Original Research Article

Optimizedhybridblockmethods with high efficiencyforthe solutionoffirst-orderordinarydifferentialequations

ABSTRACT

Thisarticle presents optimizedhybridblockmethodsfor solvingfirst-orderordinarydifferentialequations.The derivation employed theinterpolationandcollocationtechniquesusingathree-parameterapproximation.Thehybridpointswereobtainedby minimizingthelocaltruncationerrorofthe mainmethod.The obtainedschemeswerereformulatedtoreducethenumberoffunction evaluations.The discrete schemes were derived as a by-product of the continuous scheme and used simultaneously to solve first-order initial value problems (IVPs).Theschemesareself-starting,consistent,zero-stable, andconvergent.Thenumericalresultswerecompared withsome existingtechniques and found to be more accurateandefficient.

Keywords: Linear stability,Localtruncationerror (LTE),Parameterapproximations,Initialvalueproblems(IVPs),Ordinarydifferentialequations (ODEs)

1. INTRODUCTION

Differential equationsoccur naturally in the mathematical formulation of physical phenomena in the scientific and technical domains. Finding analytical solutions to most differential equations is often challenging. The utilization of numerical techniques was necessary in order to obtain an approximate solution. Various approaches, such as collocation, interpolation, integration, and interpolation polynomials, have been thoroughly investigated in academic literature to construct continuous linear multistep methods (LMMs) for the direct solution of initial value problems in ordinary differential equations see [1,2,3,4,5,6,7,8,11] and the literature therein.

The study conducted by the author in [13] proposed a two-step method that involved the selection of two intermediate locations through the optimization of the LTEs. The method was reformulated as an R-K method, but its implementation required a greater computational cost. However, the most optimal formulation was attained through the process of reformulating the method in a manner that decreases the frequency of instances of the source term f. Upon conducting a comparative analysis between the proposed economic reformulation and the existing methods documented in the literature, it was observed that the former demonstrated a higher level of performance. In [10], the authors presented a novel optimized one-step hybrid block technique that is specifically tailored for the optimization of first-order initial value problems (IVPs). The method entailed the careful selection of three hybrid points to optimize the LTEs of the basic equations governing the behavior of the block. The technique displayed zero-stability, therefore showcasing a level of algebraic accuracy that is fifth-order. The validation of the approach's efficacy and precision was accomplished through the use of numerical illustrations. Furthermore, [20] introduced a novel one-step implicit block approach that incorporates three intra-step grid points. The major goal of the LTE was to minimize the principal term in order to attain one of the three optimal intra-step points. A revision of the methodology led to a significant decrease in computing costs while maintaining the same degrees of consistency, zero-stability, A-stability, and convergence. The method was utilized in order to tackle practical concerns, and a comparison analysis was carried out with current approaches in the literature to determine the superiority of the innovative approach. Several scholarly studies have been conducted to explore the enhancement of hybrid points by minimizing the LTE. Notable contributions in this area include the research conducted by [12,14,15,16,17,18,19,20,21].

The study conducted in our research utilizes a novel class of hybrid block techniques that contains three off-step points and employs three-parameter approximations. By implementing optimization techniques for LTE, it is possible to attain optimal hybrid points. The main aim of this work is to present an efficient method for solving initial value problems that adhere to the prescribed form.

$$x' = f(t,x), x(t_0) = x_0 \qquad (1)$$

where, $t \in [t_0, T]$, $f: [t_0, T] \times \Re \to \Re$. It isassumedthatequation(1)satisfiestheconditionsoftheexistenceanduniquenesstheoremfor initialvalueproblems (see [11,13]).

2. MATERIAL AND METHODS

This section outlines the derivation of a one-step optimized hybrid block method incorporating three optimal points for solving first-order ODEs.

To derive the method, the exact solution $x(t)$ is approximated using a polynomial Q(t) expressed as

$$x(t) \approx Q(t) = \sum_{j=0}^{k} b_j t^j \qquad (2)$$

where $b_j \in R$ arerealunknowncoefficientstobedetermined, $k = (I + C) - 1$, I and C denotethenumberofinterpolationandcollocationpointsrespectively.

