Some Fixed Point Results for Integral Type Mappings in b-Metric Space

Abstract

Singh[15] obtained some results on fixed point theorems for Lebesgue integrable mapping satisfying b-(E.A.) property in b- metric spaces. In this manuscript, we prove some common fixed point theorems for generalized b-(E.A.) property in b- metric spaces. We have given an example to support our results.

Keywords: Fixed Point, Generalized b-(E.A) property, Weakly Compatible Mapping, b-metric Space.

Mathematical Subject Classification (2000): Primary 47H10, Secondary 54H25.

1. Introduction

Many fixed point results have been established over the past 95 years and we discover that the majority of these results are based on the Banach contraction principle.There are numerous ways to generalize the idea of metric spaces. Czerwik introduced the idea of a b- metric space in [7, 8] and in the following few years, other writers have proved numerous fixed point theorems in b-metric spaces. Jungck [11] first proposed the idea of compatible mapping in 1986 and he used it to strengthen the commutativity requirements in standard fixed point theorems.

Aamri and Moutawakil [1] and Liu *et al.* [17] have defined the property (E.A) and the common property (E.A), respectively. Later on, authors such as Ali *et al.* [3], Babu and Sailaja [5], Nazir and Abbas [12], Oztirk and Radanovic [13] and Ozturk and Turkoglu [14] published new fixed point results based on this idea. Sequential requires the following definitions.

2. Preliminaries

Definition 2.1[7] Let Ẋ be a non empty set. A mapping ḍ: Ẋ×Ẋ→ $[0,\infty)$ is called b-metric if there exists a real number $b \geq 1$ such that for every $x, y, z \in$ Ẋ, we have

(i) ḍ$(x,y) = 0$ if and only if $x = y$.

(ii) ḍ$(x,y) =$ ḍ(y,x) if and only if $x = y$.

(iii) ḍ$(x,z) \leq b[$ḍ$(x,y) +$ ḍ$(y,z)]$

In this case $(\dot{X}, ḍ)$ is called a b-metric space. There exists so many examples in literature see [3,4,5] showing that every metric space is a b- metric space with $b = 1$, while the converse need not be true i.e the class of b-metric space is effectively larger than that of ordinary metric spaces.

Definition 2.2 [10] Let $\{x_n\}$ be a sequence in a b-metric space $(\dot{X}, ḍ)$.

(i) $\{x_n\}$ is called b-convergent if and only if there exists $x \in \dot{X}$ such that $ḍ(x_n, x) \to 0$ as $n \to \infty$.

(ii) $\{x_n\}$ is called b-Cauchy sequence if and only if there exists $x \in \dot{X}$ such that $ḍ(x_n, x_m) \to 0$ as $n, m \to \infty$.

A b-metric space $(\dot{X}, ḍ)$ is said to be complete if and only if each b-Cauchy sequence in $\dot{X}$ is b-convergent.

Definition 2.3 [9] Let $(\dot{X}, ḍ)$ be a b-metric space. A subset $Y \subset \dot{X}$ is called closed if and only if for each sequence $\{x_n\}$ in Y which is b-converges to an element x, we have $x \in$ Y.

Definition 2.4 [11] Let $(\dot{X}, ḍ)$ be a b-metric space and f and g are self maps on $\dot{X}$

(i) f and g are said to be compatible if whenever a sequence $\{x_n\}$ in $\dot{X}$ such that $\{gx_n\}$ and $\{fx_n\}$ are b- convergent to some $t \in \dot{X}$, then $\lim_{n\to\infty} ḍ(fgx_n, gfx_n) = 0$.

(ii) f and g are said to non-compatible if there exists at least one sequence $\{x_n\}$ in $\dot{X}$ such that $\{fx_n\}$ and $\{gx_n\}$ are b-convergent to some $t \in \dot{X}$, but $\lim_{n\to\infty} ḍ(fgx_n, gfx_n)$ is either non zero or does not exist.

Definition 2.5 [14] f and g are said to satisfy the b-(E.A) property if there exists a sequence $\{x_n\}$ such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t$, for some $t \in \dot{X}$.

Remarks 2.6 Non compatibility implies b-(E.A)-property.

