**EXPONENTIAL-GAMMA-RAYLEIGH DISTRIBUTION: THEORY AND PROPERTIES**

**ABSTRACT**

**The use of traditional probability models to forecast real-world events is causing growing dissatisfaction among scholars. One of the motives could be the tail characteristics and goodness of fit metrics has a constraining tendency. Subsequently, there has been a significant increase in the generalisation of well-known probability distributions in recent years. The challenge is finding families versatile enough to fit both skewed and symmetric data. It is essential to understand that most generalised distributions described in the literature were developed using the generalised transformed transformer (T-X) method. This method was proposed by Alzaatreh et al. (2013). Also, Adewusi et al. (2019) showed that this generalization approach is beneficial by transforming the Exponential-Gamma distribution developed by Ogunwale et al. (2019) to a family of distribution known as the Exponential-Gamma-X. Therefore, in this study, we focused on developing a new family of continuous distributions called the Exponential-Gamma-Rayleigh distribution by transforming the newly generated continuous T-X family of distribution called the Exponential-Gamma-X distribution using the traditionally existing Rayleigh distribution as a transformer “X”. Several expressions for the new distribution’s theory and properties were explored and obtained; the maximum likelihood estimation approach was used to estimate the distributions' parameters,** **and finally, simulations studies were conducted to assess the asymptotic behaviour of the estimates.**

**Keywords: *Exponential-Gamma-Rayleigh, Statistical properties, Maximum Likelihood Estimation***

 **INTRODUCTION**

A statistical distribution describes how values in a given field are distributed. In another way, the statistical distribution reveals which values are common and uncommon. e. Statistical methods are used in almost every applied sector, including healthcare sciences, reliability engineering, and related fields of hydrological events, sports sciences, information technology, economics, and finance, banking sector, business management, agriculture, and mesosphere and lower thermosphere weather observations. Several researchers have generated new adaptable distributions from existing distributions using various modification techniques to increase their flexibility in modeling data. These adaptable distributions are created by adding extra parameters to the baseline distribution with generators or combining two distributions (Ali, *et al*., 2021). These modified distributions can model data sets efficiently and in most cases, provide the best fit to data sets. When applied because they have more parameters and are more adaptable than their baseline distributions. Introducing an extra parameter to an existing distribution has proven beneficial in studying tail properties and improving the goodness of fit of the compound distribution. Many applications, such as lifetime analysis require extended forms of these distributions. Most statistical distribution modelling approaches are concerned with determining which probability distribution best represents the data. No single probability distribution, however, can fit all types of data. As a result, new classical distributions must be developed or created (Nasiru, 2018). Many researchers as work on developing new distribution examples are Fayomi, *et al*., (2022), introduced a generalization of the Gumbel distribution based on the T-X model. They used a linear combination of the generalized exponential distributions to characterize the density of a new family; they used simulated assessment for the suggested distribution and also applied three real-world data sets in modelling the proposed model in order to ensure its authenticity and superiority over the existing ones. Muhammad, *et al*., (2022), suggested a new life-time model called modified Frechet–Rayleigh distribution (MFRD) using the modified Frechet technique. Their results indicated that the new model performed better than various versions of Rayleigh distribution. Proloy and Shreya (2022) in their study introduced a generalization known as Exponential Transformed Inverse Rayleigh distribution They concluded that their new distribution performs better than the existing models regarding model fit. Albalawi, *et al*., (2022), in their study introduced a classical generalized logarithmic transformation exponential distribution. Alfred and Aliyu (2022), proposed and studied a new generalization of Maxwell and Lomax distributions using the T-X method. They found that the proposed distribution offers a better fit than the other competing extensions of Lomax distributions considered in the study. Idika, *et al*., (2021), in their used the standard transformation of a random variable technique to obtain a partially bounded one-parameter version of the bounded three-parameter power function distribution. Ibeh, *et al*., (2021), in their study, introduced a new three parameters distribution in the family of generalized exponential distributions generated using the transformed-transformer technique. Therefore Adewusi et al. (2019) show that this generalization approach is beneficial by transforming the Exponential-Gamma distribution developed by Ogunwale et al. (2019) to a family of distribution known as the Exponential-Gamma-X. Therefore, in this study, we aimed to develop a new family of continuous distributions called the Exponential-Gamma-Rayleigh distribution, raising from transforming the Exponential-Gamma-X using the Rayleigh distribution.

**Methodology**

In this study, we proposed and explored a new generalization of the Exponential-Gamma and Rayleigh distributions using the T-X method.

