

# Optimized hybrid block methods with high efficiency indices for the solution of first-order ordinary differential equations

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## ABSTRACT

In this article, optimized hybrid block methods are proposed for the solution of first-order ordinary differential equations. The techniques of interpolation and collocation were adopted for the derivation of the methods using a three-parameter approximation. The hybrid points were obtained by minimizing the local truncation error of the main method. The schemes obtained were reformulated to reduce the number of function evaluation. The discrete schemes were produced as a by-product of the continuous scheme and used to simultaneously solve initial value problems (IVPs) in block mode. The resulting schemes are self-starting, do not require the creation of individual predictors, consistent, zero-stable, and convergent. The accuracy and efficiency of the methods were ascertained using several numerical experiments. The numerical results were favorably compared to some techniques from the cited literature.

*Keywords:* Linear stability, Local truncation error (LTE), Parameter approximations, Initial value problems (IVPs), Ordinary differential equations (ODEs)

## 1. INTRODUCTION

A system of differential equations is derived via mathematical modelling of physical phenomena in the scientific and technical domains, specifically in epidemiological systems characterized by many interactions among separate compartments. Finding analytical solutions to most differential equations is often challenging. The utilization of numerical techniques was necessary in order to obtain an approximate solution. Various approaches, such as collocation, interpolation, integration, and interpolation polynomials, have been thoroughly investigated in academic literature to construct continuous linear multistep methods (LMMs) for the direct solution of initial value problems in ordinary differential equations see [1,2,3,4,5,6,7,8,11] and the literature therein.

The study conducted by the author in [13] proposed a two-step methodology that involved the selection of two intermediate locations through the optimization of the LTEs. The method was reformulated as an R-K method, but its implementation required a greater computational cost. However, the most optimal formulation was attained through the process of reformulating the method in a manner that decreases the frequency of instances of the source term  $f$ . Upon conducting a comparative analysis between the proposed economic reformulation and the existing methodologies documented in the literature, it was observed that the former demonstrated a higher level of performance. In [10], the authors presented a novel optimized one-step hybrid block technique that is specifically tailored for the optimization of first-order initial value problems (IVPs). The methodology entailed the careful selection of three hybrid points to optimize the LTEs (Local Truncation Errors) of the basic equations governing the behavior of the block. The technique displayed zero-stability, therefore showcasing a level of algebraic correctness that is fifth-order. The validation of the approach's efficacy

and precision was accomplished through the use of numerical illustrations. Furthermore, [20] introduced a novel one-step implicit block approach that incorporates three intra-step grid points. The major goal of the LTE was to minimize the principal term in order to attain one of the three optimal intra-step positions. A revision of the methodology led to a significant decrease in computing costs while maintaining the same degrees of consistency, zero-stability, A-stability, and convergence. The methodology was utilized in order to tackle practical concerns, and a comparison analysis was carried out with current approaches in the literature to determine the superiority of the innovative approach. Several scholarly studies have been conducted to explore the enhancement of hybrid points by minimizing the LTE. Notable contributions in this area include the research conducted by [12,14,15,16,17,18,19,20,21].

The study conducted in our research utilizes a novel class of hybrid block techniques that contains three off-step points and employs three-parameter approximations. By implementing optimization techniques for LTE, it is possible to attain optimal hybrid points. The main aim of this work is to present an efficient methodology for solving initial value problems that adhere to the prescribed form.

$$x' = f(t, x), x(t_0) = x_0 \quad (1)$$

where,  $t \in [t_0, T]$ ,  $f : [t_0, T] \times \mathfrak{R} \rightarrow \mathfrak{R}$ . It is assumed that equation (1) satisfies the conditions of the existence and uniqueness theorem for initial value problems (see [11,13]).

## 2. MATERIAL AND METHODS

In this section, we provide the derivation of the **proposed** optimal hybrid block method. This method incorporates three intra-step points and is derived by the reduction of the major term of the LTE.

Let us consider the polynomial  $p(t)$  as an approximation for the exact solution  $x(t)$  of equation (1). The coefficients of the polynomial function  $p(t)$  are determined by utilizing approximate values of  $x$  and  $f$  at various grid and off-step points.

