Some Inequalities via Functional Type Generalization of Cauchy-Bunyakovsky-Schwartz Inequality

Original Research Article

ABSTRACT

The Cauchy-Bunyakovsky-Schwartz inequality and its various refinements are very important in mathematical analysis. In this work, we first introduce an inequality of the form

$$[f^{(n)}(x)]^2 \leq k(x) \sum_{k=0}^{m} a_k\, f^{(m-k)} \left(\frac{p}{r}x + q\right) \sum_{k=0}^{l} b_k\, f^{(l-k)} \left(\left(\frac{2}{r} - \frac{p}{r}\right) x - q\right)$$

and by using a functional type generalization of the Cauchy-Bunyakovsky-Schwartz inequality we get some inequalities for derivatives of a one-parameter deformation of the Gamma function to satisfy the introduced inequality. Also, we show that the established results are generalizations of some previous results.

Keywords: Cauchy-Bunyakovsky-Schwartz inequality; Gamma function; v-Gamma function; inequality

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1 INTRODUCTION

The functional type Cauchy-Bunyakovsky-Schwartz inequality is given in [15] as

$$\left(\int_a^b f(t)g(t)dt\right)^2 \leq \int_a^b f^2(t)dt \int_a^b g^2(t)dt \tag{1.1}$$

on the space of continious real valued functions $C[a, b]$.It is one of the fundamental mathematical inequalities used in different branches of mathematics, as well as in physics, engineering, and statistics. In recent years, many generalizations of the inequality (1.1) have been given, for example, see [1, 2, 8, 17, 18]. One of the generalization of the equation (1.1) is given in [12] as

$$\int_a^b F_m(f_1, f_2, \ldots, f_m)G_k(g_1, g_2, \ldots, g_k)\, dx$$
$$\leq \left(\int_a^b F_m^2(f_1, f_2, \ldots, f_m)\, dx\right)^{\frac{1}{2}} \left(\int_a^b G_k^2(g_1, g_2, \ldots, g_k)\, dx\right)^{\frac{1}{2}} \tag{1.2}$$

for $\{f_i\}_{i=1}^m, \{g_j\}_{j=1}^k \in C[a,b]$. Let $\alpha_1, \alpha_2, \ldots, \alpha_m \in \mathbb{R}$. Then a subclass of the inequality (1.2) is

$$F_m(f_1, f_2, \ldots, f_m) = f_1^{\frac{1+\alpha_1}{2}} f_2^{\frac{1+\alpha_2}{2}} \ldots f_m^{\frac{1+\alpha_m}{2}}, \; G_m(g_1, g_2, \ldots, g_m) = g_1^{\frac{1-\alpha_1}{2}} g_2^{\frac{1-\alpha_2}{2}} \ldots g_m^{\frac{1-\alpha_m}{2}} \quad (1.3)$$

for $m = k$. In particular, when $m = 2$ and $m = 3$ it gives the following inequalities respectively

$$\left(\int_a^b f(t)g(t)\,dt\right)^2 \leq \int_a^b f^{1+\alpha}(t)g^{1+\beta}(t)\,dt \int_a^b f^{1-\alpha}(t)g^{1-\beta}(t)\,dt, \quad (1.4)$$

$$\left(\int_a^b f(t)g(t)h(t)\,dt\right)^2 \leq \int_a^b f^{1+\alpha}(t)g^{1+\beta}(t)h^{1+\gamma}(t)\,dt \int_a^b f^{1-\alpha}(t)h^{1-\beta}(t)h^{1-\gamma}(t)\,dt \quad (1.5)$$

for $\alpha, \beta, \gamma \in \mathbb{R}$ and f, g, h are real integrable functions such that the integrals in the inequalities (1.4) and (1.5) exist.
In [13], the author gives the inequalities for some well-known special functions in order to get new inequalities of the form

$$f^2(x) \leq k(x)\,f(px+q)\,f((2-p)x-q) \quad (p, q \in \mathbb{R},\; k(x) > 0). \quad (1.6)$$

Perhaps, the most used of the special functions is the Gamma function. One can come across wildly different usage of it. For example, it is used to define Hadamard fractional integral, Riemann-Lioville fractional integral, nonlinear fractional implicit integro-differential equations of Hadamard-Caputo type with fractional boundary conditions or abr-fractional Volterra-Fredholm system; see for example [3, 9, 10, 16].
Numerous extensions and deformations of Euler's classical Gamma function are discussed in the literature; see for example, [4, 11, 14]. A one-parameter deformation of the classical Gamma function, namely v-Gamma function, is defined in [7] as

$$\Gamma_v(x) = \int_0^\infty \left(\frac{t}{v}\right)^{\frac{x}{v}-1} e^{-t}\,dt \quad (x, v > 0). \quad (1.7)$$