Example 2.7 [14] Let $\dot{X} = [0, 1]$ and define $ḍ : \dot{X} \times \dot{X} \to [0, \infty)$ as follows

$$ḍ(x, y) = (x - y)^2.$$

Let $f, g: \dot{X} \to \dot{X}$ be defined as

$$f(x) = \begin{cases} \frac{1}{6}, & x \in \left[0, \frac{1}{2}\right] \\ \frac{x+1}{9}, & x \in \left[\frac{1}{2}, 1\right] \end{cases} \quad \text{and} \quad g(x) = \begin{cases} \frac{1-x}{3}, & x \in \left[0, \frac{1}{2}\right] \\ \frac{x}{3}, & x \in \left[\frac{1}{2}, 2\right] \end{cases}$$

for a sequence $\{x_n\}$ in $\dot{X}$ such that $x_n = \frac{1}{2} - \frac{1}{n}, n = 0, 1, 2 ....$

and $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = \frac{1}{6}$

So f and g are satisfying the b-(E.A) property.

$\lim_{n\to\infty} \text{ḍ}(fgx_n, gfx_n)$ exists and it is not equal to 0. Thus f and g are non-compatible.

Definition 2.8 [3] Let f and g be self-maps of a set Ẋ. If $w = fx = gx$ for some x in Ẋ then x is called a coincidence point of f and g and w is called a point of coincidence of f and g.

Definition 2.9[3] Let f and g be self-maps of a set Ẋ. Then f and g are said to be weakly compatible if they commute at their coincidence point.

Proposition 2.10[3] Let f and g be weakly compatibility self-maps of a set Ẋ. If f and g have a unique point of coincidence $w = fx = gx$, then w is called unique common fixed point of f and g.

Definition 2.11 [3] A function ϕ is said to be integral sub additive, if for each $\alpha, \beta > 0$,

$$\int_0^{\alpha+\beta} \varphi(t)dt \leq \int_0^{\alpha} \varphi(t)dt + \int_0^{\beta} \varphi(t)dt$$

Lemma 2.12[17] Let $\phi: [0,\infty) \to [0,\infty)$ be lebesgue integrable mapping which is summable on each compact subset of $[0,\infty)$, non negative and such that for each $\epsilon > 0$, $\int_0^{\epsilon} \varphi(t)dt > 0$ and $\{a_n\}$ be a sequence of nonnegative numbers with $\lim_{n\to\infty} a_n = a$. Then

$$\lim_{n\to\infty} \int_0^{a_n} \varphi(t)dt = \int_0^{a} \varphi(t)dt$$

Lemma 2.13 [17] Let $\phi: [0,\infty) \to [0,\infty)$ be lebesgue integrable mapping which is summable on each compact subset of $[0,\infty)$, non negative and such that for each $\epsilon > 0$, $\int_0^{\epsilon} \varphi(t)dt > 0$

and $\{a_n\}$ be a sequence of non negative numbers with $\lim_{n\to\infty} a_n = a$. Then

$$\lim_{n\to\infty} \int_0^{a_n} \varphi(t)dt \Leftrightarrow \int_0^{a} \varphi(t)dt$$

We define ψ and ϕ as follows:

$\psi = \{\psi: [0,\infty) \to [0,\infty), \psi$ is upper semi-continuous, sequence $\psi^n(t)$ converges to 0 as $n \to \infty$ for all $t > 0$ and $\psi(t) < t$ for any $t > 0\}$

$\phi = \{\phi: [0,\infty) \to [0,\infty), \phi$ is lebesgue integrable, summable on each compact subset of $[0,\infty)$, non negative and for each $\epsilon > 0$, $\int_0^{\epsilon} \varphi(t)dt > 0\}$

3. Main Result

In 2020, Amarjeet Singh Saluja[15] proved the following fixed point theorem:

"Let (X, d) be b-metric space with $b > 1$ and $f, g, S, T: X \to X$ be mappings with $fX \subseteq TX$ and $gX \subseteq SX$ such that

(3.1.1) $\int_0^{b^{\varepsilon} d(fx,gy)} \varphi(t)dt \leq \int_0^{M_b(x,y)} \varphi(t)dt \quad for\ all\ x, y \in X$

Where $\epsilon > 1$ is a constant and $\varphi: [0, \infty) \to [0, \infty)$ is Lebesque integrable mapping which is summable on each compact subset of $[0, \infty)$, non-negative and such that for $c > 0, \int_0^c \varphi(t)dt > 0$ and

(3.1.2) $M_b(x, y) = max \left\{ \begin{array}{c} d(Sx, Ty), d(fx, Sx), d(gy, Ty), \\ \frac{d(gy,Ty)+d(fx,Sx)}{2b}, \frac{d(Sx,gy)+d(fx,Ty)}{2b} \end{array} \right\}$

Suppose that one of the pair (f, S) and (g, T) satisfy the b- (E.A)- property and that one of the subspaces $f(X),\ g(X),\ T(X) and\ S(X)$ is b-closed in X. Then the pair (f, S) and (g, T) have a point of coincidence in X, Moreover, if the pair (f, S) and (g, T) are weakly compatible, then f, g, S and T have a unique common fixed point."