Let  $x = x(t_j)$  and  $x'_j = f_j = f(t_j, x)$  be the approximate values of  $x$  and  $f$  respectively at  $t_j$ . And  $t_n$  is the grid point given by  $t_{n+j} = t_n + jh, h = t_j - t_{j-1}$ . Then

$$x(t) \approx Q(t) = \sum_{j=0}^{\infty} b_j t^j \quad (2)$$

where  $b_j \in R$  are real unknown coefficients to be determined. Thus the  $m$  partial sum of equation (2) is obtained as

$$x(t) \approx Q(t) = \sum_{j=0}^k b_j t^j . \quad (3)$$

The first derivative of (3) is obtained as

$$x'(t) \approx Q'(t) = \sum_{j=0}^m j b_j t^{j-1} , \quad (4)$$

where  $k = (I + C) - 1$ ,  $I$  and  $C$  denote the number of interpolation and collocation points respectively. Let  $p, q, r$  be the off-step points such that  $0 < p < q < r < 1$ . Interpolating equation (3) at  $t_{n+j}, j = 0$ , and collocating equation (4) at  $t_{n+j}, j = 0, p, q, r, 1$  yield the optimized hybrid block method (OHBM) which can be written in matrix form as

$$\begin{pmatrix} 1 & t_n & t_n^2 & t_n^3 & t_n^4 & t_n^5 \\ 0 & 1 & 2t_n & 3t_n^2 & 4t_n^3 & 5t_n^4 \\ 0 & 1 & 2t_{n+p} & 3t_{n+p}^2 & 4t_{n+p}^3 & 5t_{n+p}^4 \\ 0 & 1 & 2t_{n+q} & 3t_{n+q}^2 & 4t_{n+q}^3 & 5t_{n+q}^4 \\ 0 & 1 & 2t_{n+r} & 3t_{n+r}^2 & 4t_{n+r}^3 & 5t_{n+r}^4 \\ 0 & 1 & 2t_{n+1} & 3t_{n+1}^2 & 4t_{n+1}^3 & 5t_{n+1}^4 \end{pmatrix} \begin{pmatrix} b_0 \\ b_1 \\ b_2 \\ b_3 \\ b_4 \\ b_5 \end{pmatrix} = \begin{pmatrix} x_n \\ f_n \\ f_{n+p} \\ f_{n+q} \\ f_{n+r} \\ f_{n+1} \end{pmatrix} \quad (5)$$

Solving equation (5) by **Gaussian Elimination method** to obtain the coefficients  $b_j$ 's,  $j = 0, 1, \dots, 5$  and putting back into equation (3), then the implicit continuous scheme can be written in the form

$$Q(t) = \alpha_0(t)x_n + h(\beta_0(t)f_n + \beta_p(t)f_{n+p} + \beta_q(t)f_{n+q} + \beta_r(t)f_{n+r} + \beta_1(t)f_{n+1}). \quad (6)$$

Where  $\alpha_0(t), \beta_0(t), \beta_p(t), \beta_q(t), \beta_r(t), \beta_1(t)$  are continuous coefficients. Evaluating equation (6) at the points  $t = t_{n+p}, t_{n+q}, t_{n+r}, t_{n+1}$ , yield the following

$$\begin{aligned} x_{n+p} = x_n + & \frac{hu_1(-3p^3 + 30qr + 5p^2)(1 + q + r) - 10p(q + r + qr))f_n}{60qr} \\ & + \frac{hp^3(3p^2 + 10qr - 5p(q + r))f_{n+1}}{60(-1 + p)(-1 + q)(-1 + r)} \quad (7) \\ & + \frac{hp(12p^3 - 30qr + 5p^2(1 + q + r) + 20p(q + r + qr))f_{n+p}}{60(-1 + p)(p - q)(q - r)} \\ & + \frac{hp^3(3p^2 + 10r - 5p(1 + r))f_{n+q}}{60(p - q)(-1 + q)q(q + r)} + \frac{hp^3(3p^2 + 10q - 5q(1 + q))f_{n+r}}{60(p - r)(-1 + r)r(-q + r)} \end{aligned}$$

$$\begin{aligned} x_{n+q} = x_n + & \frac{hq(5u_1(q^2 + 6r - q(1 + r)) + q(-3q^2 - 10r + 5q(1 + r)))f_n}{60pr} \\ & + \frac{hq^3(q(3q - 5r) - 5p(q - 2r))f_{n+1}}{60(-1 + p)(-1 + q)(-1 + r)} - \frac{hq^3(3q^2 + 10r - 5q(1 + r))f_{n+p}}{60(-1 + p)p(p - q)(p - r)} \\ & + \frac{hq(5u_1(3q^2 + 6r - 4q(1 + r)) + s(-12q^2 - 20r + q(1 + r)))f_{n+q}}{60(p - q)(-1 + q)(q - r)} \quad (8) \\ & - \frac{hq^3(5p(-2 + q) + (5 - 3q)q)f_{n+r}}{60(p - r)(-1 + r)(-q + r)} \end{aligned}$$

