An Almost Exact Algorithm for the General Solution of Second Order Linear and Nonlinear Fredholm Integro-Differential Equations

ABSTRACT:

In this paper, Fredholm integro-differential equations are solved using the derivative of the Lucas polynomials in matrix form. The equation is first transformed into systems of nonlinear algebraic equations using the Lucas polynomials. The unknown parameters required for approximating the solution of Fredholm integro-differential equations are then determined using Gaussian elimination. The method has proven to be an active and dependable technique for solving many Fredholm integro-differential equations of different order. The novelty in this technique is that it is capable of solving Fredholm integro differential equation of any order by simply updating the matrix of derivative of the Lucas polynomials also surprisingly the technique was tried on mix Fredholm-Volterra integro differential equation and the result obtained was amazing. Some test problems contained in the literature were solved using the developed technique and the results confirmed the applicability, validity and efficiency of the method. The accuracy of the method was observed to be better when compared with some existing methods.

Keywords**:** Fredholm integro-differential equations; matrix of derivative; Lucas's polynomials, algorithm.

1 INTRODUCTION

The Fredholm integro-differential equation is the result of converting boundary value problems in differential equations to integro-differential equations with limits of integration considered as constant. This class of problems has gained importance in the literature with a variety of applications such as epidemiology, the mathematical modeling of epidemics, particularly when the models contain age-structure or describe spatial epidemics. These equations are characterized by the existence of one or more of the derivatives $z'(x)$, $z''(x)$, outside the integral sign. The linear Fredholm integro differential equation is of the form

$$z^{(k)}(x) = f(x) + \lambda \int_a^b \mathcal{H}(x,t) z(t)dt\,, z^{(m)}(0) = \beta_m, 0 \leq m \leq k-1 \;\; x \,\epsilon\, [a,b] \qquad (1.0)$$

where the function $f(x)$ and the kernel $\mathcal{H}(x,t)$ are known $z^{(k)}(x) = \frac{d^k z}{dx^k}$. Since (1.0) combines differential and integral operators, it is important to define initial conditions $z(0)$**,** $z^{(1)}(0), z^{(2)}(0), \dots, z^{(k-1)}(0)$ for the determination of the particular solution $z(x)$.

The expectation here is to develop an efficient numerical method as opposed to proving theoretical concepts of convergence and existence. Presented over the last few decades are varieties of powerful methods, such as the numerical solution of two-dimensional Fredholm integro-differential equations by Chebyshev integral operational matrix method [1], Two-dimensional Chebyshev polynomials for solving two-dimensional integro-differential equations [2], Operational matrices of Bernstein polynomials and their applications [3], Sixth-kind Chebyshev and Bernoulli polynomial numerical methods for solving nonlinear mixed partial integro differential equations with continuous kernels [4], [5] The numerical solution of non-linear 2D Volterra-Fredholm integro-differential equations using two-dimensional triangular function by [6], A computational technique for solving three-dimensional mixed volterra–fredholm integral equations [4], The Legendre Galerkin method for solving fractional integro-differential equations of Fredholm type by [7], Comparison of some numerical methods for the solutions of first and second orders linear integro-differential equations [8], The numerical solution of Fredholm integro-differential equations using hybrid function operational matrix of differentiation [9]. These methods generated impressive numerical results for the model problems considered as experiment.

This paper proposes a method based on the derivative of Lucas polynomials similar to [9], [10], , [11] and [12] but with a divergent approach for the numerical solution of (1.0). This approach, to the best of our knowledge has not been discussed by any researcher to date. It is our strong believe that others will find the method appealing and convincing as an improvement to many existing methods for the numerical solution of (1.0).

