Fixed Point for Compatible Mappings Satisfying Cubic Type Generalized $\psi - \phi$ Weak Contraction

Abstract

The goal of this paper is to introduce cubic type generalized $\psi - \phi$ weak contraction condition in b-metric spaces and prove common fixed point theorem for compatible mappings. Present results generalize the results of Kumar *et al.* [20].

Keywords and Phrases: common fixed point, weak contraction, compatible mappings, b-metric spaces.

2010 Mathematical Subject Classification: 47H10, 54H25.

1 Introduction and Preliminaries

Fixed point theory is one of the most progressive and fascinating research area in nonlinear functional analysis and it is useful for demonstrating the existence theorems for nonlinear differential and integral equations. The Banach contraction principle [7] is the crucial result in fixed point theory, which has numerous applications in different branches of mathematics such as differential and integral equation, numerical analysis etc. Several researchers established some new type contraction and proved numerous fixed point theorems in order to generalize the Banach fixed point theorem (see [11],[5],[21],[22],[23]). In 1976, for generalization of Banach's fixed point theorem, Jungck [14] used the notion of commuting maps to prove a common fixed point theorem. In 1982, Sessa [27] generalized the notion of commutativity to weak commutativity and proved some common fixed point theorems for weakly commuting mappings.

In 1986, Jungck [16] extended the notion of weakly commuting mappings to a larger class of mappings known as compatible mappings.

Definition 1.1. [16] A pair of self mappings (ξ, ζ) on a metric space $(\mathfrak{M}, \Delta)$ is said to be compatible if $\lim_{n\to\infty} \Delta(\xi\zeta\omega_n, \zeta\xi\omega_n) = 0$, whenever $\{\omega_n\} \in \mathfrak{M}$ is a sequence such that $\lim_{n\to\infty} \xi\omega_n = \lim_{n\to\infty} \zeta\omega_n = \varkappa$, for some $\varkappa \in \mathfrak{M}$.

In 1969, Boyd and Wong [10] introduced ϕ-contraction condition of the form $d(\xi u, \xi v) \leq \phi(d(u, v))$, for all $u, v \in \mathcal{M}$, where ξ is a self map on a complete metric space $\mathcal{M}$ and $\phi : [0, \infty) \to [0, \infty)$ is an uppper semi continuous function from right such that $0 \leq \phi(t) < t$, for all $t > 0$. In 1977, Alber and Guerre-Delabriere [4] generalized ϕ-contraction to ϕ-weak contraction in Hilbert spaces, which was further extended and proved by Rhoades [26] in complete metric spaces .
A self map ξ on a complete metric space is said to be a weak contraction if for each $u, v \in \mathcal{M}$, there exists a continuous non-decreasing function $\phi : [0, \infty) \to [0, \infty)$ satisfying $\phi(t) > 0$, for all $t > 0$ and $\phi(t) = 0$ if and only if $t = 0$ such that

$$d(\xi u, \xi v) \leq d(u, v) - \phi(d(u, v)).$$

The function ϕ in the above inequality is known as control function or altering distance function. The notion of control function was given by Khan *et al.* [19] as follows.

Definition 1.2. [19] An altering distance is a function $\phi : [0, \infty) \to [0, \infty)$ satisfying the following:

(i) ϕ is an increasing and continuous function,

(ii) $\phi(t) = 0$ if and only of $t = 0$.

In 2021, Kumar *et al.* [20] introduced weak contraction condition involving cubic terms of distance functions and proved the following theorem:

Theorem 1.3. [20] Assume that f, g, h and k are self mappings defined on a complete metric space $(\mathcal{X}, d)$ satisfying the following conditions:

$$(C1) \quad d^3(fu, gv) \leq \rho \max\Big\{\frac{[d^2(hu, fu)d(kv, gv) + d(hu, fu)d^2(kv, gv)]}{2}, d(hu, fu)d(hu, gv)d(kv, fu), d(hu, gv)d(kv, fu)d(kv, gv)\Big\} - \phi(m(hu, kv)),$$

for all $u, v \in \mathcal{X}$, where

$$m(hu, kv) = \max\Big\{d^2(hu, kv), d(hu, fu)d(kv, gv), d(hu, gv)d(kv, fu), \frac{1}{2}[d(hu, fu)d(hu, gv) + d(kv, fu)d(kv, gv)]\Big\}.$$

Further ρ is a real number such that $0 < \rho < 1$ and $\phi : [0, \infty) \to [0, \infty)$ is a continuous function with $\phi(0) = 0$ and $\phi(t) > 0$ for each $t > 0$.

(C2) $f(\mathcal{X}) \subset k(\mathcal{X})$, $g(\mathcal{X}) \subset h(\mathcal{X})$;

(C3) there is one continuous mapping among f, g, h and k.