The first derivative of (2) is obtained as

$$x'(t) \approx Q'(t) = \sum_{j=0}^{k} j b_j t^{j-1}, \qquad (3)$$

Let p, q, r be the off-steppoints suchthat $0 < p < q < r < 1$. To determine the coefficients b_j, equation(2) is interpolatedat t_n, and equation (3) is collocatedat $t_{n+j}, j = 0, p, q, r, 1$. This setup leads to a system of six equations with six unknown coefficients. This system may be written in matrix form as

$$\begin{pmatrix} 1 & t_n & t_n^2 & t_n^3 & t_n^4 & t_n^5 \\ 0 & 1 & 2t_n & 3t_n^2 & 4t_n^3 & 5t_n^4 \\ 0 & 1 & 2t_{n+p} & 3t_{n+p}^2 & 4t_{n+p}^3 & 5t_{n+p}^4 \\ 0 & 1 & 2t_{n+q} & 3t_{n+q}^2 & 4t_{n+q}^3 & 5t_{n+q}^4 \\ 0 & 1 & 2t_{n+r} & 3t_{n+r}^2 & 4t_{n+r}^3 & 5t_{n+r}^4 \\ 0 & 1 & 2t_{n+1} & 3t_{n+1}^2 & 4t_{n+1}^3 & 5t_{n+1}^4 \end{pmatrix} \begin{pmatrix} b_0 \\ b_1 \\ b_2 \\ b_3 \\ b_4 \\ b_5 \end{pmatrix} = \begin{pmatrix} x_n \\ f_n \\ f_{n+p} \\ f_{n+q} \\ f_{n+r} \\ f_{n+1} \end{pmatrix} \qquad (4)$$

Equation(4)is solved by theGaussianEliminationmethodtoobtainthecoefficients$b_j, j = 0,1, \ldots, 5$andputtingbackintoequation(3),thentheimplicitcontinuousschemecan be written intheform

$$Q(t) = \alpha_0(t)x_n + h(\beta_0(t)f_n + \beta_p(t)f_{n+p} + \beta_q(t)f_{n+q} + \beta_r(t)f_{n+r} + \beta_1(t)f_{n+1}\,. \qquad (5)$$

Where $\alpha_0(t), \beta_0(t), \beta_p(t), \beta_q(t), \beta_r(t), \beta_1(t)$ are continuous coefficients. Evaluatingequation(5)atthepoints$t = t_{n+p}, t_{n+q}, t_{n+r}, t_{n+1}$,yieldthefollowing

$$\begin{aligned} x_{n+p} = x_n &+ \frac{hu_1(-3p^3 + 30qr + 5p^2)(1 + q + r) - 10p(q + r + qr))f_n}{60qr} \\ &+ \frac{hp^3(3p^2 + 10qr - 5p(q + r))f_{n+1}}{60(-1 + p)(-1 + q)(-1 + r)} \qquad (6) \\ &+ \frac{hp\left(12p^3 - 30qr + 5p^2(1 + q + r) + 20p(q + r + qr))f_{n+p}\right)}{60(-1 + p)(p - q)(q - r)} \\ &+ \frac{hp^3(3p^2 + 10r - 5p(1 + r))f_{n+q}}{60(p - q)(-1 + q)q(q + r)} + \frac{hp^3\left(3p^2 + 10q - 5q(1 + q)\right)f_{n+r}}{60(p - r)(-1 + r)r(-q + r)} \end{aligned}$$

$$\begin{aligned} x_{n+q} = x_n &+ \frac{hq(5u_1\left(q^2 + 6r - q(1 + r)\right) + q(-3q^2 - 10r + 5q(1 + r))f_n}{60pr} \\ &+ \frac{hq^3\left(q(3q - 5r) - 5p(q - 2r)\right)f_{n+1}}{60(-1 + p)(-1 + q)(-1 + r)} - \frac{hq^3\left(3q^2 + 10r - 5q(1 + r)\right)f_{n+p}}{60(-1 + p)p(p - q)(p - r)} \\ &+ \frac{hq(5u_1\left(3q^2 + 6r - 4q(1 + r)\right) + s(-12q^2 - 20r + q(1 + r))f_{n+q}}{60(p - q)(-1 + q)(q - r)} \qquad (7) \\ &- \frac{hq^3(5p(-2 + q) + (5 - 3q)q)f_{n+r}}{60(p - r)(-1 + r)(-q + r)} \end{aligned}$$

