx_4^{'}(t-\tau) + x_1^{'}(t-\tau), \\ x_3^{''}(t) = \alpha(p^2 - x_3^2)x_3^{'} - x_3 + ax_3^{'}(t-\tau) + bx_4^{'}(t-\tau) + cx_1^{'}(t-\tau) + x_2^{'}(t-\tau), \\ x_4^{''}(t) = \alpha(p^2 - x_4^2)x_4^{'} - x_4 + ax_4^{'}(t-\tau) + bx_1^{'}(t-\tau) + cx_2^{'}(t-\tau) + x_3^{'}(t-\tau). \end{cases} \tag{4}$$

The multiple periodic solutions of spatiotemporal patterns of the system (4) were obtained by using symmetric Hopf bifurcation theory. The normal form of the system on the central manifold and numerical simulations were also derived[4]. Sabarathinam and Thamilmaran proposed the following coupled hamiltonian Duffing oscillators:

$$\begin{cases} x_1^{''}(t) + bx_1^{'}(t) + wx_1(t) + \beta x_1^3(t) = \epsilon a_{12}(x_2(t) - x_1(t)) + \epsilon a_{13}(x_3(t) - x_1(t)), \\ x_2^{''}(t) + bx_2^{'}(t) + wx_2(t) + \beta x_2^3(t) = \epsilon a_{21}(x_1(t) - x_2(t)) + \epsilon a_{23}(x_3(t) - x_2(t)), \\ x_3^{''}(t) + bx_3^{'}(t) + wx_3(t) + \beta x_3^3(t) = \epsilon a_{31}(x_1(t) - x_3(t)) + \epsilon a_{32}(x_2(t) - x_3(t)). \end{cases} \tag{5}$$

The stability and transient chaos for model (5) were investigated [5]. In [6], the authors concerned the synchronization in a ring of four mutually coupled van der Pol oscillators. Brechtl et al. investigated the chaos and memory effects in the Bonhoeffer-van der Pol oscillator with a non-ideal capacitor[7]. Sysoev considered the reconstruction of ensembles of generalized van der Pol oscillators from vector time series[8]. The existence of islands of quasiperiodic regimes on the parameter plane of period and amplitude of the external force was considered for a pulse driven coupled van der Pol oscillators, and a number of different types of oscillations in this system were illustrated[9]. The oscillatory behavior of a van der Pol oscillator powered by a DC excitation source was shown numerically and experimentally[10]. Stability and bifurcation analysis in the delay-coupled van der Pol oscillators were studied by Zhang et al. [11, 12]. The two coupled van der Pol oscillators system with attractive and repulsive interactions indicated competitive tendencies of being complete synchronization and anti-synchronization resulting in the stabilization of the fixed point[13]. The coupled bi-stable van der Pol oscillators revealed regimes of nonconventional synchronization[14]. The pitchfork bifurcation and Hopf bifurcation for different van der Pol-Duffing oscillators were studied[15-19]. Qualitative analysis has been shown in a delayed van der Pol oscillator[20]. Spiral and arget wave chimeras in a coupled van der Pol oscillator were discussed[21]. A novel variational formulation of Duffing equation using the extended framework of Hamilton's principle was provided, it recovered all the governing differential equations as its Euler–Lagrange equation[22]. The stability and instability of rapidly oscillating solutions for the hard spring delayed Duffing oscillator were explored[23]. By introducing the concept of the discriminant for the Duffing equation, one can solve the equation in three cases depending on sign of the discriminant and apply it in soliton theory[24]. To suppress the nonlinearity vibration in an excited van der Pol–Duffing oscillator, a supplemental time delay was added[25]. In this paper, we shall concern the following coupled multiple time delays nonlinear model:

