EXPONENTIAL-GAMMA- RAYLEIGH DISTRIBUTION THEORY

ABSTRACT

The use of traditional probability models to forecast real-world events is causing growing dissatisfaction among scholars. One of the motives could be that the tail characteristics and goodness of fit metrics have a constraining tendency. Subsequently, there has been a significant increase in the generalisation of well-known probability distributions in recent years. This study focussed on the development of a family of continuous probability distribution from the extension of EG distribution. This is achieved by modifying the newly generated continuous Exponential-Gamma-X family of distributions. Explicit expressions for the probability density function cumulative distribution function, moments, and moment generating function, characteristics function of the EGRD distribution are derived. Various statistical properties, including the coefficient of variation, Also, the study are focussed on the estimation of the proposed model using the technique of maximum likelihood estimation.

Keywords: ***Exponential Gamma-X, Statistical properties, Maximum Likelihood Estimation***

Comment in abstract: EG and EGRD is not clear, the author should write it more clear. My comment , it must be improved

INTRODUCTION

Several researchers have generated new adaptable distributions from existing distributions using various modification techniques to increase their flexibility in modeling data. These adaptable distributions are created by adding extra parameters to the baseline distribution with generators or combining two distributions (Ali, *et al*., 2021). These modified distributions can model data sets efficiently and in most cases, provide the best fit to data sets. When applied because they have more parameters and are more adaptable than their baseline distributions. Introducing an extra parameter to an existing distribution has proven beneficial in studying tail properties and improving the goodness of fit of the compound distribution. Recent emerging data of interest exhibit non-normal characteristics such as high or moderate skewness and high kurtosis, which existing distributions cannot accurately model. As a result, researchers are attempting to develop models that account for the limitations of these distributions. So, in this study, we aim to develop a model that might efficiently fit such data. This purpose is achieved by adding an extra parameter using generators or by the existing models; therefore, in this study, we intend to develop a new probability distribution called the Exponential-Gamma-Rayleigh distribution by using the pdf of the new Exponential-Gamma-X distributions developed by Adewusi, *et al*., (2019).

Comment: Introduction is too short, and it should describe the previous research and some citation related to the topic. My comment , it must be improved.

Methods

In this study, we proposed and explored a new generalization of the Exponential-Gamma and Rayleigh distributions using the T-X method.

Theorem 1: Let X be continuous independent random variable such that; $X \sim RD(x, \sigma)$ follows an Rayleigh distribution and, let $f(x)$ and $F(x)$ be the probability density function and cumulative distribution function of Rayleigh distribution given as:

$$f(x) = \frac{x}{\sigma^2} \mathrm{e}^{-x^2/2\sigma^2}, \; x > 0, \sigma > 0 \tag{1}$$

and $F(x) = 1 - e^{-x^2/2\sigma^2}, \; x > 0, \sigma > 0$ (2)

Similarly, Adewusi, *et al*., (2019), defined a generalized T-X family of Exponential-Gamma-X distribution as:

$$g(x) = \frac{\lambda^{\alpha+1} f(x)}{\Gamma(\alpha)\left(1 - F(x)\right)} \left[-\log\left(1 - F(x)\right)\right]^{\alpha-1} \exp\left[-2\lambda\left(-\log\left(1 - F(x)\right)\right)\right], \; x, \lambda, \alpha > 0, \tag{3}$$

where $g(x)$ in (3) is the PDF of Exponential-Gamma-X family of distribution, $f(x)$ and $1 - F(x)$ are the pdf and the survival function of the baseline distribution.