$$\begin{aligned} x_{n+r} = x_n &+ \frac{hr\left(r(5q(-2 + r) + (5 - 3r)r) + 5p(-2q(-3 + r) + (-2 + r)r)\right)f_n}{60pq} \\ &+ \frac{hr^3(10pq - 5pr - 5qr + 3r^2)f_{n+1}}{60(-1 + p)(-1 + q)(-1 + r)} + \frac{hr^3(5q(-2 + r) + (5 - r)r)f_{n+p}}{60(-1 + p)p(p - q)(p - r)} \\ &- \frac{hr^3(5p(-2 + r) + (5 - 3r)r)f_{n+q}}{60(p - q)(-1 + q)(q - r)} \qquad (8) \\ &+ \frac{hr(r(3(5 - 4r)r + 5q(-4 + 3r)) + 5q\left(q(6 - 4r) + r(-4 + 3r)\right))f_{n+r}}{60(p - r)(-1 + r)(-q + r)} \end{aligned}$$

$$x_{n+1} = x_n + \frac{h(-3 + q(5-10r) + 5q + 5p(1-2r + q(-2+6r)))f_n}{60pqr} + \frac{h(12 + 15q + 15r - 20qr + 5p(3 - 4r + q(-4+6r)))f_{n+1}}{60(-1+p)(-1+q)(-1+r)} + \frac{h(3 - 5r + 5q(-1+2q))f_{n+p}}{60(-1+p)p(p-q)(p-r)} + \frac{h(3-5r+5p(-1+2q))f_{n+q}}{60(p-q)(-1+q)(q-r)} + \frac{h(3-5q+5p(-1+2q))f_{n+r}}{60(p-r)(-1+r)(-q+r)} \quad (9)$$

where, $f_{n+j} = f(t_{n+j}, x_{n+j})$, for $j = p, q, r, 1$, and $x_{n+j} \approx x(t_n + jh)$ are approximations of the exact solution. Expanding the main formula $x(t_{n+1})$ in the Taylor series around t_n.

$$\mathcal{L}(x(t_{n+1}); h) = \frac{1}{7200}(-2 + 3p + 3q - 5pq + 3r - 5pr - 5qr + 10pqr)x^6[t_n]h^6 + \frac{1}{302400}(-24 + 21p + 21p^2 + 21q - 14pq - 35p^2q + 21p^2 - 35pp^2 + 21r)x^7[t_n]h^7 + \frac{1}{302400}(-14pr - 35p^2r - 14qr + 70p^2qr - 35q^2r + 70pq^2r)x^7[t_n]h^7 + \frac{1}{302400}(21r^2 - 35pr^2 - 35qr^2 + 70pqr^2)x^7[t_n]h^7 + O(h)^8. \quad (10)$$

Setting the principal term of the LTE in (10) to zero yields the following equation in three unknowns:

$$\frac{1}{7200}(-2 + 3p + 3q - 5pq + 3r - 5pr - 5qr + 10pqr) = 0 \quad (11)$$

$$q = \frac{2 - 3p - 3r + 5pr}{3 - 5p - 5r + 10pr} \quad (12)$$

while the other two parameters are given as

$$p = \frac{1}{10}(5 - \sqrt{5}); r = \frac{1}{10}(5 + \sqrt{5}) \quad (13)$$

Substituting equation (13) into equation (12), we get $q = \frac{1}{2}$.

The LTE of the main formula in equation (9) is computed by substituting the values of the parameters p, q, r into equation (10) to obtain

$$\mathcal{L}(x(t_{n+1}); h) = -\frac{x^7[t_n]h^7}{1512000} + O(h)^8. \quad (14)$$

Lastly, putting the values of the parameters p, q, r into equations (6) - (9) we get the following one-step optimal hybrid block method:

$$x_{n+p} = x_n + \frac{h}{3000}\Big((275+\sqrt{5})f_n + (625+95\sqrt{5})f_{n+p} - 192\sqrt{5}f_{n+q} + (625-205\sqrt{5})f_{n+r} + (-25+\sqrt{5})f_{n+1}\Big),$$

$$x_{n+q} = x_n + \frac{h}{192}\big(17f_n + (40+15\sqrt{5})f_{n+p} + (40-15\sqrt{5})f_{n+r} - f_{n+1}\big), \qquad (15)$$

$$x_{n+r} = x_n + \frac{h}{3000}\Big((275-\sqrt{5})f_n + (625+205\sqrt{5})f_{n+p} + 192\sqrt{5}f_{n+q} + (625-95\sqrt{5})f_{n+r} - (25+\sqrt{5})f_{n+1}\Big),$$