$$\begin{aligned}
x_{n+r} = x_n + & \frac{hr(r(5q(-2+r) + (5-3r)r) + 5p(-2q(-3+r) + (-2+r)r))f_n}{60pq} \\
& + \frac{hr^3(10pq - 5pr - 5qr + 3r^2)f_{n+1}}{60(-1+p)(-1+q)(-1+r)} + \frac{hr^3(5q(-2+r) + (5-r)r)f_{n+p}}{60(-1+p)p(p-q)(p-r)} \\
& - \frac{hr^3(5p(-2+r) + (5-3r)r)f_{n+q}}{60(p-q)(-1+q)(q-r)} \\
& + \frac{hr(r(3(5-4r)r + 5q(-4+3r)) + 5q(q(6-4r) + r(-4+3r)))f_{n+r}}{60(p-r)(-1+r)(-q+r)}
\end{aligned} \tag{9}$$

$$\begin{aligned}
x_{n+1} = x_n + & \frac{h(-3+q(5-10r) + 5q + 5p(1-2r+q(-2+6r)))f_n}{60pqr} \\
& + \frac{h(12+15q+15r-20qr+5p(3-4r+q(-4+6r)))f_{n+1}}{60(-1+p)(-1+q)(-1+r)} \tag{10} \\
& + \frac{h(3-5r+5q(-1+2q))f_{n+p}}{60(-1+p)p(p-q)(p-r)} + \frac{h(3-5r+5p(-1+2q))f_{n+q}}{60(p-q)(-1+q)(q-r)} \\
& + \frac{h(3-5q+5p(-1+2q))f_{n+r}}{60(p-r)(-1+r)(-q+r)}
\end{aligned}$$

where,  $f_{n+j} = f(t_{n+j}, x_{n+j})$ , for  $j = p, q, r, 1$ , and  $x_{n+j} \approx x(t_n + jh)$  are approximations of the exact solution. Expanding the main formula  $x(t_{n+1})$  in the Taylor series around  $t_n$ .

$$\begin{aligned}
\mathcal{L}(x(t_{n+1}); h) = & \frac{1}{7200}(-2+3p+3q-5pq+3r-5pr-5qr+10pqr)x^6[t_n]h^6 \\
& + \frac{1}{302400}(-24+21p+21p^2+21q-14pq-35p^2q+21p^2-35pp^2 \\
& + 21r)x^7[t_n]h^7 \\
& + \frac{1}{302400}(-14pr-35p^2r-14qr+70p^2qr-35q^2r+70pq^2r)x^7[t_n]h^7 \\
& + \frac{1}{302400}(21r^2-35pr^2-35qr^2+70pqr^2)x^7[t_n]h^7 \\
& + O(h)^8. \tag{11}
\end{aligned}$$

Setting the principal term of the LTE in (11) to zero yields the following equation in three unknowns:

$$\frac{1}{7200}(-2+3p+3q-5pq+3r-5pr-5qr+10pqr) = 0 \tag{12}$$

$$q = \frac{2-3p-3r+5pr}{3-5p-5r+10pr} \tag{13}$$

while the other two parameters are given as

$$p = \frac{1}{10}(5 - \sqrt{5}); r = \frac{1}{10}(5 + \sqrt{5}) \quad (14)$$

Substituting equation (14) into equation (13), we get  $q = \frac{1}{2}$ .

The LTE of the main formula in equation (14) is computed by substituting the values of the parameters  $p, q, r$  into equation (15) to obtain

$$\mathcal{L}(x(t_{n+1}); h) = -\frac{x^7[t_n]h^7}{1512000} + O(h)^8. \quad (15)$$

