Review Form 3

Journal Name:	
Manuscript Number:	**Ms_JAMCS_128007**
Title of the Manuscript:	**On Certain Results On The Diophantine Equation:** $\sum_{r=1}^{n} w_r^2 + \frac{n}{3}d^2 = 3\left(\frac{nd^2}{3} + \sum_{r=1}^{\frac{n}{3}} w_{3r-1}^2\right)$
Type of the Article	**Original Research Article**

<mark>**PART 1:**</mark> **Comments**

	Reviewer's comment	Author's Feedback *(Please correct the manuscript and highlight that part in the manuscript. It is mandatory that authors should write his/her feedback here)*

Please write a few sentences regarding the importance of this manuscript for the scientific community. A minimum of 3-4 sentences may be required for this part.	**Reviewer Report**
	
	 Summary:
	
	 This paper investigates a Diophantine equation involving sequences in arithmetic progression and aims to extend existing results on such equations. The authors seek integer solutions for the equation in the context of sums of squares and their relations to a variable dd. While the general approach is valuable and potentially impactful, the manuscript requires significant revisions to improve clarity, rigor, and completeness.
	
	 General Comments:
	
	 1. **Clarity of the Mathematical Formulation:**
	    ○ The formulation of the Diophantine equation is not sufficiently clear in many places. The equations are quite complex and not presented in a way that can be easily followed. It is difficult for a reader to understand the structure and derivation of the main equation without extensive background or additional context. Please consider breaking down the equations and steps more clearly, explaining each part of the derivation carefully.
	 2. **Presentation and Organization:**
	    ○ The manuscript seems to jump between ideas, and the flow of the argument is not very smooth. The explanations are scattered, and the connection between different results is not always obvious. I recommend reorganizing the paper into a more coherent structure:
	       ■ **Introduction:** Briefly introduce the Diophantine equation in context. Clearly state the problem and its significance. Mention the previous research and where your work contributes.
	       ■ **Main Results:** Present the key theorems and definitions in a more structured manner. Clearly state the main results before going into the detailed proof.
	       ■ **Proofs:** Present the proofs in a step-by-step manner, breaking down complex manipulations into smaller, digestible parts.
	 3. **Notation and Terminology:**
	    ○ The notation used for the sequences and terms in the equations is inconsistent and difficult to follow. For instance, w w_r is used in several contexts, but it's unclear what exactly each index represents and how the terms in the sequence are related to each other. Clarifying the notation at the outset is essential. Additionally, the use of $w_1, w_2, \dots, w_{30}$ $w_1, w_2, \dots, w_{30}$ and dd could be better explained in terms of their roles in the equation and their interrelationships.	

4. **Typographical and Formatting Issues:**
   - There are several typographical issues in the manuscript, such as inconsistent spacing, missing punctuation, and awkward sentence structures. These need to be corrected to improve readability.
   - The formatting of mathematical equations in some sections is unclear, particularly when they span multiple lines. Equations should be presented clearly with proper alignment and formatting, especially when defining sequences or summations.

Specific Comments:

1. **Theorem 1.1 (and subsequent theorems):**
   - The presentation of Theorem 1.1 and its proof is unclear. The sequence wrw_r is introduced but is not fully explained. It would be beneficial to provide a clear definition of wrw_r, its role in the equation, and how it is structured. Additionally, the step where the equation is simplified to its final form requires more explanation. For instance, why are terms like $702w_1d$ and $6210d^2$ generated, and how are they derived?
   - Similarly, the subsequent theorems (1.2 and 1.3) present similar issues. The same advice applies: clarify the notation, present the results step by step, and explain each term's significance.
2. **Factorization and Polynomial Decomposition:**
   - The paper mentions decomposing and factoring polynomials, but it lacks detailed steps. It would help if you showed intermediate steps and provided a clearer explanation of how the factorizations lead to the final solution. Consider providing explicit details about the factorization techniques used, and ensure that the reader can follow the logic from one step to the next.
3. **Integer Solutions and Existence Conditions:**
   - The conditions for the existence of integer solutions are mentioned, but the reasoning behind them is not sufficiently detailed. Specifically, the requirement that differences between successive w_r's be equal to d (i.e., $wr+1-wr=dw_{r+1} - w_r = d$) seems to play a crucial role in the existence of integer solutions. It would be helpful to discuss why this condition is necessary and how it contributes to the solution space.
4. **Context of Previous Work:**
   - While the paper refers to existing literature on Diophantine equations, it would benefit from a more thorough discussion of the state-of-the-art work. For instance, comparisons with known results in Diophantine equations, such as Fermat's Last Theorem or the Ramanujan-Nagell equation, would help contextualize the study.
   - Including citations to relevant prior work and explicitly highlighting how your approach advances or extends existing knowledge would strengthen the paper's contribution.