In this paper, we prove the above-mentioned theorem proved by Singh[15]by using generalized b- (E.A)- property as follows:

Theorem 3.1 Let $(\dot{X}, \underset{\cdot}{d})$be b-metric space with $b > 1$ and $f, g, S, T: \dot{X} \to \dot{X}$ be mappings with $f\dot{X} \subseteq T\dot{X}$ and $g\dot{X} \subseteq S\dot{X}$ such that

(3.1.3) $\int_0^{b^{\varepsilon} \underset{\cdot}{d}(fx,gy)} \varphi(t)dt \leq \int_0^{M_b(x,y)} \varphi(t)dt \quad for\ all\ x, y \in \dot{X}$

where $\epsilon > 1$ is a constant and $\varphi: [0, \infty) \to [0, \infty)$ is Lebesque integrable mapping which is summable on each compact subset of $[0, \infty)$, non-negative and such that for

$c > 0, \int_0^c \varphi(t)dt > 0$ and

(3.1.4) $M_b(x, y) = \{\underset{\cdot}{d}(Sx, Ty), \underset{\cdot}{d}(fx, Sx), \underset{\cdot}{d}(gy, Ty), \frac{\underset{\cdot}{d}(Sx,gy)+\underset{\cdot}{d}(fx,Ty)}{2b},$

$$\underset{\cdot}{d}(Sx, fx)\left[\frac{1 + \underset{\cdot}{d}(Sx, Ty)}{1 + \underset{\cdot}{d}(Ty, fy)}\right], \underset{\cdot}{d}(Ty, gy)\left[\frac{1 + \underset{\cdot}{d}(Sx, Ty)}{1 + \underset{\cdot}{d}(Sx, fx)}\right],$$

$$\left.\frac{\underset{\cdot}{d}^2(Sx, fx)}{1 + \underset{\cdot}{d}(fx, gy)}, \frac{\underset{\cdot}{d}^2(Ty, gy)}{1 + \underset{\cdot}{d}(fx, gy)}\right\}$$

Suppose that one of the pair (f, S) and (g, T) satisfy the b- (E.A)- property and that one of the subspaces $f(\dot{X}), g(\dot{X}), T(\dot{X}) and\ S(\dot{X})$ is b-closed in X. Then the pair (f, S) and (g, T)

have a point of coincidence in $\dot{X}$, Moreover if the pair (f,S) and (g,T) are weakly compatible, then f,g,S and T have a unique common fixed point.

Proof If the pair (f,S) satisfies the b-(E.A)-property, then there exists a sequence $\{x_n\}$ in $\dot{X}$ satisfying $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} Sx_n = q$, for some $q \in \dot{X}$.

As $f\dot{X} \subseteq T\dot{X}$ there exists a sequence $\{y_n\}$ in $\dot{X}$ such that

$$fx_n = Ty_n.$$

Hence

$$\lim_{n\to\infty} Ty_n = q$$

Now, we will show that

$$\lim_{n\to\infty} gy_n = q,$$

From (3.1.3), we have

(3.1.5) $$\int_0^{b^{\varepsilon}ḍ(fx_n,gy_n)} \varphi(t)dt \le \int_0^{M_b(x_n,y_n)} \varphi(t)dt,$$