$$x_{n+1} = x + \frac{h}{12}\big(f_n + 5f_{n+p} + 5f_{n+r} + f_{n+1}\big).$$

The hybridblockmethod in(15) is reformulated to reduce the frequency of f.This procedure is believed to reduce the number of function evaluation and hence the computing time. Thus, we obtain the modified optimal hybrid block method (MOHBM) as given in (16) below:

$$hf_{n+p} = -\frac{1}{10}\Big(2hf_n + (21+\sqrt{5})x_n + (-25+15\sqrt{5})x_{n+p} + \big(32-32\sqrt{5}\big)x_{n+q} + (-25+15\sqrt{5})x_{n+r} + (-3+\sqrt{5})x_{n+1},$$

$$hf_{n+q} = \frac{1}{16}\big(2hf_n + 20x_n + (-25-25\sqrt{5})x_{n+p} + 32x_{n+q} + (-25+25\sqrt{5})x_{n+r} - 2x_{n+1}\big), \qquad (16)$$

$$hf_{n+r} = \frac{1}{10}\Big(-2hf_n + (-21+\sqrt{5})x_n + (25+15\sqrt{5})x_{n+p} - \big(32+32\sqrt{5}\big)x_{n+q} + (25+15\sqrt{5})x_{n+r} - (3+\sqrt{5})x_{n+1},$$

$$hf_{n+1} = hf_n + 9x_n - 25x_{n+p} + 32x_{n+q} - 25x_{n+r} + 9x_{n+1}.$$

3.Analysisofthe basic properties of the methods

In what follows,thebasicpropertiesoftheOHBM(15)(equivalently, theMOHBM(16))which includeaccuracy,consistency,zero-stability,convergence,linearstability, and*A*-stability are examined.

3.1 Orderofaccuracyandconsistency

RewritingtheOHBM(15)inthematrixdifferenceformyields

$$A_1X_n = A_0X_{n-1} + h(B_0F_{n-1} + B_1F_n), \qquad (17)$$

Where $A_0, A_1, B_0,$ and B_1 are 4×4 matrices given by

$$A_0 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}; A_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}; B_0 = \begin{pmatrix} 0 & 0 & 0 & \frac{275+\sqrt{5}}{3000} \\ 0 & 0 & 0 & \frac{17}{192} \\ 0 & 0 & 0 & \frac{275-\sqrt{5}}{3000} \\ 0 & 0 & 0 & \frac{1}{12} \end{pmatrix} \qquad (18)$$

$$B_1 = \begin{pmatrix} \frac{625+95\sqrt{5}}{3000} & \frac{-192\sqrt{5}}{3000} & \frac{625-205\sqrt{5}}{3000} & \frac{275+\sqrt{5}}{3000} \\ \frac{40+15\sqrt{5}}{192} & 0 & \frac{40-15\sqrt{5}}{192} & \frac{-1}{192} \\ \frac{625-205\sqrt{5}}{3000} & 0 & \frac{625-95\sqrt{5}}{3000} & -\frac{(25+\sqrt{5})}{3000} \\ \frac{5}{12} & 0 & \frac{5}{12} & \frac{1}{12} \end{pmatrix} \quad (19)$$

$$X_n = (x_{n+p}, x_{n+q}, x_{n+r}, x_{n+1})^T,$$

$$X_{n-1} = (x_{n-1+p}, x_{n-1+q}, x_{n-1+r}, x_n)^T,$$

$$F_n = (f_{n+p}, f_{n+q}, f_{n+r}, f_{n+1})^T, \quad (20)$$

$$F_{n-1} = (f_{n-1+p}, f_{n-1+q}, f_{n-1+r}, f_n)^T.$$

Forasufficientlydifferentiabletestfunction$m(t_n)$intheinterval$[0, T]$,Letthedifferenceoperator $\bar{D}$fortheOHBMin(17) be givenas

$$\bar{D}(m(t_n); h) = \sum_{j=0,p,q,r,1} \left[\bar{\xi}_j(t_n + jh) - h\bar{\mu}_j m'(t_n + jh)\right], \quad (21)$$

Where,$\bar{\xi}_j$ and $\bar{\mu}_j$ are column vectors of the matrices A_0 and A_1, respectively. The Taylor series expansion about $t_n$for $x(t_n + jh)$ and $x'(t_n + jh)$ yield

$$\bar{\mathcal{L}}(m(t_n); h) = c_0 x(t_n) + c_1 h x'^{(t_n)} + c_2 h^2 x^{(2)}(t_n) + \cdots + c_p h^p x^{(p)}(t_n) + \cdots \quad (22)$$

where $c_i, i = 0,1,2, \ldots$ are vectors. From equation (22), the order of the OHBM is $p = (5,5,5,6)^T$ with the error constant

$$c_{p+1} = \frac{1}{180000}, \frac{1}{180000}, \frac{1}{230400}, \frac{-1}{1512000} \quad (23)$$

Showing that the OHBM has at least fifth order accuracy.