Some results and inequalities associated with the v-Gamma function are presented in [5, 6].
Differentiating the equation (1.7) with respect to x we have

$$\Gamma_v^{(n)}(x) = \frac{1}{v^n} \int_0^\infty \left(\frac{t}{v}\right)^{\frac{x}{v}-1} \ln^n\left(\frac{t}{v}\right) e^{-t}\,dt \quad (x, v > 0). \quad (1.8)$$

Note that when $v = 1$, we have $\Gamma_v^{(n)}(x) = \Gamma^{(n)}(x)$ for $n \in \mathbb{N} = \{0, 1, 2, \ldots\}$.

In this presented paper we introduce a generalization form of the inequality (1.6) as

$$[f^{(n)}(x)]^2 \leq k(x) \sum_{k=0}^{m} a_k\, f^{(m-k)}\,(px+q) \sum_{k=0}^{l} b_k\, f^{(l-k)}\,((2-p)\,x-q) \quad (1.9)$$

for $l, m, n \in \mathbb{N}$, $p, q, a_k, b_k \in \mathbb{R}$ and $k(x) > 0$, and show that the inequalities we obtained are satisfied the inequality (1.9).

2 MAİN RESULTS

In this section, we prove some inequalities which involve the derivatives of the v-Gamma function by using the inequalities (1.4) and (1.5).

Theorem 2.1. *Let $x, v > 0$. Then the inequality*

$$[\Gamma_v^{(n)}(x)]^2 \leq \Gamma_v^{(n)}(x+\alpha x-\alpha v)\Gamma_v^{(n)}(x-\alpha x+\alpha v) \tag{2.1}$$

is valid for $x+\alpha x-\alpha v>0$, $x-\alpha x+\alpha v>0$, $n \in 2\mathbb{N}$,
and the inequality

$$\begin{aligned}[\Gamma_v^{(n)}(x)]^2 \leq{}& \frac{1}{(1+\beta)^{\frac{x}{v}+\frac{\alpha x}{v}-\alpha}(1-\beta)^{\frac{x}{v}-\frac{\alpha x}{v}+\alpha}} \sum_{k=0}^{n(1+\beta)} (-1)^k v^{-k} \binom{n(1+\beta)}{k} \ln^k(1+\beta) \\ &\times \Gamma_v^{((n(1+\beta)-k)}(x+\alpha x-\alpha v) \sum_{k=0}^{n(1-\beta)} (-1)^k v^{-k} \binom{n(1-\beta)}{k} \ln^k(1-\beta) \\ &\times \Gamma_v^{(n(1-\beta)-k)}(x-\alpha x+\alpha v) \end{aligned} \tag{2.2}$$

is valid for $x+\alpha x-\alpha v>0$, $x-\alpha x+\alpha v>0$ and some $\beta \in (-1,1)/\{0\}$ such that $n(1+\beta)$, $n(1-\beta) \in 2\mathbb{N}$.

Proof. By substituting $[a,b]=[0,\infty)$, $f(t)=\left(\frac{t}{v}\right)^{\frac{x}{v}-1}$, $g(t)=\ln^n\left(\frac{t}{v}\right)e^{-t}$ in the inequality (1.4) we have

$$\begin{aligned}\left(\int_0^\infty \left(\frac{t}{v}\right)^{\frac{x}{v}-1} \ln^n\left(\frac{t}{v}\right) e^{-t}\,dt\right)^2 \leq{}& \int_0^\infty \left(\frac{t}{v}\right)^{\left(\frac{x}{v}-1\right)(1+\alpha)} \left(\ln^n\left(\frac{t}{v}\right)e^{-t}\right)^{1+\beta} dt \\ &\times \int_0^\infty \left(\frac{t}{v}\right)^{\left(\frac{x}{v}-1\right)(1-\alpha)} \left(\ln^n\left(\frac{t}{v}\right)e^{-t}\right)^{1-\beta} dt. \end{aligned} \tag{2.3}$$