$$\begin{cases} x_1^{''} - \beta_1 x_1 + \gamma_1 x_1^3 - (\alpha_1 - \beta_{11} x_1^2 - \beta_{12} x_1^4) x_1' + (k_1 - r_1 x_1^2) x_1 + \sum_{j=1}^5 a_{1j} x_j'(t-\tau_j) \\ \qquad\qquad = \sum_{i=2,}^5 b_{1i}[x_i(t-\tau_i) - x_1(t-\tau_1)], \\ x_2^{''} - \beta_2 x_2 + \gamma_2 x_2^3 - (\alpha_2 - \beta_{21} x_2^2 - \beta_{22} x_2^4) x_2' + (k_2 - r_2 x_2^2) x_2 + \sum_{j=1}^5 a_{2j} x_j'(t-\tau_j) \\ \qquad\qquad = \sum_{i=1, i\neq 2}^5 b_{2i}[x_i(t-\tau_i) - x_2(t-\tau_2)], \\ x_3^{''} - \beta_3 x_3 + \gamma_3 x_3^3 - (\alpha_3 - \beta_{31} x_3^2 - \beta_{32} x_3^4) x_3' + (k_3 - r_3 x_3^2) x_3 + \sum_{j=1}^5 a_{3j} x_j'(t-\tau_j) \\ \qquad\qquad = \sum_{i=1, i\neq 3,}^5 b_{3i}[x_i(t-\tau_i) - x_3(t-\tau_3)], \\ x_4^{''} - \beta_4 x_4 + \gamma_4 x_4^3 - (\alpha_4 - \beta_{41} x_4^2 - \beta_{42} x_4^4) x_4' + (k_4 - r_4 x_4^2) x_4 + \sum_{j=1}^5 a_{4j} x_j'(t-\tau_j) \\ \qquad\qquad = \sum_{i=1, i\neq 4,}^5 b_{4i}[x_i(t-\tau_i) - x_4(t-\tau_4)], \\ x_5^{''} - \beta_5 x_5 + \gamma_5 x_5^3 - (\alpha_5 - \beta_{51} x_5^2 - \beta_{52} x_5^4) x_5' + (k_5 - r_5 x_5^2) x_5 + \sum_{j=1}^5 a_{5j} x_j'(t-\tau_j) \\ \qquad\qquad = \sum_{i=1,}^4 b_{5i}[x_i(t-\tau_i) - x_5(t-\tau_5)], \end{cases} \tag{6}$$

where $\gamma_i, \alpha_i, \beta_{i1}, \beta_{i2}, k_i, a_{ij}$, and b_{ij} are system parameters. It is convenient to write (6) as an equivalent ten dimensional first order system:

$$
\begin{cases}
x_1'(t) = x_2(t), \\
x_2'(t) = (\beta_1 - k_1)x_1 + \alpha_1 x_2 - \sum_{j=1}^{5} a_{2,2j} x_{2j}(t-\tau_{2j}) + \sum_{i=2,}^{5} b_{1,2i-1}[x_{2i-1}(t-\tau_{2i-1}) \\
\qquad\qquad - x_1(t-\tau_1)] - \gamma_1 x_1^3 - \beta_{11} x_1^2 x_2 - \beta_{12} x_1^4 x_2 + r_1 x_1^3, \\
x_3'(t) = x_4(t), \\
x_4'(t) = (\beta_2 - k_2)x_3 + \alpha_2 x_4 - \sum_{j=1}^{5} a_{4,2j} x_{2j}(t-\tau_{2j}) + \sum_{i=1,i\neq 2}^{5} b_{3,2i-1}[x_{2i-1}(t-\tau_{2i-1}) \\
\qquad\qquad - x_3(t-\tau_3)] - \gamma_2 x_3^3 - \beta_{21} x_3^2 x_4 - \beta_{22} x_3^4 x_4 + r_2 x_3^3, \\
x_5'(t) = x_6(t), \\
x_6'(t) = (\beta_3 - k_3)x_5 + \alpha_3 x_6 - \sum_{j=1}^{5} a_{6,2j} x_{2j}(t-\tau_{2j}) + \sum_{i=1,i\neq 3}^{5} b_{5,2i-1}[x_{2i-1}(t-\tau_{2i-1}) \\
\qquad\qquad - x_5(t-\tau_5)] - \gamma_3 x_5^3 - \beta_{31} x_5^2 x_6 - \beta_{32} x_5^4 x_6 + r_3 x_5^3, \\
x_7'(t) = x_8(t), \\
x_8'(t) = (\beta_4 - k_4)x_7 + \alpha_4 x_8 - \sum_{j=1}^{5} a_{8,2j} x_{2j}(t-\tau_{2j}) + \sum_{i=1,i\neq 4}^{5} b_{7,2i-1}[x_{2i-1}(t-\tau_{2i-1}) \\
\qquad\qquad - x_7(t-\tau_7)] - \gamma_4 x_7^3 - \beta_{41} x_7^2 x_8 - \beta_{42} x_7^4 x_8 + r_4 x_7^3, \\
x_9'(t) = x_{10}(t), \\
x_{10}'(t) = (\beta_5 - k_5)x_9 + \alpha_5 x_{10} - \sum_{j=1}^{5} a_{10,2j} x_{2j}(t-\tau_{2j}) + \sum_{i=1}^{4} b_{9,2i-1}[x_{2i-1}(t-\tau_{2i-1}) \\
\qquad\qquad - x_9(t-\tau_9)] - \gamma_5 x_9^3 - \beta_{51} x_9^2 x_{10} - \beta_{52} x_9^4 x_{10} + r_5 x_9^3,
\end{cases}
\tag{7}
$$

where $a_{ij} = a_{2i,2j}, b_{ij} = b_{2i-1,2j-1}, \tau_{2i} = \tau_i, \tau_{2j-1} = \tau_j, i, j = 1, 2, \cdots, 5$. The matrix form of the system (7) is as the following:

$$x'(t) = Cx(t) + Dx(t-\tau) + f(x(t)) \tag{8}$$

where $x(t) = [x_1(t), x_2(t), \cdots, x_{10}(t)]^T, x(t-\tau) = [x_1(t-\tau_1), x_2(t-\tau_2), \cdots, x_{10}(t-\tau_{10})]^T$, $f(x(t))$ is a 10×1 vector, C and D both are 10×10 matrices as the following: $f(x(t)) = [0, -\gamma_1 x_1^3 - \beta_{11} x_1^2 x_2 - \beta_{12} x_1^4 x_2 + r_1 x_1^3, 0, -\gamma_2 x_3^3 - \beta_{21} x_3^2 x_4 - \beta_{22} x_3^4 x_4 + r_2 x_3^3, \cdots, 0, -\gamma_5 x_9^3 -$

$\beta_{51}x_9^2x_{10}-\beta_{52}x_9^4x_{10}+r_5x_9^3]^T$,

$$C=(c_{ij})_{10\times 10}=\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ c_{21} & \alpha_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & c_{43} & \alpha_2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & c_{65} & \alpha_3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & c_{87} & \alpha_4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & c_{10,9} & \alpha_5 \end{pmatrix},$$

$$D=(d_{ij})_{10\times 10}=\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ n_{21} & a_{22} & b_{13} & a_{24} & b_{15} & a_{26} & b_{17} & a_{28} & b_{19} & a_{2,10} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ b_{31} & a_{42} & n_{43} & a_{44} & b_{35} & a_{46} & b_{37} & a_{48} & b_{39} & a_{4,10} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ b_{51} & a_{62} & b_{53} & a_{64} & n_{65} & a_{66} & b_{57} & a_{68} & b_{59} & a_{6,10} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ b_{71} & a_{82} & b_{73} & a_{84} & b_{75} & a_{86} & n_{87} & a_{88} & b_{79} & a_{8,10} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ b_{91} & a_{10,2} & b_{93} & a_{10,4} & b_{95} & a_{10,6} & b_{97} & a_{10,8} & n_{10,9} & a_{10,10} \end{pmatrix},$$

where $c_{21}=\beta_1-k_1, c_{43}=\beta_2-k_2, c_{65}=\beta_3-k_3, c_{87}=\beta_4-k_4, c_{10,9}=\beta_5-k_5, n_{21}=-\sum_{i=2,}^{5}b_{1,2i-1}, n_{43}=-\sum_{i=1,i\neq 2}^{5}b_{3,2i-1}, n_{65}=-\sum_{i=1,i\neq 3}^{5}b_{5,2i-1}, n_{87}=\sum_{i=1,i\neq 4}^{5}b_{7,2i-1}, n_{10,9}=-\sum_{i=1}^{4}b_{9,2i-1}$. The linearized system of (8) is as the following:

$$x'(t)=Cx(t)+Dx(t-\tau) \tag{9}$$

2 Preliminaries

Lemma 1 If matrix $M(=C+D)$ is a nonsingular matrix for selected parameters, then there exists a unique zero equilibrium point for system (6) (or (7)).

Proof Obviously, system (9) has a trivial solution. Since $f(0)=0$, so the system (8) has a trivial solution. Note that M is a nonsingular matrix for selected parameters, implying that system (9) has a unique trivial solution. Ihis suggests that the system (6) or (7) has a unique trivial solution.

Lemma 2 All solutions of system (6) (or (7)) are bounded assuming that $\beta_{i2} > 0.i = 1, 2, \cdots, 5$.

Proof To prove the boundedness of the solutions in system (7), we construct a Lyapunov function $V(t) = \sum_{i=1}^{10} \frac{1}{2} x_i^2(t)$. Calculating the derivative of $V(t)$ through system (7) we have

$$\begin{aligned} & V'(t)|_{(7)} = \sum_{i=1}^{10} x_i'(t) x_i(t) \\ \leq \quad & B_1 \sum_{i=1}^{9} |x_i||x_{i+1}| + B_2 \sum_{i=1}^{5} x_{2i}^2 - \sum_{i=1}^{5} (\gamma_i - r_i) x_{2i-1}^3 x_{2i} - \sum_{i=1}^{5} \beta_{i1} x_{2i-1}^2 x_{2i}^2 \\ & - \sum_{i=1}^{5} \beta_{i2} x_{2i-1}^4 x_{2i}^2 \end{aligned} \tag{10}$$

where B_1, B_2 are same positive constants. Obviously, when $x_{2i-1} \to +\infty, x_{2i} \to +\infty (1 \leq i \leq 5,)$ $x_{2i-1}^4 x_{2i}^2$ are higher order infinity than $x_{2i-1}^2 x_{2i}^2, x_{2i-1}^3 x_{2i}$ and $|x_i||x_{i+1}|$, respectively. Since $\beta_{i2} > 0.i = 1, 2, \cdots, 5$, therefore, there exists suitably large $K > 0$ such that $V'(t)|_{(7)} < 0$ as $|x_{2i-1}| > K, |x_{2i}| > K (i = 1, 2, \cdots, 5)$. This means that all solutions of the system (7) are bounded.