Lastly, putting the values of the parameters  $p, q, r$  into equations (10) - (14) we get the following one-step optimal hybrid block method:

$$\begin{aligned} x_{n+p} &= x_n + \frac{h}{3000} \left( (275 + \sqrt{5})f_n + (625 + 95\sqrt{5})f_{n+p} - 192\sqrt{5}f_{n+q} + (625 - 205\sqrt{5})f_{n+r} \right. \\ &\quad \left. + (-25 + \sqrt{5})f_{n+1} \right), \\ x_{n+q} &= x_n + \frac{h}{192} (17f_n + (40 + 15\sqrt{5})f_{n+p} + (40 - 15\sqrt{5})f_{n+r} - f_{n+1}), \\ x_{n+r} &= x_n + \frac{h}{3000} \left( (275 - \sqrt{5})f_n + (625 + 205\sqrt{5})f_{n+p} + 192\sqrt{5}f_{n+q} + (625 - 95\sqrt{5})f_{n+r} \right. \\ &\quad \left. - (25 + \sqrt{5})f_{n+1} \right), \\ x_{n+1} &= x_n + \frac{h}{12} (f_n + 5f_{n+p} + 5f_{n+r} + f_{n+1}). \end{aligned} \quad (16)$$

The hybrid block method in (16) is reformulated to reduce the frequency of  $f$ . This procedure is believed to reduce the number of function evaluation and hence the computing time. Thus, we obtain the modified optimal hybrid block method (MOHBM) as given in (17) below:

$$\begin{aligned} hf_{n+p} &= -\frac{1}{10} \left( 2hf_n + (21 + \sqrt{5})x_n + (-25 + 15\sqrt{5})x_{n+p} + (32 - 32\sqrt{5})x_{n+q} \right. \\ &\quad \left. + (-25 + 15\sqrt{5})x_{n+r} + (-3 + \sqrt{5})x_{n+1} \right), \\ hf_{n+q} &= \frac{1}{16} (2hf_n + 20x_n + (-25 - 25\sqrt{5})x_{n+p} + 32x_{n+q} + (-25 + 25\sqrt{5})x_{n+r} \\ &\quad - 2x_{n+1}), \\ hf_{n+r} &= \frac{1}{10} \left( -2hf_n + (-21 + \sqrt{5})x_n + (25 + 15\sqrt{5})x_{n+p} - (32 + 32\sqrt{5})x_{n+q} \right. \\ &\quad \left. + (25 + 15\sqrt{5})x_{n+r} - (3 + \sqrt{5})x_{n+1} \right), \\ hf_{n+1} &= hf_n + 9x_n - 25x_{n+p} + 32x_{n+q} - 25x_{n+r} + 9x_{n+1}. \end{aligned} \quad (17)$$

### 3. Analysis of the basic properties of the methods

In what follows, the basic properties of the OHBM (16) (or equivalently MOHBM (17)) including accuracy, consistency, zero-stability, convergence, linear stability, and A-stability are investigated.

#### 3.1 Order of accuracy and consistency

Rewriting the OHBM (16) in the matrix difference form yields

$$A_1 X_n = A_0 X_{n-1} + h(B_0 F_{n-1} + B_1 F_n), \quad (26)$$

Where  $A_0, A_1, B_0,$  and  $B_1$  are  $4 \times 4$  matrices given by

$$A_0 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}; A_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}; B_0 = \begin{pmatrix} 0 & 0 & 0 & \frac{275+\sqrt{5}}{3000} \\ 0 & 0 & 0 & \frac{17}{192} \\ 0 & 0 & 0 & \frac{275-\sqrt{5}}{3000} \\ 0 & 0 & 0 & \frac{1}{12} \end{pmatrix} \quad (18)$$

$$B_1 = \begin{pmatrix} \frac{625 + 95\sqrt{5}}{3000} & \frac{-192\sqrt{5}}{3000} & \frac{625 - 205\sqrt{5}}{3000} & \frac{275 + \sqrt{5}}{3000} \\ \frac{40 + 15\sqrt{5}}{192} & 0 & \frac{40 - 15\sqrt{5}}{192} & \frac{-1}{192} \\ \frac{625 - 205\sqrt{5}}{3000} & 0 & \frac{625 - 95\sqrt{5}}{3000} & \frac{-(25 + \sqrt{5})}{3000} \\ \frac{3000}{5} & 0 & \frac{3000}{5} & \frac{3000}{1} \\ \frac{1}{12} & 0 & \frac{1}{12} & \frac{1}{12} \end{pmatrix} \quad (19)$$

$$X_n = (x_{n+p}, x_{n+q}, x_{n+r}, x_{n+1})^T,$$

$$X_{n-1} = (x_{n-1+p}, x_{n-1+q}, x_{n-1+r}, x_n)^T,$$

$$F_n = (f_{n+p}, f_{n+q}, f_{n+r}, f_{n+1})^T, \quad (20)$$

$$F_{n-1} = (f_{n-1+p}, f_{n-1+q}, f_{n-1+r}, f_n)^T.$$

For a sufficiently differentiable test function  $m(t_n)$  in the interval  $[0, T]$ , Let the difference operator  $\bar{D}$  for the OHBM in (20) be given as

$$\bar{D}(m(t_n); h) = \sum_{j=0,p,q,r,1} [\bar{\xi}_j(t_n + jh) - h\bar{\mu}_j m'(t_n + jh)], \quad (21)$$