For simplicity let

$$I_1 = \int_0^\infty \left(\frac{t}{v}\right)^{\frac{x}{v}-1} \ln^n\left(\frac{t}{v}\right) e^{-t}\,dt,$$

$$I_2 = \int_0^\infty \left(\frac{t}{v}\right)^{\left(\frac{x}{v}-1\right)(1+\alpha)} \ln^{n(1+\beta)}\left(\frac{t}{v}\right) e^{-t(1+\beta)}\,dt,$$

and

$$I_3 = \int_0^\infty \left(\frac{t}{v}\right)^{\left(\frac{x}{v}-1\right)(1-\alpha)} \ln^{n(1-\beta)}\left(\frac{t}{v}\right) e^{-t(1-\beta)}\,dt.$$

If $\beta=0$ we have

$$I_1 = v^n\Gamma_v^{(n)}(x), \quad I_2 = v^n\Gamma_v^{(n)}(x+\alpha x-\alpha v), \quad I_3 = v^n\Gamma_v^{(n)}(x-\alpha x+\alpha v), \tag{2.4}$$

for $x+\alpha x-\alpha v>0$, $x-\alpha x+\alpha v>0$, and the inequality (2.1) follows for $n \in 2\mathbb{N}$.
Now, for the inequality (2.2) let $t(1+\beta)=u$ and $\beta \neq 0$ in I_2. Then we get

$$\begin{aligned} I_2 &= \int_0^\infty \left(\frac{u}{(1+\beta)v}\right)^{\frac{x}{v}+\frac{\alpha x}{v}-\alpha-1} \ln^{n(1+\beta)}\left(\frac{u}{(1+\beta)v}\right) e^{-u}\frac{du}{1+\beta} \\ &= \left(\frac{1}{1+\beta}\right)^{\frac{x}{v}+\frac{\alpha x}{v}-\alpha} \sum_{k=0}^{n(1+\beta)} (-1)^k \binom{n(1+\beta)}{k} \ln^k(1+\beta) \\ &\qquad \times \int_0^\infty \left(\frac{u}{v}\right)^{\frac{x}{v}+\frac{\alpha x}{v}-\alpha-1} \ln^{n(1+\beta)-k}\left(\frac{u}{v}\right) e^{-u}\,du. \end{aligned}$$

By using the equation (1.8) we have

$$I_2 = \left(\frac{1}{1+\beta}\right)^{\frac{x}{v}+\frac{\alpha x}{v}-\alpha} \sum_{k=0}^{n(1+\beta)} (-1)^k \binom{n(1+\beta)}{k} \ln^k(1+\beta)$$
$$\times v^{n(1+\beta)-k} \Gamma_v^{((n(1+\beta)-k)} (x+\alpha x-\alpha v) \tag{2.5}$$

for $x+\alpha x-\alpha v>0$, $\beta>-1$ and $n(1+\beta)\in\mathbb{N}$.
Similarly, let $t(1-\beta)=y$ and $\beta\neq 0$ in I_3. Then

$$\begin{aligned}
I_3 &= \int_0^\infty \left(\frac{y}{(1-\beta)v}\right)^{\frac{x}{v}-\frac{\alpha x}{v}+\alpha-1} \ln^{n(1-\beta)}\left(\frac{y}{(1-\beta)v}\right) e^{-y}\frac{dy}{1-\beta} \\
&= \left(\frac{1}{1-\beta}\right)^{\frac{x}{v}-\frac{\alpha x}{v}+\alpha} \sum_{k=0}^{n(1-\beta)} (-1)^k \binom{n(1-\beta)}{k} \ln^k(1-\beta) \\
&\qquad \times \int_0^\infty \left(\frac{y}{v}\right)^{\frac{x}{v}-\frac{\alpha x}{v}+\alpha-1} \ln^{n(1-\beta)-k}\left(\frac{y}{v}\right) e^{-y}\, dy.
\end{aligned}$$

By using the equation (1.8) we get

$$I_3 = \left(\frac{1}{1-\beta}\right)^{\frac{x}{v}-\frac{\alpha x}{v}+\alpha} \sum_{k=0}^{n(1-\beta)} (-1)^k \binom{n(1-\beta)}{k} \ln^k(1-\beta)$$
$$\times v^{n(1-\beta)-k} \Gamma_v^{(n(1-\beta)-k)} (x-\alpha x+\alpha v) \tag{2.6}$$

for $x-\alpha x+\alpha v>0$, $\beta<1$ and $n(1-\beta)\in\mathbb{N}$.
Hence by using the equations (2.4), (2.5) and (2.6) and taking $n(1+\beta)\in 2\mathbb{N}$, $n(1-\beta)\in 2\mathbb{N}$ to guarantee the positivity of the right-hand side of the inequality (2.3), we get the desired result (2.2). ☐