2. MATERIALS AND METHODS

Lucas Polynomial

The Lucas polynomials denoted by $z_n(x)$ are defined using the recurrence formula;

$$z_n(x) = 2^{-n}\left(x - \sqrt{x^2} + 4\right)^n \left(x + \sqrt{x^2} + 4\right)^n$$

The first few Lucas polynomials are given below;

$$z_0(x) = 2$$
$$z_1(x) = x$$
$$z_2(x) = x^2 + 2$$
$$z_3(x) = x^3 + 3x$$
$$z_4(x) = x^4 + 4x^2 + 2$$
$$z_5(x) = x^5 + 5x^3 + 5x$$
$$z_6(x) = x^6 + 6x^4 + 9x^2 + 2$$

Approximate Solution

This is an expression obtained after the unknown parameters have been found and substituted back into the assumed solution. It is referred to as an approximate solution since it is a reasonable replacement of the exact solution. It is denoted by $z_N(x)$, and taken as an inexact representation of the exact solution, where N is the degree of the approximant used in the calculation. Methods of approximate solution are usually adopted because complete information needed to arrive at the exact solution may not be available. In this paper, the approximate solutions used are given as;

$$z_N(x) = \sum_{i=0}^{N} \delta_i z_i(x)$$

where x represents the independent variables in the problem, $\delta_i (i \geq 0)$are the unknown parameters to be determined and $z_i(x), (i \geq 0)$ is the Lucas Polynomials basis function. Note that the approximate solution above can be written in matrix form as;

$$z_N(x) = \sum_{i=0}^{N} \delta_i z_i(x) = Z(x)\partial^T$$

Where the Lucas coefficients vector ∂ and the Lucas vector $Z(x)$ are defined as;

$$\left.\begin{array}{c} \partial = \delta_0, \delta_1, \delta_2, \ldots, \delta_N \\ Z(x) = z_0(x), z_1(x), z_2(x), \ldots, z_N(x) \end{array}\right\}$$

Similarly, the derivative of the approximate solution above can be written in matrix form as;

$$z_N^{(k)}(x) = \sum_{i=0}^{N} \delta_i z_i^{(k)}(x) = Z(x)(\Omega^T)^{(k)}\partial^T$$

where Ω is an $(N+1)\times(N+1)$ square matrix of the derivative of the Lucas polynomial.

2.1 Methodology

We consider a technique for the numerical solution of the linear Fredholm integro-differential equations of the form (1.0). First, we consider the approximate solution of (1.0) in the form of Lucas series given below as;

$$z(x) \cong z_N(x) = \sum_{i=0}^{N} \delta_i z_i(x) = Z(x)\partial^T \tag{2.0}$$

where $z_i(x)$ are the Lucas polynomials of degree i and $\delta_i's$ are the unknown parameters to be sought for. We approximate equation (1.0) by substituting (2.0) into (1.0) with $\lambda = 1$ to get;

$$Z(x)(\Omega^T)^{(k)}\partial^T = f(x) + \int_a^b \mathcal{H}(x,t)Z(x)\partial^T dt \tag{3.0}$$

Expanding the integral in (3.0), we obtain the following expression;

$$Z(x)(\Omega^T)^{(k)}\partial^T = f(x) + \left(\delta_0 \int_a^b \mathcal{H}(x,t)z_0(t)dt + \cdots + \delta_N \int_a^b \mathcal{H}(x,t)z_n(t)dt\right) \tag{4.0}$$

Replacing x with x_i in (4.0), we obtain the expression below as;

$$Z(x_i)(\Omega^T)^{(k)}\partial^T = f(x_i) + \left(\delta_0 \int_a^b \mathcal{H}(x_i,t)z_0(t)dt + \cdots + \delta_N \int_a^b \mathcal{H}(x_i,t)z_n(t)dt\right) \tag{5.0}$$

Collocating equation (5.0) at the points

$$x_i = \frac{i}{N-k}, i = 0,1,\dots,N-k, \quad x\epsilon[a,b]$$

We obtain $(N+1)$ by $(N+1)$ system of nonlinear algebraic equations. Since we need to solve a nonlinear system with a large number of equations, we have to rely on some iterative type method. Here, we apply Gaussian elimination for the unknown parameters $\delta_0,\ \delta_1,\ \delta_2,\dots,\delta_N$. Replacing the calculated parameters δ_0, $\delta_1,\ \delta_2,\dots,\delta_N$ into (2.0), we obtain the approximate solution to (1.0).