Where,  $\bar{\xi}_j$  and  $\bar{\mu}_j$  are column vectors of the matrices  $A_0$  and  $A_1$ , respectively. The Taylor series expansion about  $t_n$  for  $x(t_n + jh)$  and  $x'(t_n + jh)$  yield

$$\bar{L}(m(t_n); h) = c_0 x(t_n) + c_1 h x'(t_n) + c_2 h^2 x^{(2)}(t_n) + \dots + c_p h^p x^{(p)}(t_n) + \dots \quad (22)$$

where  $c_i, i = 0, 1, 2, \dots$  are vectors. From equation (22), the order of the OHBM is  $p = (5, 5, 5, 6)^T$  with the error constant

$$c_{p+1} = \frac{1}{180000}, \frac{1}{180000}, \frac{1}{230400}, \frac{-1}{1512000} \quad (23)$$

Showing that the OHBM has at least fifth order accuracy.

### 3.2 Zero-stability and convergence

The concept of zero-stability pertains to the characteristics exhibited by a procedure when the value of  $h$  approaches zero. In the context of a homogeneous equation  $x' = 0$  and the discretized form is

$$A_1 X_n - A_0 X_{n-1} = 0 \quad (24)$$

where  $W_0$  and  $W_1$  are given in equations (27) and (34). The first characteristic polynomial  $\rho(\sigma) = \det(\sigma A_1 - A_0) = \sigma^3(\sigma - 1) = 0$ . This implies that  $\sigma_1 = \sigma_2 = \sigma_3 = 0, \sigma_4 = 1$ .

Since the OHBM and the MOHBM satisfy the properties of consistency and zero-stability, then the methods are convergent according to [9].

### 3.3 Linear stability and order stars

The concept of linear stability focuses on the performance of a method in real-world scenarios, where it is crucial to ascertain if the approach will produce desirable outcomes for a given positive value of  $h$ . To validate this concept, commonly known as linear stability, we employ the methodology on a linearized test problem.

$$x(t) = \sigma x(t), \text{Re}(\sigma) < 0 \quad (25)$$

Applying the proposed block method to the trial problem (39), we obtain the recurrence relation

$$X_n = H(\hbar)X_{n-1}, \hbar = \sigma h. \quad (26)$$

where the matrix  $H(\hbar)$  is given by  $(A_1 - rB_0)^{-1}(A_0 - rB_0)$ . The stability property of this matrix's eigenvalues, which governs how the numerical solution behaves, is the spectral radius,  $H(\hbar)$ , which is used in the method to define the region of absolute stability  $S$ . The method is A-stable if

$$S = \{\hbar \in \mathbb{C} : |\rho[H(\hbar)]| < 1\} \quad (27)$$

Upon performing various calculations, it becomes evident that the predominant eigenvalue can be expressed as a quotient function.

$$\rho[H(\hbar)] = \frac{\hbar^4 + 16\hbar^3 + 132\hbar^2 + 600\hbar + 1200}{\hbar^4 - 16\hbar^3 + 132\hbar^2 - 600\hbar + 1200} \quad (28)$$

which has a modulus of less than one in  $\mathbb{C}^-$  (see Figure 1). Hence, the OHBM (16) is A-stable.