where

$$M_b(x_n,y_n) = max\,\Bigg\{ḍ(Sx_n,Ty_n), ḍ(fx_n,Sx_n), ḍ(gy_n,Ty_n), \frac{ḍ(Sx_n,gy_n)+ḍ(fx_n,Ty_n)}{2b},$$
$$ḍ(Sx_n,fx_n)\left[\frac{1+ḍ(Sx_n,Ty_n)}{1+ḍ(Ty_n,fy_n)}\right], ḍ(Ty_n,gy_n)\left[\frac{1+ḍ(Sx_n,Ty_n)}{1+ḍ(Sx_n,fx_n)}\right],$$
$$\frac{ḍ^2(Sx_n,fx_n)}{1+ḍ(fx_n,gy_n)}, \frac{ḍ^2(Ty_n,gy_n)}{1+ḍ(fx_n,gy_n)}\Bigg\}$$
$$= \Bigg\{ḍ(Sx_n,Ty_n), ḍ(fx_n,Sx_n), ḍ(gy_n,Ty_n), \frac{ḍ(Sx_n,gy_n)+ḍ(fx_n,Ty_n)}{2b},$$
$$ḍ(Sx_n,fx_n)\left[\frac{1+ḍ(Sx_n,Ty_n)}{1+ḍ(Ty_n,fy_n)}\right], ḍ(Ty_n,gy_n)\left[\frac{1+ḍ(Sx_n,Ty_n)}{1+ḍ(Sx_n,fx_n)}\right],$$
$$\frac{ḍ^2(Sx_n,fx_n)}{1+ḍ(fx_n,gy_n)}, \frac{ḍ^2(Ty_n,gy_n)}{1+ḍ(fx_n,gy_n)}\Bigg\}$$

$$\lim_{n\to\infty} M_b(x_n,y_n) = \lim_{n\to\infty} \max\Big\{d(q,q), d(q,q), d(q,gy_n), \tfrac{1}{2b}\big(d(q,gy_n) +$$
$$d(q,q)\big), d(q,q)\left[\tfrac{1+d(q,q)}{1+d(q,fy_n)}\right],$$
$$d(q,gy_n)\left[\frac{1+d(q,q)}{1+d(q,q)}\right], \frac{d^2(q,q)}{1+d(q,gy_n)}, \frac{d^2(q,gy_n)}{1+d(q,gy_n)}\Big\}$$

$$= \lim_{n\to\infty} \max\left\{0, 0, d(q, gy_n), \frac{1}{2b} d(q, gy_n), 0, d(q, gy_n), 0, \frac{d^2(q, gy_n)}{1 + d(q, gy_n)}\right\}$$

$$= \lim_{n\to\infty} d(q, gy_n).$$

Now, on taking limit and using lemma 2.13, we get

$$\lim_{n\to\infty} \int_0^{b^{\varepsilon} d(fx_n, gy_n)} \varphi(t)dt \leq \lim_{n\to\infty} \int_0^{d(q, gy_n)} \varphi(t)dt$$

Since $b^{\varepsilon} > b > 1$, we have

$$\lim_{n\to\infty} gy_n = q,$$

If $T\dot{X}$ is closed subspace of $\dot{X}$, then there exists $r \in \dot{X}$, such that $Tr = q$.

Now, we shall show that $gr = q$. Indeed, we have

From (3.1.3), we have

(3.1.6) $$\int_0^{bd(fx_n, gr)} \phi(t)dt \leq \int_0^{M_b d(x_n, r)} \phi(t)dt,$$

where

$$M_b(x_n, r) = \{d(Sx_n, Tr), d(fx_n, Sx_n), d(gr, Tr), \frac{d(Sx_n, gr) + d(fx_n, Tr)}{2b},$$
$$d(Sx_n, fx_n)\left[\frac{1 + d(Sx_n, Tr)}{1 + d(Tr, fr)}\right], d(Tr, gr)\left[\frac{1 + d(Sx_n, Tr)}{1 + d(Sx_n, fx_n)}\right],$$
$$\left.\frac{d^2(Sx_n, fx_n)}{1 + d(fx_n, gr)}, \frac{d^2(Tr, gr)}{1 + d(fx_n, gr)}\right\}.$$

Taking limit as $n \to \infty$, we have

$$\lim_{n\to\infty} M_b(x_n, r) = \max\left\{d(q, q), d(q, q), d(q, gr), \frac{1}{2b}\big(d(q, gr) + d(q, q)\big),\right.$$
$$\left. d(q, q)\left[\frac{1 + d(q, q)}{1 + d(q, fr)}\right], d(q, gr)\left[\frac{1 + d(q, q)}{1 + d(q, q)}\right], \frac{d^2(q, q)}{1 + d(q, gr)}, \frac{d^2(q, gr)}{1 + d(q, gr)}\right\}$$
$$= \max\left\{0, 0, d(q, gr), \frac{1}{2} d(q, gr), 0, d(q, gr), 0, \frac{d^2(q, gr)}{1 + d(q, gr)}\right\}$$
$$= d(q, gr).$$

Hence from (3.1.6), we have

$$\int_0^{bd(q, gr)} \phi(t)dt \leq \int_0^{d(q, gr)} \phi(t)dt,$$

which is a contradiction, since $b > 1$.