3.2 Zero-stabilityandconvergence

The concept of zero-stability pertains to the characteristics exhibited by a procedure when the value of h approaches zero. In the context of a homogeneous equation$x' = 0$,thediscretizedformis

$A_1 X_n - A_0 X_{n-1} = 0$(24)

where A_0andA_1aregiveninequations(18).Thefirstcharacteristicpolynomial$\rho(\sigma) = det(\sigma A_1 - A_0) = \sigma^3(\sigma - 1) = 0$.Thisimpliesthat $\sigma_1 = \sigma_2 = \sigma_3 = 0, \sigma_4 = 1$.

Since the OHBM and the MOHBMsatisfythepropertiesofconsistencyandzero-stability,then the methods areconvergentaccordingto [9].

3.3 Linearstabilityandorderstars

The concept of linear stability focuses on the performance of a method in real-world scenarios, where it is crucial to ascertain if the method will produce desirable outcomes for a given positive value of h. To validate this concept, commonly known as linear stability, we applythe proposedblock method on a linearized test problem.

$x(t) = \sigma x(t), Re(\sigma) < 0$(25)

Yieldingarecurrencerelation given as

$$X_n = H(\hbar)X_{n-1}, \hbar = \sigma h. \quad (26)$$

wherethematrix$H(\hbar)$isgivenby$(A_1 - rB_0)^{-1}(A_0 - rB_0)$.Thestabilitypropertyofthismatrix'seigenvalues,whichgovernshowthenumericalsolutionbehaves,isthespectralradius,$H(\hbar)$,usedinthemethodtodefinetheregionofabsolute stability$S$.ThemethodisA-stableif

$$S = \{\hbar \in C: |\rho[H(\hbar)]| < 1\} \quad (27)$$

Upon performing various calculations, it becomes evident that the predominant eigenvalue can be expressed as a quotient function.

$$\rho[H(\hbar)] = \frac{\hbar^4 + 16\hbar^3 + 132\hbar^2 + 600\hbar + 1200}{\hbar^4 - 16\hbar^3 + 132\hbar^2 - 600\hbar + 1200} \quad (28)$$

which hasa modulus of lessthanoneinC$^-$(seeFigure1).Hence,theOHBM(16)isA-stable.

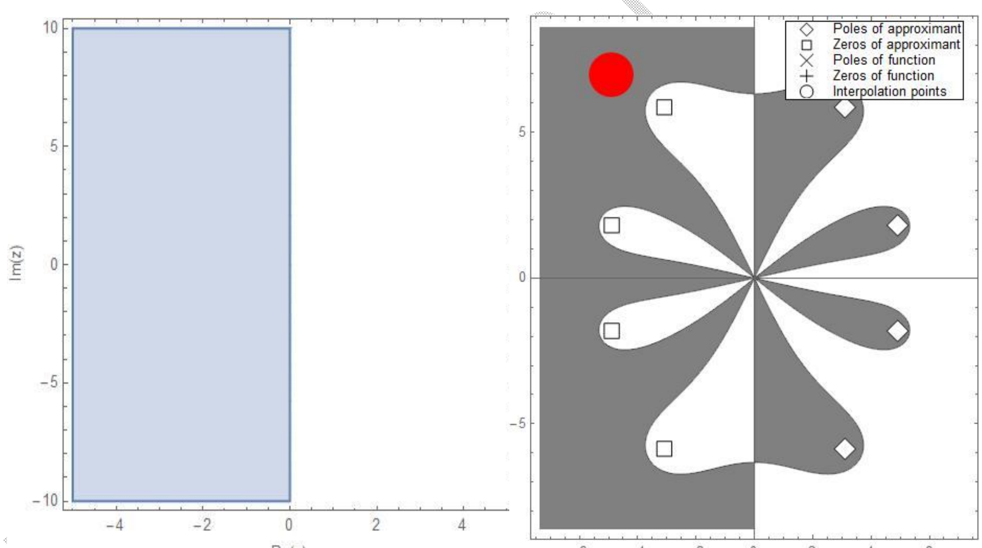


Fig. 1: (a) Region of absolute stability (b) Order star for OHBM

4. RESULTS AND DISCUSSION

Inthe sequel,theaccuracyoftheproposedmethodswillbedemonstratedbyimplementationinsolvingsomepopularappliedproblemsoftheformin equation (1).ThemethodsbeingcomparedaretheOHBM (16),theMOHBM (17),the OSBM in [10] and BHMO and RBHMO in [13].