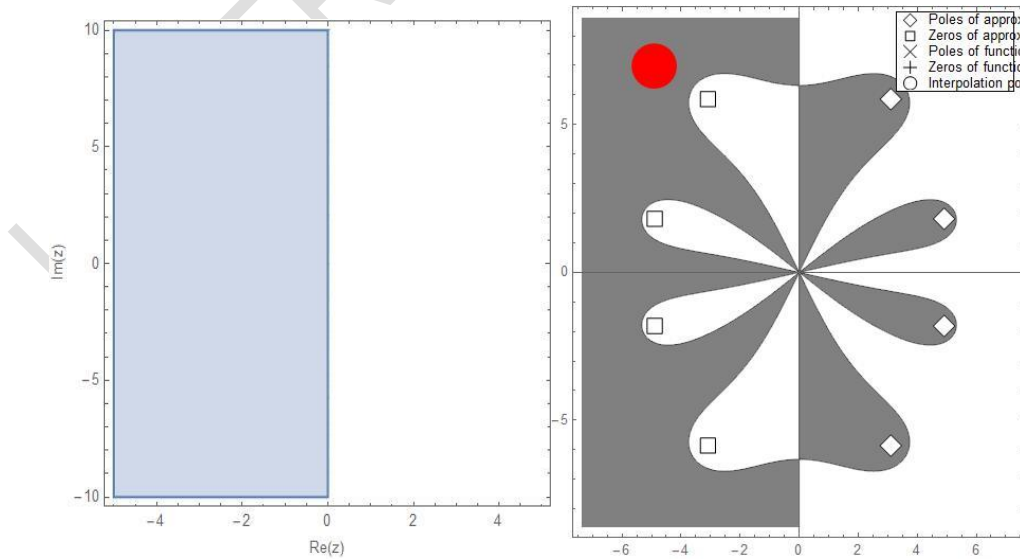


Fig. 1: (a) Region of absolute stability

(b) Order star for OHBM

## 4. RESULTS AND DISCUSSION

In the sequel, the accuracy of the proposed methods will be demonstrated by implementation in solving some popular applied problems of the form (1) in literature. The methods being compared are the OHBM (16), the MOHBM (17), the OSBM in [10] and BHMO and RBHMO in [13].

To measure the performance of each of the aforementioned methods, maximum global absolute error (MAbErr), absolute error at the final grid point (AbErrF), and the CPU time in seconds are computed.

### Problem 4.1

Given the first-order ODE which has appeared in [8,10]:

$$x'(t) = -10(x - 1)^2, \quad x(0) = 2. \quad (29)$$

The exact solution is  $x(t) = 1 + \frac{1}{1+18x}$ . The problem is solved in the interval  $[0,0.1]$  taking  $n = 10, 20, 40$ . The MAbErr, AbErrF, and CPU time are computed using the methods OHBM, MOHBM, and OSBM, and results presented in Table 1. The efficiency curves of MAbErr and CPU time are represented in Fig 3a. The figure indicates that the OHBM and MOHBM outperform existing methods with respect to accuracy and computing time.

### Problem 4.2

Given the first-order ODE which has appeared in [6,10]:

$$x'(t) = tx, \quad x(0) = 1 \quad (30)$$

The exact solution  $x(t) = e^{\frac{1}{2}t^2}$ . The problem is solved in the interval  $[0,1]$  for step sizes  $n = 20, 40, 80$ , with the MAbErr, AbErrF, and CPU time computed using the methods OHBM, MOHBM, BHMO and RBHMO, and results presented in Table 2. The efficiency curves of AbErr and CPU time are represented in Fig 4. The figure reveals that the MOHBM outperform existing method with respect to accuracy and computing time.

### Problem 4.3

Given the nonlinear problem investigated by Akinfenwa and Jator. (2011):

$$x'(t) = -\frac{x^3}{2}, \quad x(0) = 1, \quad (31)$$

with exact solution  $x(t) = 1/\sqrt{t+1}$ , The problem is solved in the interval  $[0,4]$  taking  $n = 20, 40, 80, 100$ . The MAbErr, AbErrF, and CPU time are computed using the methods OHBM, MOHBM, and OSBM, and results presented in Tables 3. The efficiency curves of MAbErr and CPU time are represented in Fig 4a. As revealed by the figure, the OHBM and MOHBM outperform existing method with respect to accuracy and computing time.