Hence $q = gr$ or $Tr = gr = q$, which implies that r is coincident point of the pair (g, T). As $g\dot{X} \subseteq S\dot{X}$, there exists $z \in \dot{X}$, such that $q = Sz$

From (3.1.3), we have

(3.1.7) $$\int_0^{b^{\epsilon}ḍ(fz,gr)} \varphi(t)dt \leq \int_0^{M_b(z,r)} \varphi(t)dt \quad for\ all\ x, y \in \dot{X},$$

where

$$M_b(z,r) = max\ \{ḍ(Sz,Tr), ḍ(fz,Sz), ḍ(gr,Tr), \frac{ḍ(Sz,gr) + ḍ(fz,Tr)}{2b},$$
$$ḍ(Sz,fz)\left[\frac{1 + ḍ(Sz,Tr)}{1 + ḍ(Tr,fr)}\right], ḍ(Tr,gr)\left[\frac{1 + ḍ(Sz,Tr)}{1 + ḍ(Sz,fz)}\right],$$
$$\left.\frac{ḍ^2(Sz,fz)}{1 + ḍ(fz,gr)}, \frac{ḍ^2(Tr,gr)}{1 + ḍ(fz,gr)}\right\}$$
$$= max\{ḍ(q,q), ḍ(fz,q), ḍ(q,q), \frac{ḍ(q,q) + ḍ(fz,q)}{2b},$$
$$ḍ(q,fz)\left[\frac{1 + ḍ(q,q)}{1 + ḍ(q,q)}\right], ḍ(q,q)\left[\frac{1 + ḍ(q,q)}{1 + ḍ(q,fz)}\right],$$
$$\left.\frac{ḍ^2(q,fz)}{1 + ḍ(fz,q)}, \frac{ḍ^2(q,q)}{1 + ḍ(fz,q)}\right\}$$
$$= max\left\{0, d(fz,q), 0, \frac{1}{2b}ḍ(q,fz), ḍ(q,fz), \frac{ḍ^2(q,fz)}{1 + ḍ(fz,q)}, 0\right\}$$
$$= ḍ(q,fz).$$

From (3.1.7), we have

$ḍ(q,fz) = 0$, as $b^{\epsilon} > b > 1$.

Therefore, $Sz = fz = q$.

Thus, $fz = Sz = gr = Tr = q$

By weak compatibility of the pair (f, S) and (g, T), we obtain

$fq = Sq$ and $gq = Tq$.

Now we shall show that q is the common fixed point of f, g, S and T.

From (3.1.3), we have

(3.1.8) $$\int_0^{b^{\epsilon}ḍ(fq,q)} \varphi(t)dt = \int_0^{b^{\epsilon}ḍ(fq,gr)} \varphi(t)dt \leq \int_0^{M_b(q,r)} \varphi(t)dt \quad for\ all\ x, y \in \dot{X},$$

where

$$M_b(q,r) = max\,\{ḍ(Sq,Tr), ḍ(fq,Sq), ḍ(gr,Tr), \frac{ḍ(Sq,gr) + ḍ(fq,Tr)}{2b},$$

$$ḍ(Sq,fq)\left[\frac{1 + ḍ(Sq,Tr)}{1 + ḍ(Tr,fr)}\right], ḍ(Tr,gr)\left[\frac{1 + ḍ(Sq,Tr)}{1 + ḍ(Sq,fq)}\right],$$

$$\left.\frac{ḍ^2(Sq,fq)}{1 + ḍ(fq,gr)}, \frac{ḍ^2(Tr,gr)}{1 + ḍ(fq,gr)}\right\}$$

$$= max\{ḍ(fq,q), ḍ(fq,fq), ḍ(q,q), \frac{ḍ(fq,q) + ḍ(fq,q)}{2b},$$

$$ḍ(Sq,Sq)\left[\frac{1 + ḍ(fq,q)}{1 + ḍ(q,q)}\right], ḍ(q,q)\left[\frac{1 + ḍ(fq,q)}{1 + ḍ(fq,fq)}\right],$$

$$\left.\frac{ḍ^2(Sq,fq)}{1 + ḍ(fq,q)}, \frac{ḍ^2(q,q)}{1 + ḍ(fq,q)}\right\}$$

$$= max\left\{ḍ(fq,q), 0, 0, \frac{1}{2b}ḍ(fq,q), 0, 0, 0, 0\right\}$$

$$= ḍ(fq,q).$$

Using (3.1.8), we have

$fq = q = Sq$.