3. ILLUSTRATION OF THE METHOD

In demonstrating the present technique's simplicity and computational efficiency, four sample problems contained in the literature are considered. In each of these sample problems, we compare our results with the exact solution and the solutions obtained by other methods in literature. All calculations are performed using Scientific Workplace 5.5 Software; the detailed procedure is outline below. Also, the absolute errors in tables are the values of $|z(x) - z_N(x)|$ at selected points.

Problem 3.1

We first consider the mix linear third order Fredholm-Volterra integro-differential equation considered by [13]

$$z^{(3)}(x) = x - 2x^3 + 3x^4 + \int_0^1 \mathrm{x}z(t)dt + \int_0^x \mathrm{t}z(t)dt,\ 0 \le x \le 1, \tag{6.0}$$

with the initial conditions $z(0) = 0$, $z^{(1)}(0) = 6$, $z^{(2)}(0) = -24$ and the exact solution $z(x) = 6x - 12x^2$.

Applying the present technique on (6.0) with $N = 4$, the given problem becomes;

$$Z(x)(\Omega^T)^{(2)}\partial^T = x - 2x^3 + 3x^4 + \int_0^1 xZ(t)\partial^T dt + \int_0^x \mathrm{t}Z(t)\partial^T dt \tag{7.0}$$

with the initial condition

$$Z(0)\partial^T = 0, \qquad Z(0)(\Omega^T)^{(1)}\partial^T = 6, \qquad Z(0)(\Omega^T)^{(2)}\partial^T = -24$$

Collocating (7.0) at $x_i = \frac{i}{1}, i = 0{,}1$ and evaluating the initial condition at $x = 0$ and solving for the unknown parameters, we have

$$[\delta_0 = 12, \delta_1 = 6, \delta_2 = -12, \qquad \delta_3 = 0, \qquad \delta_4 = 0]$$

Substituting these parameters into (2.0), we get the approximate solution to problem 3.1 as;

$$z_N(x) = 6x - 12x^2$$

The approximate solution is the same as the exact solution showing that the method has higher accuracy than the method considered by [13].

Problem 3.2

Consider the model Fredholm integro-differential equation:

$$z''(x) = 32x + \int_{-1}^{1}(1 - \mathrm{x}\mathrm{t})z(t)dt\,, \quad 0 \leq x \leq 1 \tag{8.0}$$

subject to the initial conditions: $z'(0) = 0, z(0) = 1$. The exact solution to the problem is given by;

$$z(x) = 1 + \frac{3}{2}x^2 + 5x^3$$

Source: [14].

Applying the present technique with $N = 4$, the given problem becomes;

$$Z(x)(\Omega^T)^{(2)}\partial^T = 32x + \int_{-1}^{1}(1 - \mathrm{x}\mathrm{t})Z(t)\partial^T dt \tag{9.0}$$

with the initial conditions

$$Z(0)\partial^T = 1, \;\; Z(0)(\Omega^T)^{(1)}\partial^T = 0$$

Collocating (9.0) at $x_i = \frac{i}{2}, i = 0{,}1{,}2$ and evaluating the initial conditions at $x = 0$and solving for the unknown parameters, we have

$$[\delta_0 = -1, \delta_1 = -15, \delta_2 = \frac{3}{2}, \delta_3 = 5, \delta_4 = 0]$$

Substituting the calculated parameters into (2.0), we get the approximate solution to the problemas;

$$z(x) = 1 + \frac{3}{2}x^2 + 5x^3$$

The approximate solution is the same as the exact solution showing the accuracy of the method.

Table 1compared our results with [14].