Suggestions for Improvement:

1. **Clarify the Definitions:**
   - Define all variables, sequences, and terms clearly at the beginning of the paper. Include the specific relationship between wrw_r and dd, and explain why these sequences are structured the way they are.
2. **Improve Proofs:**
   - Present the proofs in a more detailed and structured way. Break down each calculation into smaller steps, and explain how each part contributes to the overall result.
3. **Enhance Readability:**
   - Revise the manuscript to improve readability, eliminating typographical errors and awkward sentences. Consider using tools like LaTeX for better presentation of mathematical equations.
4. **Add More Context:**

- o Provide more context and comparisons with existing work. A more thorough review of relevant literature and a clearer explanation of your paper's contributions would help readers understand the significance of your findings.

5. **Add More References**:

Provide more references should be added with existing work and authors can consider following list too:

1. Bellaouar Djamel, Boudaoud Abdelmadjid, ÖZER ÖZEN (2019). ON A SEQUENCE FORMED BY ITERATING A DIVISOR OPERATOR. Czechoslovak Mathematical Journal, 69(144), 1177-1196.

2. Vidhyalakshmi SA, Gopalan MA, Thangam S Aarthy, ÖZER ÖZEN* (2019). ONTERNARY BIQUADRATIC DIOPHANTINE EQUATION $11(x^2 - y^2) 3(x+y) = z^4$. Notes on Number Theory and Discrete Mathematics, 25(3), 65-71.

3. ÖZER ÖZEN, Gopalan MA (2019). On the homogeneous cone $z^2 + 2(k+1)y^2 = (k+1).(k+3).x^2$ Pioneer Journal of Mathematics and Mathematical Sciences, 25(1), 9-18.

4. Gopalan MA, Thangam S Aarthy, ÖZER ÖZEN* (2020). On the Quinary Homogeneous Bi-Quadratic Equation $x^4 + y^4 - (x+y)w^3 = 14z^2 T^2$. Journal of Fundamental and Applied Sciences, 12(2), 516-524.

5. Gopalan MA, Thangam S Aarthy, ÖZER ÖZEN* (2020). On the Quinary Homogeneous Bi-Quadratic Equation $x^4 + y^4 - (x+y)w^3 = 14z^2 T^2$. Journal of Fundamental and Applied Sciences, 12(2), 516-524.

6. Annouk Ikorong, ÖZER ÖZEN (2022). New significant results on Fermat numbers via elementary arithmetic methods, Theoretical Mathematics & Applications, 12(3), 1-10.

7. ÖZER ÖZEN , 2018. On The Some Nonextandable Regular P_{-2} Sets, Malaysian Journal of Mathematical Science (MJMS), 12(2): 255–266.

8. Gopalan MA,Vidhyalakshmi SA, **ÖZER ÖZEN**, 2018, A Collection of Pellian Equation (Solutions and Properties , Akinik Publications, Publication No:1, English.

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Conclusion:

The paper tackles an interesting and complex Diophantine equation involving sequences in arithmetic progression. While the mathematical content is potentially valuable, the manuscript requires major revisions in terms of clarity, structure, and explanation of results. After addressing these issues, the paper could be a valuable contribution to the field of number theory.

I recommend a **Major Revision** with the expectation that the authors will address the issues outlined

	above and provide a more polished and detailed manuscript.	
Is the title of the article suitable? **(If not please suggest an alternative title)**		
Is the abstract of the article comprehensive? Do you suggest the addition (or deletion) of some points in this section? Please write your suggestions here.		
Is the manuscript scientifically, correct? Please write here.		
Are the references sufficient and recent? If you have suggestions of additional references, please mention them in the review form.		
Is the language/English quality of the article suitable for scholarly communications?		
Optional/General comments	**MENTIONED AS ABOVE STATEMENTS**	

PART 2:

	Reviewer's comment	**Author's comment** *(if agreed with reviewer, correct the manuscript and highlight that part in the manuscript. It is mandatory that authors should write his/her feedback here)*
Are there ethical issues in this manuscript?	*(If yes, Kindly please write down the ethical issues here in details)*	

Reviewer Details:

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