Similarly, it can be shown $gq = Tq = q$.

To prove uniqueness of fixed point, suppose that p is an another fixed point of f, g, S, T.

From (3.1.3), we have

$$(3.1.8) \qquad \int_0^{b^{\varepsilon}ḍ(q,p)} \varphi(t)dt \leq \int_0^{b^{\varepsilon}ḍ(fq,gp)} \varphi(t)dt \leq \int_0^{M_b(q,p)} \varphi(t)dt \quad for\ all\ x, y \in Ẋ,$$

where

$$M_b(q,p) = max\,\{ḍ(Sq,Tp), ḍ(fq,Sq), ḍ(gp,Tp), \frac{ḍ(Sq,gp) + ḍ(fq,Tp)}{2b},$$

$$ḍ(Sq,fq)\left[\frac{1 + ḍ(Sq,Tp)}{1 + ḍ(Tp,fp)}\right], ḍ(Tp,gp)\left[\frac{1 + ḍ(Sq,Tp)}{1 + ḍ(Sq,fq)}\right],$$

$$\left.\frac{ḍ^2(Sq,fq)}{1 + ḍ(fq,gp)}, \frac{ḍ^2(Tp,gp)}{1 + ḍ(fq,gp)}\right\}$$

$$= max\{ḍ(q,p), ḍ(q,q), ḍ(p,p), \frac{ḍ(q,p) + ḍ(q,p)}{2b},$$

$$ḍ(q,q)\left[\frac{1 + ḍ(q,p)}{1 + ḍ(p,p)}\right], ḍ(p,p)\left[\frac{1 + ḍ(q,p)}{1 + ḍ(q,p)}\right],$$

$$\frac{ḍ^2(q,q)}{1+ḍ(q,p)}, \frac{ḍ^2(q,q)}{1+ḍ(q,p)}\bigg\}$$

$$= ḍ(q,p)$$

From (3.1.8), we have

$$\int_0^{b^{\varepsilon}ḍ(q,p)} \varphi(t)dt \leq \int_0^{ḍ(q,p)} \varphi(t)dt \quad for\ all\ x,y \in \dot{X}$$

From which it follows that $ḍ(q,p) = 0$ because $b^{\epsilon} > b > 1$.

Hence $p = q$, this proves the uniqueness of fixed point theorem.

Corollary 3.2 Let $(\dot{X}, ḍ)$ be a b- metric space with $s > 1$ and $f, T: \dot{X} \to \dot{X}$, be mappings with such that

(3.2.1) $\int_0^{b^{\epsilon}ḍ(\mathrm{fx,fy})} \varphi(t)dt \leq \int_0^{M_b(x,y)} \varphi(t)dt,$ for all $x, y \in \dot{X}$.

Where $\epsilon > 1$ is a constant and $\varphi: [0,\infty) \to [0,\infty)$ is Lebesque integrable mapping which is summable on each compact subset of $[0,\infty)$, non negative and such that for $c > 0$, $\int_0^c \varphi(t)dt > 0$ and

(3.2.2) $M_b(x,y) = \{ḍ(Tx,Ty), ḍ(fx,Tx), ḍ(fy,Ty), \frac{ḍ(Tx,fy)+ḍ(fx,Ty)}{2b},$

$$ḍ(Tx,fx)\left[\frac{1+ḍ(Tx,Ty)}{1+ḍ(Ty,fy)}\right], ḍ(Ty,gy)\left[\frac{1+ḍ(Tx,Ty)}{1+ḍ(Tx,fx)}\right],$$

$$\frac{ḍ^2(Tx,fx)}{1+ḍ(fx,fy)}, \frac{ḍ^2(Ty,fy)}{1+ḍ(fx,fy)}\bigg\}$$

Suppose the pair (f,T) satisfies the b-(E.A) property at a point of coincidence in $\dot{X}$. Moreover, if the pair (f,T) is weakly compatible, then f and T have a unique common fixed point.