3 The existence of oscillatory solutions

Theorem 1 Assume that zero is the unique equilibrium point of the system (7) for selected parameter values. Let $\gamma_1, \gamma_2, \cdots, \gamma_{10}$ and $0, \delta_2, 0, \delta_4, \cdots, 0, \delta_{10}$ are characteristic values of matrix C and matrix D, respectively. If the real part of each $\gamma_i (i = 1, 2, \cdots, 10)$ and $\delta_j (j = 2, 4, \cdots, 10)$ are nonpositive, then the trivial solution of system (7) is stable. If each γ_i has positive real part, or there exists a characteristic value, say γ_k, $Re(\gamma_k) < 0$ with $|Re(\gamma_k)| < Re(\delta_k)$, then the unique trivial solution of system (7) is unstable, implying that there exists a periodic oscillatory solution in system (7).

Proof According to the time delay basic differential equation theory, if the real part of each $\gamma_i (i = 1, 2, \cdots, 10)$ and $\delta_j (i = 2, 4, \cdots, 10)$ are nonpositive, then the trivial solution of system (9) is stable. Noting that the nonlinear term $f(z)$ of system (7) is a higher order infinitesimal as $x_i \to 0$. Therefore, the stability of the trivial solution of system (9) ensures the stability of the trivial solution of system (7). Obviously, if the trivial solution of system

(9) is unstable, then the trivial solution of system (7) is also unstable. Therefore, in order to discuss the instability of the trivial solution of system (7) we only need to deal with the instability of the trivial solution of system (9). Firstly, consider an auxiliary system of (9) as follows:

$$x'(t) = Cx(t) + Dx(t - \tau_*) \tag{11}$$

where $\tau_* = \min_{1\leq i\leq 5}\{\tau_{2i}, \tau_{2i-1}\}$ and $x(t-\tau_*) = [x_1(t-\tau_*), x_2(t-\tau_*), \cdots, x_{10}(t-\tau_*)]^T$. Since $\gamma_1, \gamma_2, \cdots, \gamma_{10}$ and $0, \delta_2, 0, \delta_4, \cdots, 0, \delta_{10}$ are characteristic values of matrix C and matrix D, respectively, then the characteristic equations corresponding to system (11) are the following:

$$\Pi_{i=1}^{10}(\lambda - \gamma_i - \delta_i e^{-\lambda\tau_*}) = 0. \tag{12}$$

Thus, we are led to an investigation of the nature of the roots for some $k, k \in \{1, 2, \cdots, 10\}$

$$\lambda - \gamma_k - \delta_k e^{-\lambda\tau_*} = 0. \tag{13}$$

Noting that there are five characteristic values of matrix D are zeros, if each γ_i has positive real part, so if $\delta_k = 0$ in equation (13) we know that system (11) has a characteristic value with positive real part, so the trivial solution of system (11) is unstable. If $Re(\gamma_k) < 0$ with $|Re(\gamma_k)| < Re(\delta_k)$, we show that there exists a characteristic value of the system (11) with positive real part. Indeed, if $Re(\gamma_k) < 0$ with $|Re(\gamma_k)| < Re(\delta_k)$, let $\lambda = \sigma + i\omega, \gamma_k = \gamma_{k1} + i\gamma_{k2}, \delta_k = \delta_{k1} + i\delta_{k2}$, where $\sigma = Re(\lambda), \gamma_{k1} = Re(\gamma_k), \delta_{k1} = Re(\delta_k)$, and $\omega = Im(\lambda), \gamma_{k2} = Im(\gamma_k), \delta_{k2} = Im(\delta_k)$, respectively. Separating the real part and imaginary part of the equation (13) we get

$$\sigma = \gamma_{k1} + \delta_{k1}e^{-\sigma\tau_*}\cos(\omega\tau_*) - \delta_{k2}e^{-\sigma\tau_*}\sin(\omega\tau_*) \tag{14}$$

$$\omega = \gamma_{k2} + \delta_{k1}e^{-\sigma\tau_*}\sin(\omega\tau_*) + \delta_{k2}e^{-\sigma\tau_*}\cos(\omega\tau_*) \tag{15}$$

Let

$$\psi(\sigma) = \sigma - \gamma_{k1} - \delta_{k1}e^{-\sigma\tau_*}\cos(\omega\tau_*) + \delta_{k2}e^{-\sigma\tau_*}\sin(\omega\tau_*) \tag{16}$$