**Theorem 1:** Let be continuous independent random variable such that; follows an Rayleigh distribution and, let  and  be the probability density function and cumulative distribution function of Rayleigh distribution given as:

**** (1)

and (2)

Similarly, Adewusi, *et al*., (2019), defined a generalized T-X family of Exponential-Gamma-X distribution as:

 (3)

Where in (3) is the PDF of Exponential-Gamma-X family of distribution,  and  are the pdf and the survival function of the baseline distribution.

Inserting (1) and (2) into (3) above, then probability density function, pdf of the Exponential-Gamma-Rayleigh distribution is given as:

 (4)

For convenient computation, let

  then,

 (5)

is the PDF of the new EGRD. The distribution has three parameters namely: (Shape), (scale) and (location) .

**B Statistical properties**

In this section, the statistical properties of EGRD, particularly the first four moments, variance, and coefficient of variation, moment generating function, characteristic function, skewness, and kurtosis are obtained.

1. **Moment**

**Theorem 2:** If  is a random variable distributed as an EGRD,having parameters then the non-central moment of X is given by:

 (6)

**Proof:**

 (7)

  (8)

Let  then, (8) reduces to:

 (9)

Since , based on gamma function, then

  (10)

Substituting r =1, 2, 3 and 4 in equation (10 ) we obtain the first (mean), second, third and the fourth moments by for EGRD: we obtain the variance by the association



 Mean =  (11)

  (12)

 (13)

 (14)

  (15)

Then the 3rd and the 4th moment is given as:

 (16)

 respectively (17)

**(ii). Moment generating function**

**Theorem 3:** If  is a continuous random variable distributed as an EGRD, then the moment generating function is given as 

  (18)

  (19)

Let, then  so that (19) is reduced to

 (20)

 (21)

1. **Characteristic Function (CF)**

**Theorem 5:** If is a random variable distributed as an EGRD, then the characteristics function  is defined as 

**Proof:**

  (22)

 (23)

Let, then  so that (3.16) is reduced to

 (24)

 (25)

1. **Coefficient of Variation (C.V)** is a standardized measure of dispersion of a probability distribution and is given as:

  (26)

  (27)



**(v) Skewness and Kurtosis**

**Skewness**is a measure of the asymmetry of the probability distribution of a real-valued random variable about its mean and is given as:

  (29)



**Kurtosis** is a descriptor of the shape of a probability distribution and is given as;

  (31)

(32)

 (33)

1. **Cumulative Distribution Function (CDF)**

The cumulative distribution function of a random variable *X* evaluated at *x* is the probability that *X* will take a value less than or equal to *x* and is defined as;

 (34)

 **Theorem 6**: If is a continuous random variable from the Exponential-Exponential distribution, the cumulative density function (CDF) is defined by

 (35)

**Proof:**

  (36)

 let   then  so (3.19) is reduced to:

 (37)

, ****  (38)

PDF



 ***Fig 1. PDF plots of EGRD***

CDF



 ***Fig 2. CDF plots for EGRD***

From figure 1The plot reveals various possible shapes of the EGR density function, including (approximately) symmetric, skewed, and unimodal were produced. It can be seen that the tail of the distribution is longer and shorter on both right side and left side for different combinations of the values of the parameters. This demonstrates the great flexibility of the EGR distribution, which makes it suitable for various real data and CDF plot shown in Figure 2 that EGRD starts from zero on the y axis and tend to 1 on x axis , which is an indication that the EGRD is a valid distribution because it satisfies the basic property of a valid probability distribution which states that the probability of any event is greater than or equal to zero and the sum of the cumulative probabilities of events is equal to one

1. **Reliability Function**

The reliability function also known as survival function is a function that measures the likelihood that a patient, device, or other object of interest will survive beyond a specific time range and is defined as:

  (39)

 whereis the cumulative distribution function of *X,* substituting,

  (40)

1. **Hazard Function**

The **hazard function** also called theforce of mortality*,*instantaneous failure rate*,*instantaneous death rate, or age-specific failure rate is the instantaneous risk that the event of interest happens, within a very narrow time frame and is defined as;

  (41)

 where  and are pdf and survival function of EGRD then,

  (42)



1. **THE MAXIMUM LIKELIHOOD ESTIMATOR**

 **Theorem 7:** Let  be a random sample of size n from Exponential-Gamma- Rayleigh distribution (EGRD) with pdf

 (44)

By taking the natural logarithm of (44), the log-likelihood function is obtained as;

 (45)

Therefore, the MLE which maximizes (3.35) must satisfy the following normal equations;

  (46)

  (47)

  (48)

Differentiating equation (45) with respect to λ and give the maximum likelihood estimates of the model parameters that generate the solution of the nonlinear system of equations. The parameters can be estimated numerically by solving (46), (47), and (48), while solving it analytically is very cumbersome and tasking. The numerical solution can also be obtained directly using some data sets in Python but there are other programming languages that could do the work.