Corollary 3.3 Let $(\dot{X}, ḍ)$ be a b- metric space with $s > 1$ and $f, T: \dot{X} \to \dot{X}$, be mappings with such that

(3.3.1) $\int_0^{b^{\epsilon}ḍ(\mathrm{fx,fy})} \varphi(t)dt \leq \int_0^{M_b(x,y)} \varphi(t)dt,$ for all $x, y \in \dot{X}$.

Where $\epsilon > 1$ is a constant and $\varphi: [0,\infty) \to [0,\infty)$ is Lebesque integrable mapping which is summable on each compact subset of $[0,\infty)$, non negative and such that for

$c > 0,$ $\int_0^c \varphi(t)dt > 0$ and

(3.3.2) $M_b(x,y) = \{ḍ(Tx,Ty), ḍ(fx,Tx), ḍ(fy,Ty), \frac{ḍ(Tx,fy)+ḍ(fx,Ty)}{2b},$

$$ḍ(Tx, fx)\left[\frac{1 + ḍ(Tx, Ty)}{1 + ḍ(Ty, fy)}\right], ḍ(Ty, gy)\left[\frac{1 + ḍ(Tx, Ty)}{1 + ḍ(Tx, fx)}\right],$$

$$\left.\frac{ḍ^2(Tx, fx)}{1 + ḍ(fx, fy)}, \frac{ḍ^2(Ty, fy)}{1 + ḍ(fx, fy)}\right\}$$

Suppose the pair (f, T) satisfied the b-(E.A) property and $T(X)$ is b-closed in Ẋ, Moreover, if the pair (f, T) is weakly compatible, then f and T have a unique common fixed point.

Example. 3.4 Let $\dot{X} = [0, 2]$, and define $d: \dot{X} \times \dot{X} \to (0, \infty)$ as follows

$$d(x, y) = \begin{cases} 0, & x = y \\ \left(\frac{x + y}{2}\right)^2 & x \neq y \end{cases}$$

Then $(\dot{X}, ḍ)$ be a b- metric space with $b = 2$. Let $f, g, S, T: \dot{X} \to \dot{X}$ are defined by

$$f(x) = \frac{3x}{4}, \quad g(x) = 0, \quad S(x) = \begin{cases} x & x \in [0, 1) \\ 1 & x = 1 \\ 3/4 & x \in (1, 2] \end{cases}, \quad T(x) = \frac{3x}{2}$$

Clearly, $g(x)$ is closed, $g(x) \subseteq S(x)$ and $f(x) \subseteq T(x)$.

Let $\{x_n\}$ be the sequence in Ẋ such that $x_n = 1 + \frac{1}{n+3}$, $n = 0,1,2,3 \ldots$.

So that the pair (f, S) is non –compatible since $\lim_{n\to\infty} d(fSx_n, Sfx_n) \neq 0$.

But satisfies the b-(E.A) property since $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} Sx_n = \frac{3}{4}$

To check the inequality (3.1.3), for all $x, y \in \dot{X}$ and for $x, y \in \dot{X}$ and $\epsilon = 2$.

If $x = 0$, (3.1.3) satisfied.

If $x \in (0, 1)$, then

$$b^{\varepsilon} d(fx, gy) = 2^2.\left(\frac{3x}{8}\right)^2 \leq \left(\frac{\frac{3x}{4} + x}{2}\right)^2 = d(fx, Sx) \leq M_b(x, y)$$

If $x = 1$, then

$$b^{\varepsilon} d(fx, gy) = 2^2.\left(\frac{3x}{8}\right)^2 \leq \left(\frac{\frac{3x}{4} + 1}{2}\right)^2 = d(fx, Sx) \leq M_b(x, y)$$

If $x \in (1, 2)$, then

$$b^{\varepsilon}d(fx,gy) = 2^2.\left(\frac{3x}{8}\right)^2 \leq \left(\frac{\frac{3x}{4}+\frac{3}{4}}{2}\right)^2 = d(fx,Sx) \leq M_b(x,y)$$

Thus (3.1.3) is satisfied for all $x, y \in \dot{X}$. The pairs (f, S) and (g, T) are weakly compatible. Hence, all the conditions of Theorem 3.1 are satisfied. Moreover,0 is the unique common fixed point of f, g, S and T.

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