Obviously, $\psi(\sigma)$ is a continuous function of σ. Noting that $Re(\gamma_k) = \gamma_{k1} < 0$ with $|Re(\gamma_k)| < Re(\delta_k) = \delta_{k1}$. If there is a whole number n such that $\omega\tau_* \approx 2n\pi$, then $\psi(0) = -\gamma_{k1} - \delta_{k1}\cos(\omega\tau_*) + \delta_{k2}\sin(\omega\tau_*) \approx |\gamma_{k1}| - \delta_{k1} < 0$. Since $\lim_{\sigma\to+\infty} e^{-\sigma\tau_*} = 0$, so there exists a suitably large σ, say $\sigma_1(>0)$ such that $\psi(\sigma_1) = \sigma_1 - \gamma_{k1} - \delta_{k1} e^{-\sigma_1 \tau_*} \cos(\omega \tau_*) + \delta_{k2} e^{\sigma_1 \tau_*} \sin(\omega \tau_*) > 0$. By the Intermediate Value Theorem, there exists a σ, say $\sigma_0 \in (0, \sigma_1)$ such that $\psi(\sigma_0) = 0$, implying that there is a positive real part characteristic value of the equation (13). This means that the trivial solution of system (11) is unstable. It is known that if the solution of a delayed equation is unstable for a small delay, then the instability of the solution will be maintained as the delays increase[26]. Therefore, the instability of the trivial solution of the system (11) implies the instability of the trivial solution of the system (9). This suggests that the unique positive equilibrium point $(x_1^*, x_2^*, x_3^*, \cdots, x_9^*, x_{10}^*)^T$ of system (7) is unstable. This instability of the unique positive equilibrium point together with the boundedness of the solutions will force system(7) to generate an oscillatory solution[27, 28]. The proof is completed.

To simplify, set $\mu(C) = \max_{1 \le j \le 10}[c_{jj} + \sum_{i=1, i \ne j}^{10} |c_{ij}|, \| D \| = \max_{1 \le j \le 10} \sum_{i=1}^{10} |d_{ij}|$. Then we have

Theorem 2 Assume that the conditions of Lemma 1 and Lemma 2 hold. If the following inequality is satisfied

$$\frac{\| D \| e\tau_*}{e^{|\mu(C)|\tau_*}} > 1. \tag{17}$$

Then the trivial solution of system (11) is unstable, implying that the system (7) has an oscillatory solution.

Proof To prove the instability of the trivial solution of system (11), let $w(t) = \sum_{i=1}^{10}(|x_i(t)|)$. Therefore, $w(t) > 0$ and

$$w'(t) \le \mu(C)w(t) + \| D \| w(t - \tau_*) \tag{18}$$

Specifically, consider equation

$$v'(t) = \mu(C)v(t) + \| D \| v(t - \tau_*) \tag{19}$$

Obviously, $w(t) \le v(t)$. If the trivial solution of the equation (19) is unstable, then the trivial solution of (18) is still unstable. The characteristic equation associated with equation (19) is given by

$$\lambda = \mu(C) + \| D \| e^{-\lambda \tau_*} \tag{20}$$

If the trivial solution of equation (19) is stable, then the equation (20) must have a real negative root say λ_*, and we have from (20)

$$|\lambda_*| + |\mu(C)| \ge \| D \| e^{-\lambda_* \tau_*} \tag{21}$$

One can prove that $e^x \geq ex$. So we have

$$1 \geq \frac{\parallel D \parallel e^{|\lambda_*|\tau_*}}{|\lambda_*| + |\mu(C)|} = \frac{\parallel D \parallel \tau_* e^{(|\lambda_*|+|\mu(C)|)\tau_*}}{(|\lambda_*| + |\mu(C)|)\tau_* \cdot e^{|\mu(C)|\tau_*}} \geq \frac{\parallel D \parallel e\tau_*}{e^{|\mu(C)|\tau_*}} \tag{22}$$

A contradiction with the equation (17), implying that the trivial solution of the equation (19) is unstable. It suggests that the trivial solution of the equation (18) is unstable. Thus, for all $\{\tau_i\} \geq \tau_*(i = 1, 2, \cdots, 10)$, the trivial solution of system (11) is unstable, implying that the equilibrium point of system (7) is unstable. Similarly to theorem 1, system (7) generates an oscillatory solution. The proof is completed.