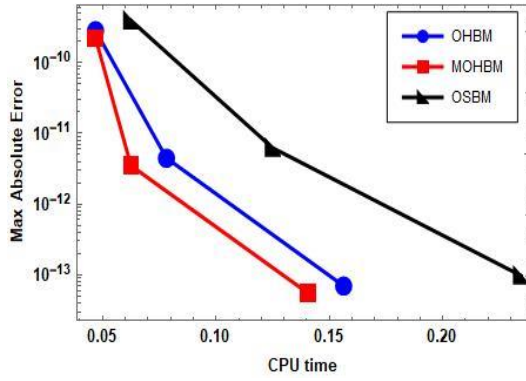


Fig.2a: Efficiency plot for Problem 4.1

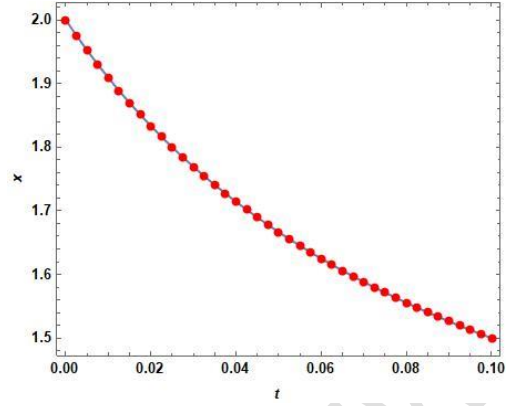


Fig.2b: Solution plot for Problem 4.1

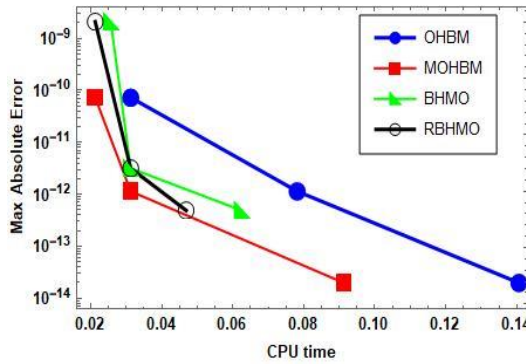


Fig.3a: Efficiency plot for Problem 4.2

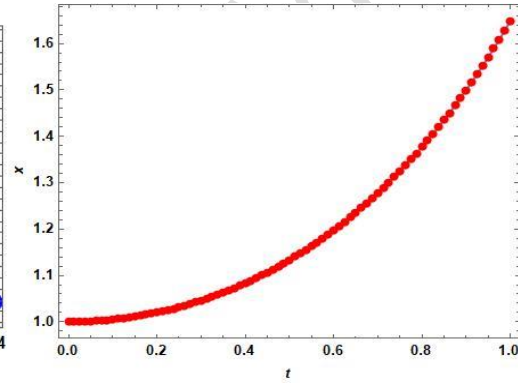


Fig.3b: Solution plot for Problem 4.2

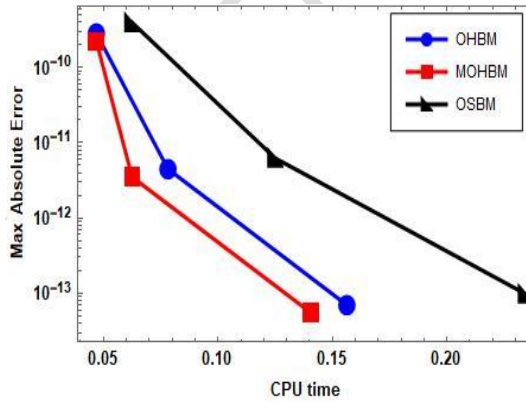


Fig.4a: Efficiency plot for Problem 4.3

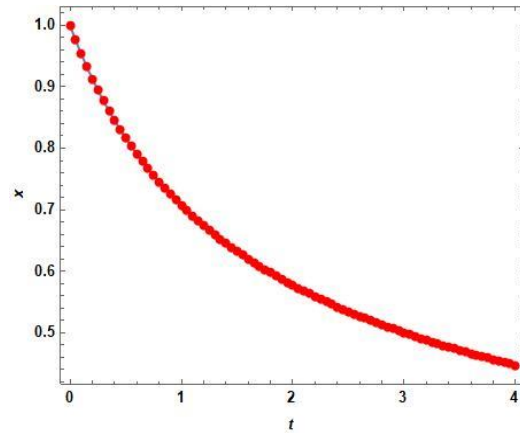


Fig.4b: Solution plot for Problem 4.3

**Table 1:** The MAbErr, AbErrF, and CPU time for Problem 4.1 using different methods and step sizes ( $n$ )

$n$	Method	MAbErr	AbErrF	MErr	Norm	CPU time
10	OHBM	2.85272E-10	1.5998E-10	2.06271E-10	7.29646E-10	4.687E-02
	MOHBM	2.22905E-10	1.25002E-10	1.61175E-10	5.70126E-10	4.687E-02
	OSBM	3.96097E-10	2.22137E-10	2.86408E-10	1.01311E-09	6.250E-02
20	OHBM	4.49196E-12	2.51776E-12	3.37380E-12	1.61640E-11	7.813E-02
	MOHBM	3.50919E-12	1.96665E-12	2.63545E-12	1.26266E-11	6.250E-02
	OSBM	6.23834E-12	3.49609E-12	4.68514E-12	2.24468E-11	1.250E-01
40	OHBM	6.99441E-14	3.97460E-14	5.34478E-14	3.54737E-13	1.563E-01
	MOHBM	5.50671E-14	3.06422E-14	4.16740E-14	2.76761E-13	1.406E-01
	OSBM	9.79217E-14	5.44009E-14	7.47315E-14	4.96271E-13	2.344E-01