Tomeasuretheperformanceofeachoftheaforementionedmethods, maximumglobalabsoluteerror(MAbErr),absoluteerroratthefinalgridpoint (AbErrF),andtheCPUtimeinseconds are computedfor each of the test problems.

Problem 4.1

Giventhefirst-order ODE which has appeared in [8,10]:

$$x'(t) = -10(x-1)^2, \quad x(0) = 2. \qquad (29)$$

The exact solution is $x(t) = 1 + \frac{1}{1+18x}$. The problem is solved in the interval [0,0.1] taking $n = 10, 20, 40$. The MAbErr, AbErrF, andCPUtime are computedusing the methods OHBM, MOHBM, and OSBM, and results presented in Table 1. The efficiency curves of MAbErr and CPU time are represented in Fig 3a. The figure indicates that the OHBM and MOHBM outperform existing methods with respect to accuracy and computing time.

Problem 4.2

Given the first-order ODE which has appeared in [6,10]:

$$x'(t) = tx, \quad x(0) = 1 \qquad (30)$$

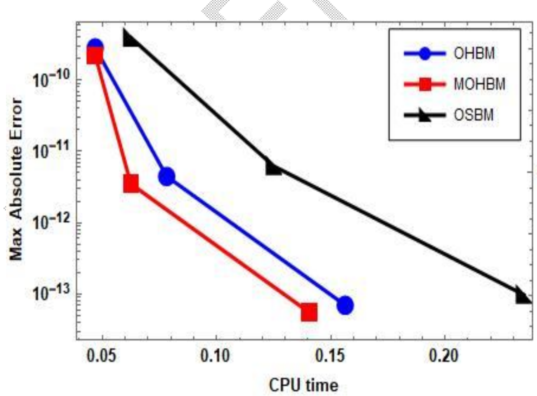
The exact solution $x(t) = e^{\frac{1}{2}t^2}$. The problem is solved in the interval [0,1] for step sizes $n = 20, 40, 80$, with theMAbErr, AbErrF, andCPUtime computedusing the methods OHBM, MOHBM, BHMO and RBHMO, and results presented in Table 2. The efficiency curves of AbErr and CPU time are represented in Fig 4. The figure reveals that the MOHBM outperform existing method with respect to accuracy and computing time.

Problem 4.3

Given the nonlinear problem investigated by Akinfenwa and Jator. (2011):

$$x'(t) = -\frac{x^3}{2}, \quad x(0) = 1, \qquad (31)$$

with exact solution $x(t) = 1/\sqrt{t+1}$, The problem is solved in the interval [0,4] taking $n = 20, 40, 80, 100$. The MAbErr, AbErrF, andCPUtime are computedusing the methods OHBM, MOHBM, and OSBM, and results presented in Tables 3. The efficiency curves of MAbErr and CPU time are represented in Fig 4a. As revealed by the figure, the OHBM and MOHBM outperform existing method with respect to accuracy and computing time.



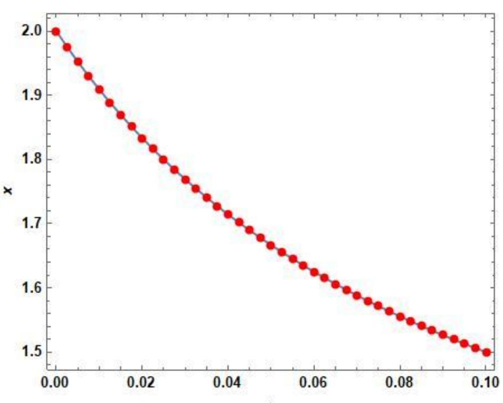
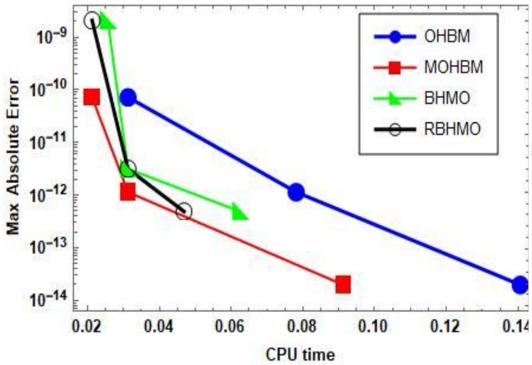