Remark 2.2. *The inequality (2.1) satisfy the inequality (1.6) for $p=1+\alpha$, $q=-\alpha v$, $k(x)=1$ and $f=\Gamma_v^{(n)}$.*

Remark 2.3. *The inequality (2.2) satisfy the generic form (1.9) for*

$$p=1+\alpha,\ \ q=-\alpha v,\ \ k(x)=\frac{1}{(1+\beta)^{\frac{x}{v}+\frac{\alpha x}{v}-\alpha}(1-\beta)^{\frac{x}{v}-\frac{\alpha x}{v}+\alpha}},$$

$$a_k=(-1)^k v^{-k}\binom{n(1+\beta)}{k}\ln^k(1+\beta),\ \ b_k=(-1)^k v^{-k}\binom{n(1-\beta)}{k}\ln^k(1-\beta)\ \ and\ \ f=\Gamma_v.$$

Example 2.1. *Let $n=2$ and $\alpha=\frac{1}{2}$ in the inequality (2.1). Then we get*

$$[\Gamma_v^{''}(x)]^2 \leq \Gamma_v^{''}(\frac{3x}{2}-\frac{v}{2})\Gamma_v^{''}(\frac{x}{2}+\frac{v}{2})$$

for $v>0$ and $x>\frac{v}{3}$.

Example 2.2. *By taking $v=1$, $n=3$, $\alpha=\frac{1}{2}$ and $\beta=\frac{1}{3}$ in the inequaliy (2.2), we get*

$$[\Gamma^{'''}(x)]^2 \leq 2^{\frac{1}{2}-\frac{7x}{2}}\, 3^{2x} \sum_{k=0}^{4}(-1)^k\binom{4}{k}\ln^k(\frac{4}{3})\Gamma^{(4-k)}\left(\frac{3x}{2}-\frac{1}{2}\right)$$
$$\times \sum_{k=0}^{2}(-1)^k\binom{2}{k}\ln^k(\frac{2}{3})\Gamma^{(2-k)}\left(\frac{x}{2}+\frac{1}{2}\right) \tag{2.7}$$

for $x>0$.

Corollary 2.4. *By taking $v = 1$ and $n = 0$ in the inequality (2.2) we get inequality for the Gamma function*

$$[\Gamma(x)]^2 \leq \frac{1}{(1+\beta)^{x+\alpha x-\alpha}(1-\beta)^{x-\alpha x+\alpha}} \Gamma\left(x+\alpha x-\alpha\right)\Gamma\left(x-\alpha x+\alpha\right) \tag{2.8}$$

for $x > 0$, $x + \alpha x - \alpha > 0$, $x - \alpha x + \alpha > 0$ and $\beta \in (-1, 1)$ given in [13].

Now we give the following theorem as an application of the inequality (1.5).

Theorem 2.5. *Let $x, v > 0$. Then the inequality*

$$[\Gamma_v^{(n)}(x)]^2 \leq \Gamma_v^{(n)}(x+\alpha x-\beta v)\Gamma_v^{(n)}(x-\alpha x+\beta v) \tag{2.9}$$

is valid for $x + \alpha x - \beta v > 0$, $x - \alpha x + \beta v > 0$, $n \in 2\mathbb{N}$,
and the inequality

$$\begin{aligned}[\Gamma_v^{(n)}(x)]^2 \leq{}& \frac{1}{(1+\gamma)^{\frac{x}{v}+\frac{\alpha x}{v}-\beta}(1-\gamma)^{\frac{x}{v}-\frac{\alpha x}{v}+\beta}} \sum_{k=0}^{n(1+\gamma)} (-1)^k v^{-k} \binom{n(1+\gamma)}{k} \ln^k(1+\gamma) \\ &\times \Gamma_v^{((n(1+\gamma)-k)}\left(x+\alpha x-\beta v\right) \sum_{k=0}^{n(1-\gamma)} (-1)^k v^{-k} \binom{n(1-\gamma)}{k} \ln^k(1-\gamma) \\ &\times \Gamma_v^{(n(1-\gamma)-k)}\left(x-\alpha x+\beta v\right)\end{aligned} \tag{2.10}$$

is valid for $x+\alpha x-\beta v > 0$, $x-\alpha x+\beta v > 0$ and some $\gamma \in (-1, 1)/\{0\}$ such that $n(1+\gamma)$, $n(1-\gamma) \in 2\mathbb{N}$.