Table 1. Numerical results of problem 3.2 compared with [14]

x	Exact solution	Approx solution of proposed method $N = 4$	Absolute error of proposed method $\lvert z(x) - z_N(x)\rvert$	[14] $\lvert z(x) - z_N(x)\rvert$
0.1000	1.0200	1.0200	0.0000	$0.0000e^{+00}$
0.2000	1.1000	1.1000	0.0000	$1.1102e^{-16}$
0.3000	1.2700	1.2700	0.0000	$8.8818e^{-16}$
0.4000	1.5600	1.5600	0.0000	$7.7716e^{-16}$
0.5000	2.0000	2.0000	0.0000	$4.4409e^{-16}$
0.6000	2.6200	2.6200	0.0000	$1.6653e^{-15}$
0.7000	3.4500	3.4500	0.0000	$2.7756e^{-15}$
0.8000	4.5200	4.5200	0.0000	$5.4401e^{-15}$
0.9000	5.8600	5.8600	0.0000	$7.2165e^{-15}$
1.0000	7.5000	7.5000	0.0000	$9.4369e^{-15}$

Problem 3.3

Consider the model Fredholm integro-differential equation given by

$$z''(x) = 10 - \frac{146}{35}x + \frac{1}{2}\int_0^1 xt\big(z(t)\big)^2 dt\,,\quad 0 \le x, t \le 1 \qquad (10.0)$$

subject to the boundary conditions: $z'(0) = 0, z(0) = 1$. The exact solution to the problem is given by;

$$z(x) = 1 + 5x^2 - x^3$$

Source: [14].

Applying the present technique with $N = 4$, the given problem becomes;

$$Z(x)(\Omega^T)^{(2)}\partial^T = 10 - \frac{146}{35}x + \frac{1}{2}\int_{-1}^1 xt(Z(t)\partial^T)^2 dt \qquad (11.0)$$

with the initial conditions

$$Z(0)\partial^T = 1,\;\; Z(0)(\Omega^T)^{(1)}\partial^T = 0$$

Collocating (11.0) at $x_i = \frac{i}{2}, i = 0,1,2$ and evaluating the initial conditions at $x = 0$and solving for the unknown parameters, we have

$$[\delta_0 = -\frac{9}{2}, \delta_1 = 3, \delta^2 = 5, \delta^3 = -1, \delta_4 = 0]$$

Substituting the calculated parameters into (2.0), we get the approximate solution to the problem as;

$$z(x) = 1 + 5x^2 - x^3$$

The approximate solution is the same as the exact solution showing the accuracy of the method.

Table 2compared our results with [14]

Table 2: Numerical results of problem 3.3 compared with [14]

x	Exact solution	Approx solution of proposed method $N = 4$	Absolute error of proposed method $\lvert z(x) - z_N(x)\rvert$	[14] $\lvert z(x) - z_N(x)\rvert$
0.1	1.049	1.049	0.000	1.1477×10^{-14}
0.2	1.192	1.192	0.000	6.7141×10^{-14}
0.3	1.423	1.423	0.000	1.8341×10^{-13}
0.4	1.736	1.736	0.000	3.3856×10^{-13}
0.5	2.125	2.125	0.000	4.8611×10^{-13}
0.6	2.584	2.584	0.000	5.7987×10^{-13}
0.7	3.107	3.107	0.000	5.9475×10^{-13}
0.8	3.688	3.688	0.000	5.3291×10^{-13}
0.9	4.321	4.321	0.000	4.1600×10^{-13}
1.0	5.000	5.000	0.000	2.7444×10^{-13}

Problem 3.4

Consider the model Fredholm integro-differential equation given by

$$z''(x) = x - sinx - \int_0^{\frac{\pi}{2}} \mathrm{xtz}(t)dt\,,\;\; 0 \le x, t \le \frac{\pi}{2} \tag{12.0}$$

subject to the boundary conditions: $z'(0) = 1, z(0) = 0$. The exact solution to the problem is given by;

$$z(x) = sinx$$

Source: [15].