Also, assume the pairs (f, h) and (g, k) to be compatible, then f, g, h and k possess a unique **CFP** in $\mathcal{X}$.

In 1989, Bakhtin [6] introduced in the theory of metric fixed point the concept of b-metric spaces, as a generalization of usual metric spaces and shows the BCP in this setting.

Definition 1.4. [6] Let $\mathcal{M}$ be a non-empty set and $s \geq 1$ be given real number. A function $\Delta : \mathcal{M} \times \mathcal{M} \to \mathbb{R}_+$ is said to b-metric if and only if for all u, v, $w \in \mathcal{M}$ the following properties are satisfied:

(i) $\Delta(u, v) = 0$ iff $u = v$,

(ii) $\Delta(u, v) = \Delta(v, u)$,

(iii) $\Delta(u, w) \leq s(\Delta(u, v) + \Delta(v, w))$.

In such a case, the pair $(\mathcal{M}, \Delta)$ is called a b-metric space and the real number $s \geq 1$ is called the coefficient of $(\mathcal{M}, \Delta)$.

Aghajani *et al.* [3] proved the following simple lemma about b-convergent sequences.

Lemma 1.5. [3] Let $(\mathcal{M}, \Delta)$ be a b-metric space with $s \geq 1$, and suppose that $\{u_n\}$ and $\{v_n\}$ are b-convergent sequences converging to u and v respectively. Then, we have,

$$\frac{1}{s^2}\Delta(u,v) \leq \lim_{n\to+\infty} \inf \Delta(u_n, v_n) \leq \lim_{n\to+\infty} \sup \Delta(u_n, v_n) \leq s^2\Delta(u,v).$$

In particular, if $u = v$, then $\lim_{n\to+\infty} \Delta(u_n, v_n)$. Moreover, for each $w \in \mathcal{M}$, we have

$$\frac{1}{s}\Delta(u,v) \leq \lim_{n\to+\infty} \inf \Delta(u_n, w) \leq \lim_{n\to+\infty} \sup \Delta(u_n, w) \leq s\Delta(u,w).$$

In this paper, we introduce cubic type generalized $\psi - \phi$ weak contraction condition in b-metric spaces and proved a fixed point theorem for compatible mappings. Assume that f, g, h and k are self mappings defined on a b-metric space $(\mathcal{M}, \Delta)$ satisfying the following condition:

$$(C4) \quad \Delta^3(fu, gv) \leq \rho\psi\Big\{[\Delta^2(hu, fu)\Delta(kv, gv), \Delta(hu, fu)\Delta^2(kv, gv)], \Delta(hu, fu)\Delta(hu, gv)\Delta(kv, fu), \Delta(hu, gv)\Delta(kv, fu)\Delta(kv, gv)\Big\} - \phi(m(hu, kv)),$$

for all $u, v \in \mathcal{X}$, where

$$m(hu, kv) = \max\Big\{\Delta^2(hu, kv), \Delta(hu, fu)\Delta(kv, gv), \Delta(hu, gv)\Delta(kv, fu), \frac{1}{2s}[\Delta(hu, fu)\Delta(hu, gv) + \Delta(kv, fu)\Delta(kv, gv)]\Big\}.$$

Further ρ is a real number such that $0 < \rho < 1$ and $\phi : [0, \infty) \to [0, \infty)$ is a continuous function with $\phi(0) = 0$ and $\phi(t) > 0$ for each $t > 0$ and $\psi \in \Psi$, where Ψ is a collection of all functions $\psi : [0, \infty)^4 \to [0, \infty)$ satisfying the following conditions:

(i) ψ is non decreasing and upper semi continuous in each coordinate variables.

(ii) $\max\{\psi(t, t, 0, 0), \psi(0, 0, 0, t), \psi(0, 0, t, 0), \psi(t, t, t, t)\} \leq t$, for each $t > 0$ and $\psi(0, 0, 0, 0) = 0$.

2 Fixed Point for Compatible Mappings

We prove common fixed point theorem for compatible mappings satisfying cubic type generalized $\psi - \phi$ weak contraction condition (C4).

Theorem 2.1. Assume that f, g, h and k are self mappings defined on a complete b-metric space $(\mathcal{M}, \Delta)$ satisfying (C4) and the following conditions:
(C5) $f(\mathcal{M}) \subset k(\mathcal{M})$, $g(\mathcal{M}) \subset h(\mathcal{M})$;
(C6) there is one continuous mapping among f, g, h and k.
Also, assume the pairs (f, h) and (g, k) to be compatible, then f, g, h and k possess a unique common fixed point in $\mathcal{M}$.