By substituting $[a, b] = [0, \infty)$, $f(t) = \left(\frac{t}{v}\right)^{\frac{x}{v}}$, $g(t) = \left(\frac{t}{v}\right)^{-1}$ $h(t) = \ln^n\left(\frac{t}{v}\right)e^{-t}$ in the inequality (1.5) we have

$$\left(\int_0^\infty \left(\frac{t}{v}\right)^{\frac{x}{v}-1} \ln^n\left(\frac{t}{v}\right) e^{-t}\, dt\right)^2 \leq \int_0^\infty \left(\frac{t}{v}\right)^{\frac{x(1+\alpha)}{v}} \left(\frac{t}{v}\right)^{-1(1+\beta)} \left(\ln^n\left(\frac{t}{v}\right)e^{-t}\right)^{1+\gamma} dt \tag{2.11}$$

$$\times \int_0^\infty \left(\frac{t}{v}\right)^{\frac{x(1-\alpha)}{v}} \left(\frac{t}{v}\right)^{-(1-\beta)} \left(\ln^n\left(\frac{t}{v}\right)e^{-t}\right)^{1-\gamma} dt. \tag{2.12}$$

Again, for simplicity let

$$J_1 = \int_0^\infty \left(\frac{t}{v}\right)^{\frac{x}{v}-1} \ln^n\left(\frac{t}{v}\right) e^{-t}\, dt,$$

$$J_2 = \int_0^\infty \left(\frac{t}{v}\right)^{\frac{x(1+\alpha)}{v}} \left(\frac{t}{v}\right)^{-1(1+\beta)} \ln^{n(1+\gamma)}\left(\frac{t}{v}\right) e^{-t(1+\gamma)}\, dt$$

and

$$J_3 = \int_0^\infty \left(\frac{t}{v}\right)^{\frac{x(1-\alpha)}{v}} \left(\frac{t}{v}\right)^{-(1-\beta)} \ln^{n(1-\gamma)}\left(\frac{t}{v}\right) e^{-t(1-\gamma)}\, dt.$$

If $\gamma = 0$ we have

$$J_1 = v^n\Gamma_v^{(n)}(x), \quad J_2 = v^n\Gamma_v^{(n)}(x+\alpha x-\beta v), \quad J_3 = v^n\Gamma_v^{(n)}(x-\alpha x+\beta v), \tag{2.13}$$

for $x+\alpha x-\beta v>0$, $x-\alpha x+\beta v>0$, and the inequality (2.9) follows for $n\in 2\mathbb{N}$.
Now, for the inequality (2.10) let $t(1+\gamma)=u$ and $\gamma\neq 0$ in J_2. Then we get

$$\begin{aligned}
J_2 &= \int_0^\infty \left(\frac{u}{(1+\gamma)v}\right)^{\frac{x(1+\alpha)}{v}-\beta-1} \ln^{n(1+\gamma)}\left(\frac{u}{(1+\gamma)v}\right) e^{-u}\,\frac{du}{1+\gamma} \\
&= \left(\frac{1}{1+\gamma}\right)^{\frac{x}{v}+\frac{\alpha x}{v}-\beta} \sum_{k=0}^{n(1+\gamma)} (-1)^k \binom{n(1+\gamma)}{k} \ln^k(1+\gamma) \\
&\qquad \times \int_0^\infty \left(\frac{u}{v}\right)^{\frac{x}{v}+\frac{\alpha x}{v}-\beta-1} \ln^{n(1+\gamma)-k}\left(\frac{u}{v}\right) e^{-u}\,du.
\end{aligned}$$

By using the equation (1.8) we get

$$\begin{aligned}
J_2 = \left(\frac{1}{1+\gamma}\right)^{\frac{x}{v}+\frac{\alpha x}{v}-\beta} \sum_{k=0}^{n(1+\gamma)} (-1)^k \binom{n(1+\gamma)}{k} \ln^k(1+\gamma) \\
\times v^{n(1+\gamma)-k}\Gamma_v^{((n(1+\gamma)-k)}(x+\alpha x-\beta v)
\end{aligned} \tag{2.14}$$

for $x+\alpha x-\beta v>0$, $\gamma>-1$ and $n(1+\gamma)\in\mathbb{N}$.
For the integral J_3 let $t(1-\gamma)=y$ and $\gamma\neq 0$. Then