Fig.2a:Efficiencyplot forProblem 4.1 Fig.2b:Solution plot forProblem 4.1



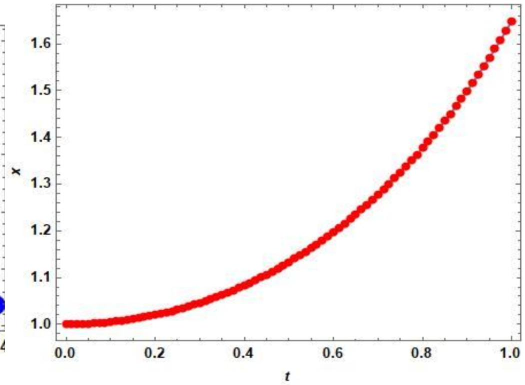
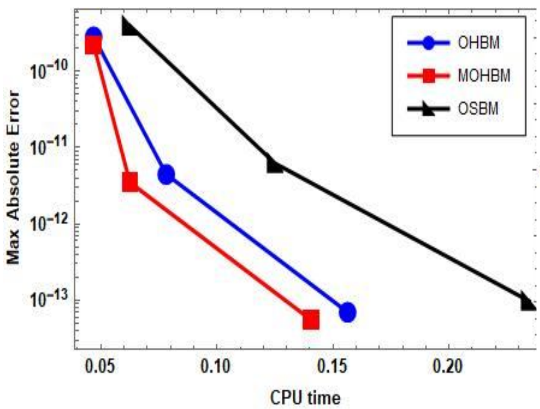


Fig.3a:Efficiencyplot forProblem 4.2 Fig.3b:Solution plot forProblem 4.2



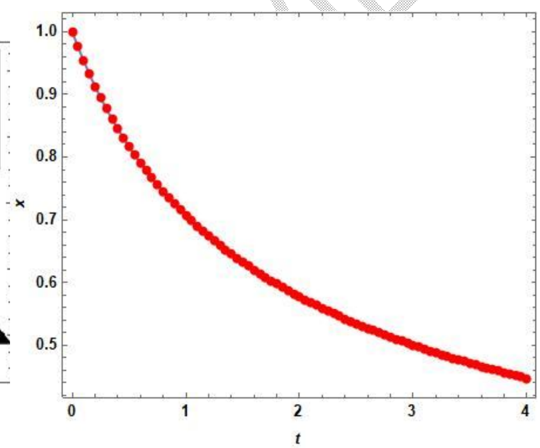


Fig.4a:Efficiencyplot forProblem 4.3 Fig.4b:Solution plot forProblem 4.3

Table1: The MAbErr, AbErrF,andCPUtimeforProblem 4.1usingdifferent methods and number of steps(n)

n	Method	MAbErr	AbErrF	MErr	Norm	CPUtime
10	OHBM	2.85272E-10	1.5998E-10	2.06271E-10	7.29646E-10	4.687E-02
	MOHBM	2.22905E-10	1.25002E-10	1.61175E-10	5.70126E-10	4.687E-02
	OSBM	3.96097E-10	2.22137E-10	2.86408E-10	1.01311E-09	6.250E-02
20	OHBM	4.49196E-12	2.51776E-12	3.37380E-12	1.61640E-11	7.813E-02
	MOHBM	3.50919E-12	1.96665E-12	2.63545E-12	1.26266E-11	6.250E-02
	OSBM	6.23834E-12	3.49609E-12	4.68514E-12	2.24468E-11	1.250E-01
40	OHBM	6.99441E-14	3.97460E-14	5.34478E-14	3.54737E-13	1.563E-01
	MOHBM	5.50671E-14	3.06422E-14	4.16740E-14	2.76761E-13	1.406E-01
	OSBM	9.79217E-14	5.44009E-14	7.47315E-14	4.96271E-13	2.344E-01