4 Simulation result

This simulation is based on the system (7). Firstly, the parameters are selected as $\beta_1 = 0.45, \beta_2 = 0.58, \beta_3 = 0.42, \beta_4 = 0.36, \beta_5 = 0.38, k_1 = 1.78, k_2 = 1.95, k_3 = 1.64, k_4 = 1.85, k_5 = 1.72, \alpha_1 = 0.014, \alpha_2 = 0.015, \alpha_3 = 0.012, \alpha_4 = 0.018, \alpha_5 = 0.015; a_{22} = 0.72, a_{24} = 0.78, a_{26} = 0.85, a_{28} = 0.75, a_{2,10} = 0.82, a_{42} = 0.76, a_{44} = 0.68, a_{46} = 0.60, a_{48} = 0.75, a_{4,10} = 0.64, a_{62} = 0.32, a_{64} = 0.38, a_{66} = 0.30, a_{68} = 0.35, a_{6,10} = 0.28, a_{82} = 0.52, a_{84} = 0.48, a_{86} = 0.50, a_{88} = 0.95, a_{8,10} = 0.16, a_{10,2} = 0.52, a_{10,4} = 0.48, a_{10,6} = 0.50, a_{10,8} = 0.65, a_{10,10} = 0.62, b_{13} = -1.78, b_{15} = 0.50, b_{17} = 0.65, b_{19} = 0.62, b_{31} = 0.52, b_{35} = -1.50, b_{37} = 0.65, b_{39} = 0.62, b_{51} = 0.42, b_{53} = 0.28, b_{57} = -1.85, b_{59} = 0.72, b_{71} = 0.32, b_{73} = -1.48, b_{75} = 0.50, b_{79} = -1.62, b_{91} = 0.36, b_{93} = 0.18, b_{95} = 1.50, b_{97} = -1.65, \beta_{11} = 0.12, \beta_{12} = 0.22, \beta_{21} = 0.20, \beta_{22} = 0.18, \beta_{31} = 0.34, \beta_{32} = 0.25, \beta_{41} = 0.54, \beta_{42} = 0.75, \beta_{51} = 0.32, \beta_{52} = 0.65, \gamma_1 = 2.52, \gamma_2 = 2.65, \gamma_3 = 2.62, \gamma_4 = 2.75, \gamma_5 = 2.42, r_1 = 0.80, r_2 = 0.82, r_3 = 0.85, r_4 = 0.92, r_5 = 0.95$. Then the characteristic values of matrix C and D in system (7) are $0.0070 \pm 0.5916i, 0.0075 \pm 0.4898i, 0.0060 \pm 0.6557i, 0.0090 \pm 0.7483i, 0.0075 \pm 0.7615i$ and $0, 2.8683, 0, -0.0515 + 0.0708i, 0, -0.0515 - 0.0708i, 0, 0.2545 + 0.0254i, 0, 0, 0.2545 + 0.0254i$, respectively. Since all characteristic values of matrix C are complex numbers, and each characteristic value has positive real part, the conditions of Theorem 1 are satisfied. When time delays are selected as $\tau_1 = 1.42, \tau_2 = 1.28, \tau_3 = 1.25, \tau_4 = 1.35, \tau_5 = 1.37, \tau_6 = 1.40, \tau_7 = 1.46, \tau_8 = 1.24, \tau_9 = 1.30, \tau_{10} = 1.32$, and $\tau_1 = 1.72, \tau_2 = 1.58, \tau_3 = 1.55, \tau_4 = 1.65, \tau_5 = 1.67, \tau_6 = 1.70, \tau_7 = 1.76, \tau_8 = 1.54, \tau_9 = 1.60, \tau_{10} = 1.62$, respectively, the system (7) generates periodic oscillations (see figure 1 and figure 2). When we change the parameters as $\beta_1 = 0.15, \beta_2 = 0.18, \beta_3 = 0.12, \beta_4 = 0.16, \beta_5 = 0.10, k_1 = 0.64, k_2 = 0.75, k_3 = 0.82, k_4 = 0.92, k_5 = 0.68, \alpha_1 =$