$$\begin{aligned}
J_3 &= \int_0^\infty \left(\frac{y}{(1-\gamma)v}\right)^{\frac{x}{v}-\frac{\alpha x}{v}+\beta-1} \ln^{n(1-\gamma)}\left(\frac{y}{(1-\gamma)v}\right) e^{-y}\,\frac{dy}{1-\gamma} \\
&= \left(\frac{1}{1-\gamma}\right)^{\frac{x}{v}-\frac{\alpha x}{v}+\beta} \sum_{k=0}^{n(1-\gamma)} (-1)^k \binom{n(1-\gamma)}{k} \ln^k(1-\gamma) \\
&\qquad \times \int_0^\infty \left(\frac{y}{v}\right)^{\frac{x}{v}-\frac{\alpha x}{v}+\beta-1} \ln^{n(1-\gamma)-k}\left(\frac{y}{v}\right) e^{-y}\,dy \\
&= \left(\frac{1}{1-\gamma}\right)^{\frac{x}{v}-\frac{\alpha x}{v}+\beta} \sum_{k=0}^{n(1-\gamma)} (-1)^k \binom{n(1-\gamma)}{k} \ln^k(1-\gamma) \\
&\qquad \times\, v^{n(1-\gamma)-k}\Gamma_v^{(n(1-\gamma)-k)}(x-\alpha x+\beta v)
\end{aligned} \tag{2.15}$$

for $x-\alpha x+\beta v>0$, $\gamma<1$ and $n(1-\gamma)\in\mathbb{N}$.
Hence by using the equations (2.13), (2.14) and (2.15) and taking $n(1+\gamma), n(1-\gamma)\in 2\mathbb{N}$, the inequality (2.10) follows.

Remark 2.6. *The inequality (2.9) satisfy the inequality (1.6) for $p=1+\alpha$, $q=-\beta v$, $k(x)=1$ and $f=\Gamma_v^{(n)}$.*

Remark 2.7. *The inequality (2.10) is a special case of the main inequality (1.9) for*

$$p=1+\alpha,\ \ q=-\beta v\ \ ,k(x)=\frac{1}{(1+\gamma)^{\frac{x}{v}+\frac{\alpha x}{v}-\beta}(1-\gamma)^{\frac{x}{v}-\frac{\alpha x}{v}+\beta}},$$

$$a_k=(-1)^k v^{-k}\binom{n(1+\gamma)}{k}\ln^k(1+\gamma),\ \ b_k=(-1)^k v^{-k}\binom{n(1-\gamma)}{k}\ln^k(1-\gamma)\ \ and\ \ f=\Gamma_v.$$

Corollary 2.8. *By taking $v=1$ in the inequality (2.9) we get the following inequality*

$$[\Gamma^{(n)}(x)]^2\leq \Gamma^{(n)}(x+\alpha x-\beta)\Gamma^{(n)}(x-\alpha x+\beta)$$

for $x>0$, $x+\alpha x-\beta>0$, $x-\alpha x+\beta>0$, $n\in 2\mathbb{N}$.

Corollary 2.9. *By taking $v = 1$ in the inequality (2.10) we get*

$$
\begin{aligned}
[\Gamma^{(n)}(x)]^2 \leq & \frac{1}{(1+\gamma)^{x+\alpha x-\beta}(1-\gamma)^{x-\alpha x+\beta}} \sum_{k=0}^{n(1+\gamma)} (-1)^k \binom{n(1+\gamma)}{k} \ln^k(1+\gamma) \\
& \times \Gamma^{((n(1+\gamma)-k)}(x+\alpha x-\beta) \\
& \times \sum_{k=0}^{n(1-\gamma)} (-1)^k \binom{n(1-\gamma)}{k} \ln^k(1-\gamma) \Gamma^{(n(1-\gamma)-k)}(x-\alpha x+\beta)
\end{aligned}
\tag{2.16}
$$

for $x > 0$, $x + \alpha x - \beta > 0$, $x - \alpha x + \beta > 0$, $\gamma \in (-1, 1)/\{0\}$ and $n(1+\gamma)$, $n(1-\gamma) \in 2\mathbb{N}$.

3 CONCLUSIONS

In this work, based on the Cauchy-Bunyakovsky-Schwartz inequality, we introduced an inequality. By getting some new inequalities, we showed that a one-parameter deformation of the Gamma function satisfies this type of inequality. We also show that the established results are generalizations of some previous ones.

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