Table 2: TheAbErr, FErr, andCPUtimeforProblem 4.2 usingdifferent methods and number of steps(n)

n	Method	MAbErr	AbErrF	MErr	Norm	CPUtime
20	OHBM	7.28471E-11	7.28471E-11	1.44385E-11	8.79531E-11	3.125E-02

	MOHBM	7.28471E-11	7.28471E-11	1.44385E-11	8.79531E-11	2.125E-02
	BHMO	2.10081E-09	2.10081E-09	3.96744E-10	2.49730E-09	2.563E-02
	RBHMO	2.10081E-09	2.10081E-09	3.96744E-10	2.49730E-09	2.125E-02
40	OHBM	1.14420E-12	1.14420E-12	2.06914E-13	1.72935E-12	7.813E-02
	MOHBM	1.14420E-12	1.14420E-12	2.06914E-13	1.72935E-12	3.125E-02
	BHMO	3.16660E-11	3.16660E-11	5.54340E-12	4.73412E-11	3.125E-02
	RBHMO	3.16660E-11	3.16660E-11	5.54340E-12	4.73412E-11	3.125E-02
80	OHBM	1.97620E-14	1.97620E-14	3.49856E-15	3.99421E-14	1.406E-01
	MOHBM	1.97627E-14	1.97627E-14	3.49856E-15	3.99421E-14	9.125E-02
	BHMO	4.89608E-13	4.89608E-13	8.34184E-14	9.76046E-13	6.250E-02
	RBHMO	4.89608E-13	4.89608E-13	8.34184E-14	9.76046E-13	4.687E-02

Table3: The MAbErr, AbErrF,andCPUtimeforProblem 4.3usingdifferent methods and number of steps(n)

n	Method	MAbErr	AbErrF	MErr	Norm	CPUtime
20	OHBM	2.59459E-09	4.73998E-10	1.1552E-09	6.21496E-09	7.813E-02
	MOHBM	2.02819E-09	3.70502E-10	9.02985E-10	4.85811E-09	7.813E-02
	OSBM	3.60018E-09	6.57769E-10	1.60301E-10	8.62403E-09	7.813E-02
40	OHBM	4.19044E-11	7.61163E-12	1.91781E-11	1.41802E-10	1.718E-01
	MOHBM	3.27428E-11	5.94735E-12	1.49850E-11	1.10798E-10	1.250E-01
	OSBM	5.8186E-11	1.05694E-11	2.66302E-11	1.96901E-10	1.718E-01
80	OHBM	6.63469E-13	1.19516E-13	3.05738E-13	3.15630E-12	3.437E-01
	MOHBM	5.18030E-13	9.34253E-14	2.38810E-13	2.46531E-12	2.500E-01
	OSBM	9.21374E-13	1.66145E-13	4.24857E-13	4.38475E-12	3.125E-01

5. CONCLUSION

The research haspresentedthe optimalhybridblockmethod, andthe modifiedoptimalhybridmethodforsolvingfirst-orderinitialvalueproblemsofODEs. Theresults inTables 1, 2, and 3 revealthat themethods OHBM (15), and MOHBM (16), are highlyefficientwithminimalerrors.Furthermore,themodifiedmethod(16)apartfromhavingminorerrorsalsoreducedthecomputationaltimewhichisanaddedadvantageto the performance of the OHBM. Thederivedmethodswereimplementedinblockmodeswiththemeritofbeingself-startingandthusrequirednostartingvalues.Themethodshavegoodaccuracypropertiesandareindeedofthehigherorderofaccuracy at the final grid point where the LTEs were optimized. This is amajor advantage of the method.

Also, the methodsdonotrequirethecreationofseparatepredictors.TheMOHBMshowedthattheefficiencyofthemethodisdependentontheimplementationstrategies. Themethod isadvantageouswhenthe major concern is toreduce the numberoffunctionevaluations,andthe computingtime.Hence,thetechniquesarestronglysuggestedforgeneraluse. The Mathematicasoftwarepackage version 12.1wasusedtodeveloptheschemes, the plotsandtheresults on Windows Operating System withProcessor Intel(R) Core (TM) i5-

4310U CPU @ 2.00GHz, 2601 Mhz, 2 Core(s), 4 Logical Processor(s) having 8.0GB installed RAM.

Disclaimer (Artificial intelligence)

Author(s) hereby declare that NO generative AI technologies such as Large Language Models (ChatGPT, COPILOT, etc.) and text-to-image generators have been used during the writing or editing of this manuscript.

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