Inserting (1) and (2) into (3) above, then probability density function, pdf of the Exponential-Gamma-Rayleigh distribution is given as:

$$g(x) = \left[\frac{\lambda}{2\sigma^2}\right]^{\alpha} \frac{2\lambda}{\Gamma(\alpha)} x^{2\alpha-1} \exp\left[-\frac{2\lambda x}{\sigma^2}\right], \; x, \lambda, \alpha, \sigma > 0 \tag{4}$$

For convenient computation, let

$$\left[\frac{\lambda}{2\sigma^2}\right]=\theta \quad \text{then,}$$

$$g(x)=\frac{\theta^{\alpha}\,2\lambda}{\Gamma(\alpha)}x^{2\alpha-1}\exp\left(-2\theta x\right),\ x,\lambda,\alpha,\theta>0 \qquad (5)$$

is the PDF of the new EGRD. The distribution has three parameters namely: α (Shape),θ (scale) and λ (location) .

Comment: the method should write in step by step (in sequence procedure), in how to derive the formula. My comment , it must be improved.

B Statistical properties

Comment: the result is not written properly, so it make the readers do not know. By the the part B till K already showed the work of resut. My comment , it must be written in well

In this section, the statistical properties of EGRD, particularly the first four moments, variance, and coefficient of variation, moment generating function, characteristic function, skewness, and kurtosis are obtained.

C Moment

Theorem 2: If X is a random variable distributed as an EGRD$(x;\theta,\alpha,\lambda,)$,having parameters $\theta,\alpha,\ \lambda$, then the r^{th} non-central moment of X is given by:

$$\mu_r^{'}=\frac{2\theta^{\alpha}\lambda\Gamma(2\alpha+r)}{\left(2\theta\right)^{2\alpha+r}\Gamma(\alpha)} \qquad (6)$$

Proof:

$$\mu_r^{'}=\int_0^{\infty}x^r f(x;\theta,\alpha,\lambda,)dx \qquad (7)$$

$$= \int_0^\infty x^r \frac{2\theta^\alpha \lambda}{\Gamma(\alpha)} x^{2\alpha-1} e^{-2\theta x} dx \quad (8)$$

Let $u = 2\theta x,\ x = \frac{u}{2\theta},\ dx = \frac{du}{2\theta}$ then, (8) reduces to:

$$= \frac{2\theta^\alpha \lambda}{\Gamma(\alpha)} \times \frac{1}{(2\theta)^{2\alpha+r}} \int_0^\infty u^{2\alpha+r-1} e^{-u} du \quad (9)$$

Since $\int_0^\infty u^{2\alpha+r-1} e^{-u} du = \Gamma(2\alpha + r)$, based on gamma function, then

$$\mu_r' = \frac{2\lambda\theta^\alpha \Gamma(2\alpha + r)}{(2\theta)^{2\alpha+r}\, \Gamma(\alpha)} \quad (10)$$

Substituting r =1, 2, 3 and 4 in equation (10) we obtain the first (mean), second, third and the fourth moments by for EGRD: we obtain the variance by the association

$$\mu_2' - (\mu_1')^2$$

Mean = $\mu_1' = \frac{2\alpha\lambda\Gamma(2\alpha)}{2^{2\alpha}\theta^{\alpha+1}\Gamma(\alpha)}$ (11)

$$\mu_2' = \frac{2\alpha\lambda(2\alpha+1)\Gamma(2\alpha)}{2^{2\alpha+1}\theta^{\alpha+2}\Gamma(\alpha)} \quad (12)$$

$$V(x) = \mu_2' - (\mu_1')^2 \quad (13)$$

$$V(x) = \frac{2\alpha\lambda(2\alpha+1)\Gamma(2\alpha)}{2^{2\alpha+1}\theta^{\alpha+2}\Gamma(\alpha)} - \left(\frac{2\alpha\lambda\Gamma(2\alpha)}{2^{2\alpha}\theta^{\alpha+1}\Gamma(\alpha)}\right)^2 \quad (14)$$

$$V(x)=\frac{\alpha\lambda(2\alpha+1)2^{2\alpha}\theta^{\alpha}-4\alpha^{2}\lambda^{2}}{2^{4\alpha}\theta^{2\alpha+2}} \quad (15)$$