Simulations were conducted to assess the accuracy of the maximum likelihood estimators (MLEs) for the parameters of the EGRD distribution. The primary goal was to determine whether the MLEs consistently converge toward the actual parameter values as the sample size increases. In this simulation study, 1000 samples were generated using sample sizes ranging from 20, 50, and 100 for EGRD distributions. The performance of estimates is evaluated based on their bias of the MLEs of the model parameter for the simulation study; the empirical means and standard deviation of the parameters were obtained as follows in Table

**Table 1: Empirical means and standard deviations (in parentheses) for EGRD distributions.**

|  |  |  |  |
| --- | --- | --- | --- |
| **Parameters** | **n= = 20** | **n = 50** | **n = 100** |
|  **, ,**  |  |  |  |  |  |  |  |  |  |
| 1,0.5,0.5,1 | **1.1628** | **0.367** | **0.5936** | **1.0871** | **0.6725** | **0.8277** | **1.0633** | **0.5324** | **0.5025** |
| (0.651) | (0.3055) | (0.2445) | (0.3557) | (0.2362) | (0.351) | (0.2411) | (0.2367) | (0.2153) |
| 0.5,0.5,0.5,0.5 | **0.6255** | **0.6425** | **0.8051** | **0.6025** | **0.6026** | **0.5365** | **0.4555** | **0.501** | **0.5222** |
| (0.2811) | (0.2261) | (0.3131) | (0.1275) | (0.1562) | (0.2655) | (0.1221) | (0.1265) | (0.216) |
| 2,2,2,2 | **2.0556** | **1.5851** | **2.0255** | **2.0225** | **1.5662** | **2.0122** | **2.0165** | **1.7632** | **1.5562** |
| (0.2565) | (0.2106) | (0.1611) | (0.2502) | (0.1725) | (0.1181) | (0.3224) | (0.1372) | (0.1182) |
| 3,0.1,2,0.1 | **3.0265** | **0.0656** | **1.6626** | **3.0055** | **0.1011** | **2.013** | **3.114** | **0.1025** | **1.6557** |
| (0.2647) | (0.1157) | (0.0325) | (0.2002) | (0.0281) | (0.0206) | (0.1157) | (0.0222) | (0.0353) |
| 1,2,3,4 | **1.2061** | **1.5588** | **2.1246** | **1.0516** | **2.0235** | **2.0262** | **1.0512** | **1.6555** | **3.0223** |
| (0.2122) | (0.2654) | (0.1512) | (0.2507) | (0.2012) | (0.1524) | (0.2077) | (0.153) | (0.1226) |
| 4,3,2,1 | **4.0175** | **3.0045** | **1.7684** | **4.015** | **3.0126** | **1.9556** | **4.0156** | **3.0162** | **1.6661** |
| (0.1576) | (0.1574) | (0.1649) | (0.0516) | (0.0515) | (0.1032) | (0.0347) | (0.0624) | (0.0556) |

the simulation study demonstrates that with larger sample sizes, the empirical means approach the true parameter values more closely and the estimates become more consistent, as indicated by the decreasing standard deviations. These findings strongly suggest that the maximum likelihood method is highly effective for estimating the Exponential Gamma Rayleigh Distribution (EGRD) parameters

 **CONCLUSION**

A new extension of the T-X family of the Exponential-Gamma-X distribution named Exponential-Gamma-Rayleigh distribution comprising three parameters has been developed and studied. We explored and generated several expressions for distribution theory and properties including the first four moment, moment generating function, characteristics function cumulative distribution function skewness, kurtosis ,Hazard function, Reliability function and the maximum likelihood estimation approach was use to estimate the parameters of the distribution, and simulation demonstrating that, as sample sizes increase, the empirical means converge to the true parameter values, and biases and mean squared errors (MSEs) approach zero. Additionally, the standard deviations decreased in all cases with larger sample sizes, confirming that the Exponential-Gamma-Rayleigh distribution provides stable and reliable parameter estimates.

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1.

2.

3.

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