Applying the present technique with $N = 10$, the given problem becomes;

$$Z(x)(\Omega^T)^{(2)}\partial^T = x - sinx - \int_0^{\frac{\pi}{2}} x\mathrm{t}(Z(x)\partial^T)dt \tag{13.0}$$

with the initial conditions

$$Z(0)\partial^T = 0,\;\; Z(0)(\Omega^T)^{(1)}\partial^T = 1$$

Collocating (11.0) at $x_i = \frac{i}{8}, i = 0,1,\ldots,8$ and evaluating the initial conditions at $x = 0$and solving for the unknown parameters, we have

$$[\delta_0 = 4.222357858\,44 \times 10^{-6}, \delta_1 = 1.590652422\,74, \delta_2 = -7.697862448\,07 \times 10^{-6}, \delta_3 = -0.212750434\,245, \delta_4 = 5.575345015\,04 \times 10^{-6}, \delta_5 = 9.830141954\,81 \times 10^{-3},$$

$$\delta_6 = -2.882400421\,5 \times 10^{-6}, \delta_7 = -2.256240440\,61 \times 10^{-4}, \delta_8 = 9.139904134\,24 \times 10^{-7}, \delta_9 = 3.059836347\,40 \times 10^{-6}, \delta_{10} = -1.314304173\,33 \times 10^{-7}]$$

Substituting the calculated parameters into (2.0), we get the approximate solution to the problemas;

$$z(x) = -1.314304173\,33 \times 10^{-7}x^{10} + 3.059836347\,4 \times 10^{-6}x^{9} - 4.003137599\,06 \times 10^{-7}x^{8} - 1.980855169\,34 \times 10^{-4}x^{7} - 1.705417207\,63 \times 10^{-7}x^{6} + 8.333389227\,76 \times 10^{-3}x^{5} - 1.077011213 \times 10^{-8}x^{4} - 0.166666665\,997x^{3} + 4.9 \times 10^{-17}x^{2} + 1.0x + 2.0 \times 10^{-18}$$

The approximate solution and the exact solution are shown in the Table 3 and comparing our results with [15]

Table 3: Numerical results of Example 3.4 compared with [15] $N = 10$

x	Exact solution	Approx solution of proposed method	Absolute error of proposed method $\lvert z(x) - z_N(x) \rvert$ $N = 10$	Absolute error by [15] $\lvert z(x) - z_N(x) \rvert$ $N = 10$
0.0	0.0	0.0	0.0	0.0
0.1	0.099833416650	0.09983341664 68	1.00667×10^{-14}	1.2×10^{-9}
0.2	0.198669330800	0.198669330 793	1.59742×10^{-12}	1.21×10^{-9}
0.3	0.295520206661	0.295520206 654	7.11440×10^{-12}	5.72×10^{-9}
0.4	0.389418342309	0.389418342 290	1.83443×10^{-11}	9.66×10^{-8}
0.5	0.479425538604	0.479425538 567	3.72407×10^{-11}	9.70×10^{-8}
0.6	0.564642473395	0.564642473 329	6.57075×10^{-11}	2.97×10^{-8}
0.7	0.644217687238	0.644217687 132	1.05649×10^{-10}	3.85×10^{-7}
0.8	0.717356090900	0.717356090 741	1.59017×10^{-10}	6.04×10^{-7}
0.9	0.783326909627	0.783326909 400	2.27620×10^{-10}	6.56×10^{-7}
1.0	0.841470984808	0.841470984 494	3.13733×10^{-10}	9.44×10^{-7}

4. CONCLUSION

In this research, we proposed a new approach for solving both linear and nonlinear Fredholm integro differential equations and surprisingly the approach is able to handled mix Fredholm-Volterra integro differential equations. The present method has an advantage over other methods because of its simplicity in the determination of the unknown parameters being the main factor in this work. Another advantage of the method is its ability to solve Fredholm integro differential equation of any order by simply updating the matrix of derivative of the Lucas polynomials unlike other methods where the derived algorithm is only designed to solve problems of a specific order only. The present work revealed that the proposed scheme is comparatively simpler to apply than many existing methods, whereas the numerical results revealed the accuracy and superiority of the presented method. The main attraction of the present technique is displayed by the superior results for different input values which further suggest the novelty of the technique against some existing methods.