Then the 3rd and the 4th moment is given as:

$$\mu_3'=\frac{2\alpha\lambda(2\alpha+1)(2\alpha+2)\Gamma(2\alpha)}{2^{2\alpha+2}\theta^{\alpha+3}\Gamma(\alpha)} \quad (16)$$

$$\mu_4'=\frac{2\alpha\lambda(2\alpha+1)(2\alpha+2)(2\alpha+3)\Gamma(2\alpha)}{2^{2\alpha+3}\theta^{\alpha+4}\Gamma(\alpha)} \text{ respectively} \quad (17)$$

D. Moment generating function

Theorem 3: If X is a continuous random variable distributed as an EGRD$(x;\theta,\alpha,\lambda)$, then the moment generating function is given as $M_x(t)=\frac{2\lambda\theta^{\alpha}}{(2\theta-t)^{2\alpha}}$

$$\mathrm{M}_x(t)=E\left(e^{tx}\right)=\int_0^{\infty}e^{tx}f(x;\theta,\alpha,\lambda)dx \quad (18)$$

$$M_x(t)=\int_0^{\infty}\frac{2\lambda\theta^{\alpha}}{\Gamma(\alpha)}x^{2\alpha+1}e^{-2\theta.x}e^{t.x}dx \quad (19)$$

Let $u=x(2\theta-t),\ x=\frac{u}{(2\theta-t)}$, then $dx=\frac{du}{(2\theta-t)}$ so that (19) is reduced to

$$\frac{2\lambda\theta^{\alpha}}{\Gamma(\alpha)}\cdot\frac{1}{(2\theta-t)^{2\alpha}}\int_0^{\infty}(u)^{2\alpha-1}e^{-u}dx \quad (20)$$

$$M_x(t)=\frac{2\lambda\theta^{\alpha}}{(2\theta-t)^{2\alpha}} \quad (21)$$

E. **Characteristic Function (CF)**

Theorem 5: If X is a random variable distributed as an EGRD$(x;\theta,\alpha,\lambda)$, then the characteristics function $\phi_x(it)$ is defined as $\phi_x(it)=\dfrac{2\lambda\theta^{\alpha}}{(2\theta-it)^{2\alpha}}$

Proof:

$$\phi_x(it)=E\left(e^{itx}\right)=\int_0^\infty e^{itx}f(x;\theta,\alpha,\lambda)dx \quad (22)$$

$$\phi_x(it)=\int_0^\infty \frac{2\lambda\theta^{\alpha}}{\Gamma(\alpha)}x^{2\alpha-1}e^{-2\theta x}e^{itx}dx \quad (23)$$

Let $u=x(2\theta-it),\ x=\dfrac{u}{(2\theta-it)}$, then $dx=\dfrac{du}{(2\theta-it)}$ so that (3.16) is reduced to

$$\frac{2\lambda\theta^{\alpha}}{\Gamma(\alpha)}\times\frac{1}{(2\theta-i\,t)^{2\alpha-1}}\int_0^\infty (u)^{2\alpha-1}e^{-u}du \quad (24)$$

$$\varphi_x(it)=\frac{2\lambda\theta^{\alpha}}{(2\theta-it)^{2\alpha}} \quad (25)$$

F. **Coefficient of Variation (C.V)** is a standardized measure of dispersion of a probability distribution and is given as:

$$C.V=\frac{\sigma}{\mu}=\frac{\sigma}{\mu_1^{/}} \quad (26)$$

$$C.V=\frac{\sqrt{\dfrac{\alpha\lambda(2\alpha+1)2^{2\alpha}\theta^{\alpha}-4\alpha^2\lambda^2}{2^{4\alpha}\theta^{2\alpha+2}}}}{\dfrac{2\alpha\lambda)}{2^{2\alpha}\theta^{\alpha+1}}} \quad (27)$$

$$C.V=\frac{\left(\alpha\lambda(2\alpha+1)2^{2\alpha}\theta^{\alpha}-4\alpha^2\lambda^2\right)^{1/2}}{2\alpha\lambda}$$