$0.22, \alpha_2 = 0.25, \alpha_3 = 0.28, \alpha_4 = 0.20, \alpha_5 = 0.26; a_{22} = 0.32,\ a_{24} = 0.38, a_{26} = 0.35, a_{28} = 0.36, a_{2,10} = 0.54, a_{42} = 0.58, a_{44} = 0.45,\ a_{46} = 0.42, a_{48} = 0.25, a_{4,10} = 0.682 a_{62} = 0.52, a_{64} = 0.14, a_{66} = 0.22, a_{68} = 0.85, a_{6,10} = 0.46, a_{82} = 0.82,\ a_{84} = 0.78, a_{86} = 0.12, a_{88} = 0.45, a_{8,10} = 0.36, a_{10,2} = 0.48,\ a_{10,4} = 0.42, a_{10,6} = 0.38, a_{10,8} = 0.15, a_{10,10} = 0.26, b_{13} = 0.78, b_{15} = 0.42, b_{17} = 0.25, b_{19} = 0.32, b_{31} = 0.72, b_{35} = 0.34, b_{37} = 0.45, b_{39} = 0.48, b_{51} = 0.12, b_{53} = 0.48, b_{57} = 0.85, b_{59} = 0.72, b_{71} = 0.62, b_{73} = 0.48, b_{75} = 0.38, b_{79} = 0.24, b_{91} = 0.36, b_{93} = 0.58, b_{95} = 1.50, b_{97} = 0.65, \beta_{11} = 0.18, \beta_{12} = 0.24, \beta_{21} = 0.20, \beta_{22} = 0.12, \beta_{31} = 0.15, \beta_{32} = 0.14, \beta_{41} = 0.24, \beta_{42} = 0.25, \beta_{51} = 0.28, \beta_{52} = 0.25, \gamma_1 = 1.52, \gamma_2 = 1.65, \gamma_3 = 1.62, \gamma_4 = 1.75, \gamma_5 = 1.42, r_1 = 0.46, r_2 = 0.42, r_3 = 0.45, r_4 = 0.32, r_5 = 0.35$. Then we have $\| D \| = 4.85$, and $\mu(C) = 1.28$. When time delays are selected as $\tau_1 = 1.64, \tau_2 = 1.68, \tau_3 = 1.70, \tau_4 = 1.72, \tau_5 = 1.75, \tau_6 = 1.77, \tau_7 = 1.76, \tau_8 = 1.62, \tau_9 = 1.58, \tau_{10} = 1.65$, then $\tau_* = 1.58$, and $\| D \| e\tau_* = 4.85 \times 1.58e = 20.8296$, $e^{|\mu(C)|\tau_*} = e^{1.28\times 1.58} = 7.5565$, the conditions of Theorem 2 are satisfied. There is a periodic oscillatory solution (see figure 3). When time delays are selected as $\tau_1 = 1.84, \tau_2 = 1.88, \tau_3 = 1.90, \tau_4 = 1.92, \tau_5 = 1.95, \tau_6 = 1.97, \tau_7 = 1.96, \tau_8 = 1.82, \tau_9 = 1.78, \tau_{10} = 1.85$, then $\tau_* = 1.78$, and $\| D \| e\tau_* = 4.85 \times 1.78e = 23.4662$, $e^{|\mu(C)|\tau_*} = e^{1.28\times 1.78} = 9.7612$, the conditions of Theorem 2 are still satisfied. There is a periodic oscillatory solution (see figure 4). However, the oscillatory solution is not smooth to enough due to the higher degree of the variables in the model.

5 Conclusion

In this paper, we have discussed the oscillatory behavior of the solutions for a five-dimensional system of coupled van der Pol-Hamiltonian-Duffing oscillator with delays. By a mathematical analysis, we have given two theorems guaranteeing the oscillation of the solutions. The instability of the trivial solution of the linear system implyies the instability of the trivial solution of the nonlinear system. Some simulations are provided to indicate the effectiveness of the criteria. We point out that the present criteria are only sufficient conditions.

Competing Interests

Author has declared that no competing interests exist.

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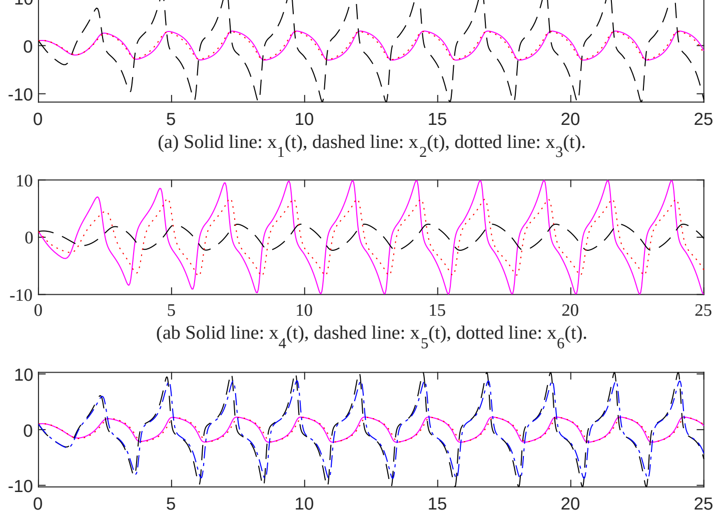


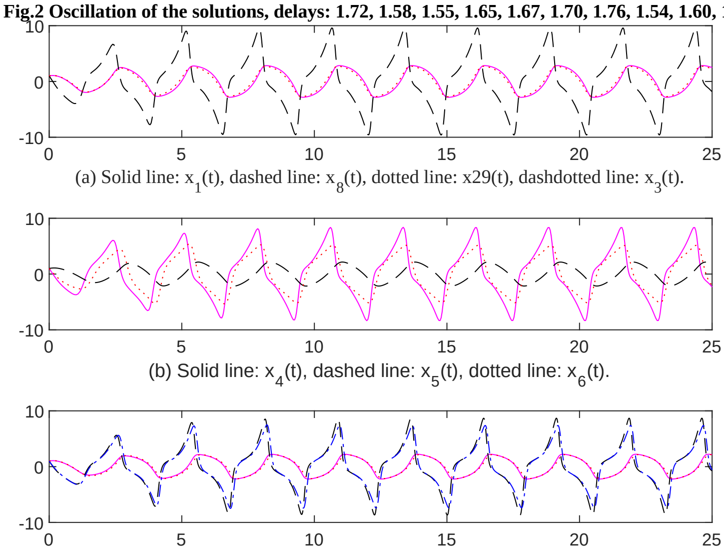
(a) Solid line: $x_1(t)$, dashed line: $x_2(t)$, dotted line: $x_3(t)$.