REFERENCES

[1] Christie Yemisi Ishola, Omotayo Adebayo Taiwo, Ajimot Folasade Adebisi and Olumuyiwa James Peter (2022) Numerical Solution of Two-Dimensional Fredholm Integro-Differential Equations by Chebyshev Integral Operational Matrix Method, *Journal of Applied Mathematics and Computational Mechanics*, 21(1), 29-40

[2] Rivaz, A., Samane J., & Yousefi, F. (2015) Two-dimensional Chebyshev polynomials for solving two-dimensional integro-differential equations, Cankaya University *Journal of Science and Engineering,* 12(2), 1-11

[3] Yousefi, S.A., & Behroozifar, M. (2010) Operational matrices of Bernstein polynomials and their applications; *International Journal of Systems Science,* 41(6), 709-716

[4] Mahdy, A. M. S, Abbas, S. N., Khaled, M. H. & Doaa, S. M. (2023) A Computational Technique for Solving Three-Dimensional Mixed Volterra–Fredholm Integral Equations, *Fractal and Fractional* 7 (2), 196

[5] Abeer M. A, Abdou, M. A., & Mahdy, A. M. S. (2023) Sixth-Kind Chebyshev and Bernoulli Polynomial Numerical Methods for Solving Nonlinear Mixed Partial Integro differential Equations with Continuous Kernels, *Journal of Function Spaces* Volume 2023, Article ID 6647649, 1-14

[6] Khajehnasiri, A. A. (2016) Numerical solution of non-linear 2D Volterra-Fredholm integro-differential equations by two-dimensional triangular function, *International Journal of Applied Computational Mathematics*, 2, 575-591

[7] Uwaheren, O.A., Adebisi, A.F., Olotu, O.T., Etuk, M.O., & Peter, O.J. (2021) Legendre Galerkin method for solving fractional integro-differential equations of Fredholm type. *The Aligarh Bulletin of Mathematics,* 40(1), 1-13.

[8] Taiwo, O.A., Jimoh, A.K., & Bello, A.K. (2014) Comparison of some numerical methods for the solutions of first and second orders linear integro-differential equations. *American Journal of Engineering Research*, 3(1), 245-250.

[9] Jafari, R., Ezzati, R. And Maleknejad, K. (2017) Numerical solution of Fredholm integro-differential equations by using hybrid function operational matrix of differentiation*. International Journal of Industrial Mathematics* Vol. 9, No. 4, 349-358

[10] Turkyilmazoglu, M. (2014), An effective approach for numerical solutions of high-order Fredholm integro-differential equations, *Applied Mathematics and Computation***,** Volume 227, 384-398

[11] Salih Yalçinbaş, Mehmet Sezer, Hüseyin Hilmi Sorkun (2009) Legendre polynomial solutions of high-order linear Fredholm integro-differential equations, *Applied Mathematics and computation*, Volume 210, Issue 2, 334-349

[12] Darania,P. and Ali Ebadian (2007) A method for the numerical solution of the integro-differential equations, *Applied Mathematics and Computation***,** Volume 188, Issue 1, 1, 657-668

[13] Padmanabha, R. A., Manjula, S. H. and Sateesha, C. (2017) Solution for n^{th} order mixed Fredholm-Volterraintegro-differential equations using Haar wavelets, Vol.09 Issue-01, (January - June, 2017), *Aryabhatta Journal of Mathematics and Informatics*, 261-271

[14] Joshua Sunday (2019) On exact finite difference scheme for the computation of second-order Fredholm Integro-differential equations. *FULafia Journal of Science and Technology*, 5(1), 113-119.

[15] Hosseini, S. M. And Shahmorad, S. (2002) Tau numerical solution of Fredholm integro-differential equations with arbitrary polynomial bases, *Applied Mathematical Modelling* 27 (2003) 145–154