G. Skewness and Kurtosis

Skewness is a measure of the asymmetry of the probability distribution of a real-valued random variable about its mean and is given as:

$$SK = \frac{E(x-\mu)^3}{\sigma^3} = \frac{\mu_3}{\sigma^3} = \frac{\mu_3^{'} - 3\mu_1^{'}\mu_2^{'} + 2\mu_1^{'3}}{\sigma^3} \quad (29)$$

$$= \frac{2\alpha\lambda(2\alpha+1)(2\alpha+2)(2\alpha+3)(2^{4\alpha}\theta^{2\alpha}) - 12\alpha^2\lambda^2(6\alpha+3)(2^{2\alpha+1}+\theta^{\alpha+1})(2\alpha+3) + 12\alpha^3\lambda^3}{2^3\theta\left(4\alpha^2\lambda + 2\alpha\lambda(2^{\alpha+1}\theta^{\alpha}) - 4\alpha^2\lambda^2\right)^{3/2}}$$

Kurtosis is a descriptor of the shape of a probability distribution and is given as;

$$K = \frac{E(x-\mu)^4}{\sigma^4} = \frac{\mu_4}{\sigma^4} = \frac{\mu_4^{'} - 4\mu_1^{'}\mu_3^{'} + 6\mu_1^{'2}\mu_2^{'} - 3\mu_1^{'4}}{\sigma^4} \quad (31)$$

$$K = \frac{\left(\frac{2\alpha\lambda(2\alpha+1)(2\alpha+2)(2\alpha+3)\Gamma(2\alpha)}{2^{2\alpha+3}\theta^{\alpha+4}\Gamma(\alpha)}\right) - 4\left(\frac{2\alpha\lambda\Gamma(2\alpha)}{2^{2\alpha}\theta^{\alpha+1}\Gamma(\alpha)}\right)\left(\frac{2\alpha(2\alpha+1)(2\alpha+2)(\Gamma(2\alpha)}{2^{2\alpha+2}\theta^{\alpha+3}\Gamma(\alpha)}\right) + 6\left(\frac{2\alpha\lambda}{2^{2\alpha}\theta^{\alpha+1}}\right)^2 - 3\left(\frac{2\alpha\lambda\Gamma(2\alpha)}{2^{2\alpha}\theta^{\alpha+2}}\right)^4}{\left(\frac{\alpha\lambda(2\alpha+1)2^{2\alpha}\theta^{\alpha} - 4\alpha^2\lambda^2}{2^{4\alpha}\theta^{2\alpha+2}}\right)^2} \quad (32)$$

$$SK = \frac{\left(\frac{2\alpha\lambda(2\alpha+1)(2\alpha+2)(\Gamma(2\alpha)}{2^{2\alpha+2}\theta^{\alpha+3}\Gamma(\alpha)}\right) - 3\left(\frac{2\alpha\lambda\Gamma(2\alpha)}{2^{2\alpha}\theta^{\alpha+1}\Gamma(\alpha)}\right)\left(\frac{2\alpha(2\alpha+1)\Gamma(2\alpha)}{2^{2\alpha+1}\theta^{\alpha+2}\Gamma(\alpha)}\right) + 2\left(\frac{2\alpha\lambda\Gamma(2\alpha)}{2^{2\alpha}\theta^{\alpha+2}}\right)^3}{\left(\frac{\alpha\lambda(2\alpha+1)2^{2\alpha}\theta^{\alpha} - 4\alpha^2\lambda^2}{2^{4\alpha}\theta^{2\alpha+2}}\right)^{3/2}}$$

(33)

Cumulative Distribution Function (CDF)