(ab Solid line: $x_4(t)$, dashed line: $x_5(t)$, dotted line: $x_6(t)$.

(c) Solid line: $x_7(t)$, dashed line: $x_8(t)$, dotted line: $x_9(t)$, dashdotted line: $x_{10}(t)$.

Fig.1 Oscillation of the solutions, delays: 1.42, 1.28, 1.25, 1.35, 1.37, 1.40, 1.46, 1.24, 1.30, 1.32.

Fig.2 Oscillation of the solutions, delays: 1.72, 1.58, 1.55, 1.65, 1.67, 1.70, 1.76, 1.54, 1.60, 1.62.



(a) Solid line: $x_1(t)$, dashed line: $x_8(t)$, dotted line: x29(t), dashdotted line: $x_3(t)$.

(b) Solid line: $x_4(t)$, dashed line: $x_5(t)$, dotted line: $x_6(t)$.

(c) Solid line: $x_7(t)$, dashed line: $x_8(t)$, dotted line: $x_9(t)$, dashdotted line: $x_{10}(t)$.

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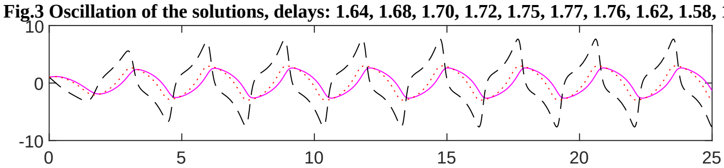
Fig.3 Oscillation of the solutions, delays: 1.64, 1.68, 1.70, 1.72, 1.75, 1.77, 1.76, 1.62, 1.58, 1.65.



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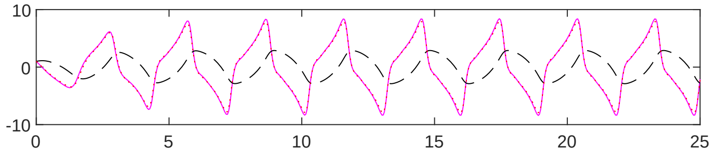

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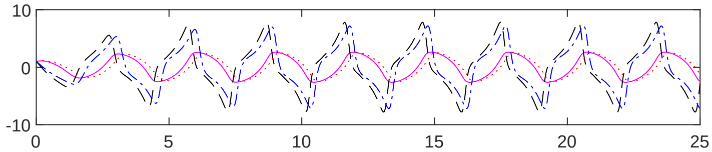
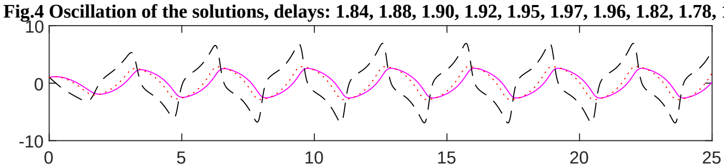


(c) Solid line: $x_7(t)$, dashed line: $x_8(t)$, dotted line: $x_9(t)$, dashdotted line: $x_{10}(t)$.

Fig.4 Oscillation of the solutions, delays: 1.84, 1.88, 1.90, 1.92, 1.95, 1.97, 1.96, 1.82, 1.78, 1.85.



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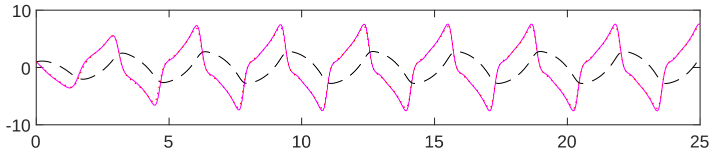

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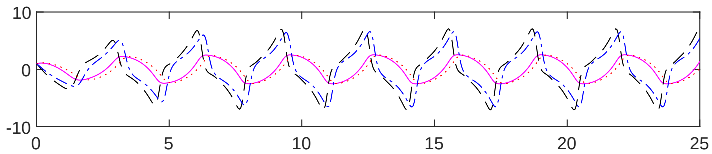

(c) Solid line: $x_7(t)$, dashed line: $x_8(t)$, dotted line: $x_9(t)$, dashdotted line: $x_{10}(t)$.