**Table 2:** The AbErr, FErr, and CPU time for Problem 4.2 using different methods and step sizes ( $n$ )

$n$	Method	MAbErr	AbErrF	MErr	Norm	CPU time
20	OHBM	7.28471E-11	7.28471E-11	1.44385E-11	8.79531E-11	3.125E-02
	MOHBM	7.28471E-11	7.28471E-11	1.44385E-11	8.79531E-11	2.125E-02
	BHMO	2.10081E-09	2.10081E-09	3.96744E-10	2.49730E-09	2.563E-02
	RBHMO	2.10081E-09	2.10081E-09	3.96744E-10	2.49730E-09	2.125E-02
40	OHBM	1.14420E-12	1.14420E-12	2.06914E-13	1.72935E-12	7.813E-02
	MOHBM	1.14420E-12	1.14420E-12	2.06914E-13	1.72935E-12	3.125E-02
	BHMO	3.16660E-11	3.16660E-11	5.54340E-12	4.73412E-11	3.125E-02
	RBHMO	3.16660E-11	3.16660E-11	5.54340E-12	4.73412E-11	3.125E-02
80	OHBM	1.97620E-14	1.97620E-14	3.49856E-15	3.99421E-14	1.406E-01
	MOHBM	1.97627E-14	1.97627E-14	3.49856E-15	3.99421E-14	9.125E-02
	BHMO	4.89608E-13	4.89608E-13	8.34184E-14	9.76046E-13	6.250E-02
	RBHMO	4.89608E-13	4.89608E-13	8.34184E-14	9.76046E-13	4.687E-02

**Table 3:** The MAbErr, AbErrF, and CPU time for Problem 4.3 using different methods and step sizes ( $n$ )

$n$	Method	MAbErr	AbErrF	MErr	Norm	CPU time
20	OHBM	2.59459E-09	4.73998E-10	1.1552E-09	6.21496E-09	7.813E-02
	MOHBM	2.02819E-09	3.70502E-10	9.02985E-10	4.85811E-09	7.813E-02
	OSBM	3.60018E-09	6.57769E-10	1.60301E-10	8.62403E-09	7.813E-02
40	OHBM	4.19044E-11	7.61163E-12	1.91781E-11	1.41802E-10	1.718E-01
	MOHBM	3.27428E-11	5.94735E-12	1.49850E-11	1.10798E-10	1.250E-01
	OSBM	5.8186E-11	1.05694E-11	2.66302E-11	1.96901E-10	1.718E-01
80	OHBM	6.63469E-13	1.19516E-13	3.05738E-13	3.15630E-12	3.437E-01
	MOHBM	5.18030E-13	9.34253E-14	2.38810E-13	2.46531E-12	2.500E-01
	OSBM	9.21374E-13	1.66145E-13	4.24857E-13	4.38475E-12	3.125E-01

## 5. CONCLUSION

the research has presented the optimal hybrid block method, and the modified optimal hybrid method for solving first-order initial value problems of ODEs. The results in

Tables 1, 2, and 3 reveal that the methods OHBM (16), and MOHBM (17), are highly efficient with minimal errors. Furthermore, the modified method (17) apart from having minor errors also reduced the computational time which is an added advantage. The derived methods were implemented in block modes with the merit of being self-starting and thus required no starting values. The methods have good accuracy properties and are indeed of the higher order of accuracy at the final grid point where the LTEs were optimized, a major advantage of the method.

Also, the methods do not require the creation of separate predictors. The MOHBM showed that the efficiency of the method is dependent on the implementation strategies. The method is advantageous when economic computations in terms of the number of function evaluations and computing times are of major concern. Hence, the techniques are strongly suggested for general use. The Mathematica software package version 12.1 was used to develop the schemes, the plots and the results on Windows Operating System with Processor Intel(R) Core (TM) i5-4310U CPU @ 2.00GHz, 2601 Mhz, 2 Core(s), 4 Logical Processor(s) having 8.0GB installed RAM.

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