The cumulative distribution function of a random variable *X* evaluated at *x* is the probability that *X* will take a value less than or equal to *x* and is defined as;

$$F(x) = P\left(X \le x\right) = \int_0^x f(x;\theta,\alpha,\lambda)dx \quad (34)$$

Theorem 6: If X is a continuous random variable from the Exponential-Exponential distribution, the cumulative density function (CDF) is defined by

$$F(x) = \frac{2\lambda\gamma(\alpha, x)}{\Gamma(\alpha)2^{2\alpha}\theta^{\alpha}},\ x,\theta,\alpha,\lambda > 0 \qquad (35)$$

Proof:

$$f(x) = \frac{2\lambda\theta^{\alpha}}{\Gamma(\alpha)} x^{2\alpha-1}e^{-2\theta x} \qquad x,\theta,\alpha,\lambda > 0 \qquad (36)$$

let $u = 2\theta x,\ \ x = \frac{u}{2\theta},$ then $dx = \frac{du}{2\theta}$ so (3.19) is reduced to:

$$= \frac{2\lambda\theta^{\alpha}}{\Gamma(\alpha)} \times \frac{1}{(2\theta)^{2\alpha}} \int_0^x u^{2\alpha-1}e^{-u}du \qquad (37)$$

$$,\ F(x) = \frac{2\lambda\gamma(2\alpha, x)}{2^{2\alpha}\theta^{\alpha}\Gamma(\alpha)} \qquad x,\theta,\alpha,\lambda > 0 \qquad (38)$$

PDF

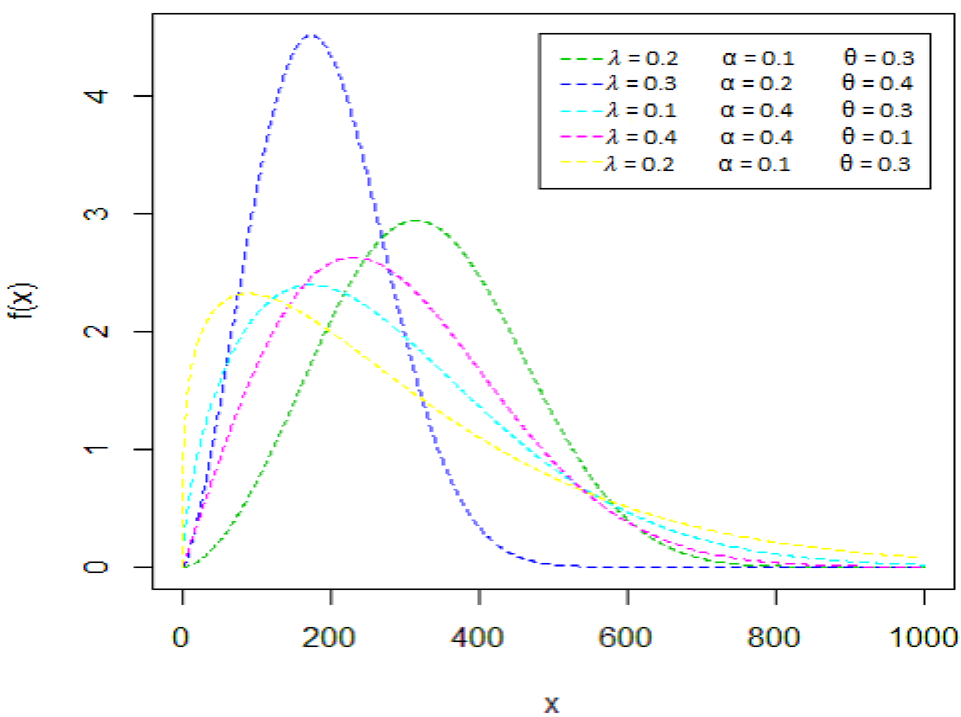

Fig 1. PDF plots of EGRD

CDF

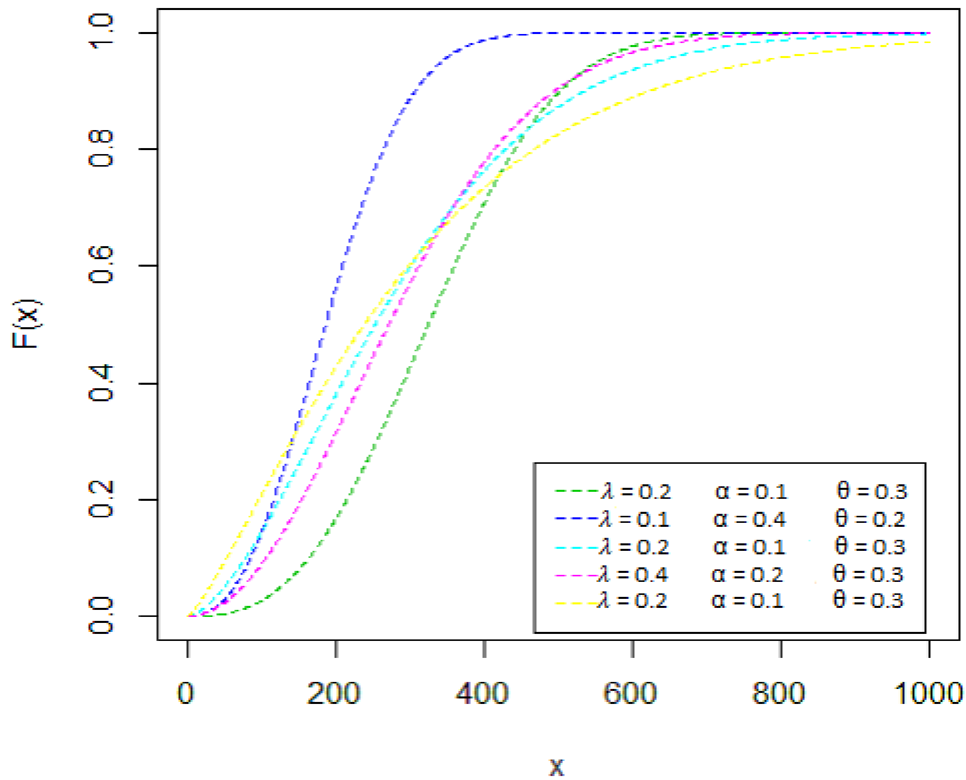

Fig 2. CDF plots for EGRD

From figure 1 The plot reveals various possible shapes of the EGR density function, including (approximately) symmetric, skewed, and unimodal were produced. It can be seen that the tail of the distribution is longer and shorter on both right side and left side for different combinations of the values of the parameters. This demonstrates the great flexibility of the EGR distribution, which makes it suitable for various real data and CDF plot shown in Figure 2 that EGRD starts from zero on the y axis and tend to 1 on x axis , which is an indication that the EGRD is a valid distribution because it satisfies the basic property of a valid probability distribution which states that the probability of any event is greater than or equal to zero and the sum of the cumulative probabilities of events is equal to one

I. **Reliability Function**

The reliability function also known as survival function is a function that measures the likelihood that a patient, device, or other object of interest will survive beyond a specific time range and is defined as:

$$S(x) = 1 - F(x) \quad (39)$$

where $F(x)$ is the cumulative distribution function of *X,* substituting,

$$S(x) = \frac{2^{2\alpha}\theta^{\alpha}\Gamma(\alpha) - 2\lambda\gamma(2\alpha, x)}{2^{2\alpha}\theta^{\alpha}\Gamma(\alpha)} \quad (40)$$

I. **Hazard Function**

The hazard function also called the *force of mortality, instantaneous failure rate, instantaneous death rate*, or *age-specific failure rate* is the instantaneous risk that the event of interest happens, within a very narrow time frame and is defined as;

$$h(x) = \frac{f(x)}{S(x)} \quad (41)$$

where $f(x)$ and $S(x)$ are pdf and survival function of EGRD then,

$$h(x) = \frac{\dfrac{2\lambda\theta^{\alpha}x^{2\alpha-1}e^{-2\theta.x}}{\Gamma(\alpha)}}{\dfrac{2^{2\alpha}\theta^{\alpha}\Gamma(\alpha) - 2\lambda\gamma(2\alpha, x)}{2^{2\alpha}\theta^{\alpha}\Gamma(\alpha)}} \quad (42)$$

$$h(x) = \frac{2^{2\alpha+1}\theta^{2\alpha}\lambda x^{2\alpha-1}e^{-2\theta.x}}{2^{2\alpha}\theta^{\alpha}\Gamma(\alpha) - 2\lambda\gamma(2\alpha, x)}$$

Comment: the **Reliability Function and Hazard Function** are not clear in relationship to this resreach, it should be improved in techni of written in deriving formula . My comment , it must be written in well

K. **THE MAXIMUM LIKELIHOOD ESTIMATOR**

Theorem 7: Let $x_1, x_2, ...x_n$ be a random sample of size n from Exponential-Gamma-Rayleigh distribution (EGRD) with pdf

$$\mathrm{L}(\alpha,\lambda,\beta;x)=\left(\frac{2\lambda\theta^{\alpha}}{\Gamma(\alpha)}\right)^{n}\prod_{i=1}^{n}x_i^{2\alpha-1}\exp\left(-2\theta\sum x_i\right) \qquad (44)$$

By taking the natural logarithm of (44), the log-likelihood function is obtained as;

$$\log_e(L)=\alpha n\log_e\theta+n\log_e 2\lambda-n\log_e\Gamma(\alpha)+\left(2\alpha-1\right)\sum\log_e x_i-2\theta\sum x_i \qquad (45)$$

Therefore, the MLE which maximizes (3.35) must satisfy the following normal equations;

$$\frac{\partial\log_e L}{\partial\alpha}=n\log_e\theta-\frac{n\Gamma^{'}(\alpha)}{\Gamma(\alpha)}+2\sum_{i=1}^{n}\log_e x_i=0 \qquad (46)$$

$$\frac{\partial\log_e L}{\partial\lambda}=\frac{n}{\lambda}=0 \qquad (47)$$

$$\frac{\partial\log_e L}{\partial\theta}=\frac{\alpha n}{\theta}-2\sum x_i=0 \qquad (48)$$

Differentiating equation (45) with respect to α , λ and θ give the maximum likelihood estimates of the model parameters that generate the solution of the nonlinear system of equations. The parameters can be estimated numerically by solving (46), (47), and (48), while solving it analytically is very cumbersome and tasking. The numerical solution can also be obtained directly using some data sets in Python but there are other programming laguages that could do the work.

CONCLUSION

A new extension of the T-X family of the Exponential-Gamma-X distribution named Exponential-Gamma-Rayleigh distribution comprising three parameters has been developed and studied. We explored and generated several expressions for distribution theory and properties including the first four moment, moment generating function, characteristics function cumulative distribution function skewness, kurtosis and the maximum likelihood estimation approach was use to estimate the parameters of the distribution.

REFERENCES

Adewusi, O. A., Ayeni, T. M., and Ogunwale, O. D. (2019). On The Exponential- Gamma-X Mixed Distribution. *International Journal of Research and Innovation in Applied Science*, 4 (11),139-141.

Ali T. M., Mayssa J. M., Muna, D. S., and Rabha W. I. (2021). The inverse exponential Rayleigh distribution and related concepts. *Italian Journal of Pure And Applied Mathematics, 47(2), 852–861.*

Comment: the **REFERENCES are only two, and it is not eligible for paper.** My comment , it must be added (at least 10 journals, and 5 bokks).

Generally, the paper has not eligible yet for publishing