Proof. Assume that $u_0 \in \mathcal{M}$ represents any point and using condition (C5), we can find $f(u_0) = k(u_1) = v_0$, for some $u_1 \in \mathcal{X}$. For this u_1, there exists $u_2 \in \mathcal{M}$ such that $g(u_1) = h(u_2) = v_1$. Approaching like this, we can derive a sequence $\{v_n\} \in \mathcal{M}$ such that

$$v_{2n} = f(u_{2n}) = k(u_{2n+1}), \quad v_{2n+1} = g(u_{2n+1}) = h(u_{2n+2}) \text{ for each } n \geq 0. \tag{2.1.1}$$

Let $\mu_n = \Delta(v_n, v_{n+1})$. First, we establish that the sequence $\{\mu_n\}$ is non-increasing and converges to zero.

Case I. If n is even, i.e., $n = 2j$, $j = 0, 1, 2 \ldots$, then on putting $u = u_{2j}$ and $v = u_{2j+1}$ in (C1), we get

$$\begin{aligned}\Delta^3(fu_{2j}, gu_{2j+1}) \leq& \rho\psi\Big\{\Delta^2(hu_{2j}, fu_{2j})\Delta(ku_{2j+1}, gu_{2j+1}),\\ &\Delta(hu_{2j}, fu_{2j})\Delta^2(ku_{2j+1}, gu_{2j+1})],\\ &\Delta(hu_{2j}, fu_{2j})\Delta(hu_{2j}, gu_{2j+1})\Delta(ku_{2j+1}, fu_{2j}),\\ &\Delta(hu_{2j}, gu_{2j+1})\Delta(ku_{2j+1}, fu_{2j})\Delta(ku_{2j+1}, gu_{2j+1})\Big\}\\ &-\phi(m(hu_{2j}, ku_{2j+1})),\end{aligned}$$

where

$$\begin{aligned}m(hu_{2j}, ku_{2j+1}) =& \max\Big\{\Delta^2(hu_{2j}, ku_{2j+1}), \Delta(hu_{2j}, fu_{2j})\Delta(ku_{2j+1}, gu_{2j+1}),\\ &\Delta(hu_{2j}, gu_{2j+1})\Delta(ku_{2j+1}, fu_{2j}), \frac{1}{2s}[\Delta(hu_{2j}, fu_{2j})\Delta(hu_{2j}, gu_{2j+1})\\ &+\Delta(ku_{2j+1}, fu_{2j})\Delta(ku_{2j+1}, gu_{2j+1})]\Big\}.\end{aligned}$$

Using (2.1.1), we obtain

$$\begin{aligned}\Delta^3(v_{2j}, v_{2j+1}) \leq& \rho\psi\Big\{[\Delta^2(v_{2j-1}, v_{2j})\Delta(v_{2j}, v_{2j+1}),\\ &\Delta(v_{2j-1}, v_{2j})\Delta^2(v_{2j}, v_{2j+1})],\\ &\Delta(v_{2j-1}, v_{2j})\Delta(v_{2j-1}, v_{2j+1})\Delta(v_{2j}, v_{2j}),\\ &\Delta(v_{2j-1}, v_{2j+1})\Delta(v_{2j}, v_{2j})\Delta(v_{2j}, v_{2j+1})\Big\}\\ &-\phi(m(v_{2j-1}, v_{2j})),\end{aligned} \tag{2.1.2}$$

where

$$m(v_{2j-1}, v_{2j}) = \max\Big\{\Delta^2(v_{2j-1}, v_{2j}), \Delta(v_{2j-1}, v_{2j})\Delta(v_{2j}, v_{2j+1}),$$
$$\Delta(v_{2j-1}, v_{2j+1})\Delta(v_{2j}, v_{2j}), \frac{1}{2s}[\Delta(v_{2j-1}, v_{2j})\Delta(v_{2j-1}, v_{2j+1})$$
$$+ \Delta(v_{2j}, v_{2j})\Delta(v_{2j}, v_{2j+1})]\Big\}.$$

Using $\mu_{2j} = \Delta(v_{2j}, v_{2j+1})$ in (2.1.2), we obtain

$$\mu_{2j}^3 \leq \rho\psi\Big\{\mu_{2j-1}^2\mu_{2j}, \mu_{2j-1}\mu_{2j}^2, 0, 0\Big\} - \phi(m(v_{2j-1}, v_{2j})), \tag{2.1.3}$$

where

$$m(v_{2j-1}, v_{2j}) = \max\Big\{\mu_{2j-1}^2, \mu_{2j-1}\mu_{2j}, 0, \frac{\mu_{2j-1}\Delta(v_{2j-1}, v_{2j+1})}{2s}\Big\}.$$

By using triangular inequality and property of ϕ, we get

$$\Delta(v_{2j-1}, v_{2j+1}) \leq s\{\Delta(v_{2j-1}, v_{2j}) + \Delta(v_{2j}, v_{2j+1})\} = s\{\mu_{2j-1} + \mu_{2j}\}$$
$$\text{and } m(v_{2j-1}, v_{2j}) \leq \max\Big\{\mu_{2j-1}^2, \mu_{2j-1}\mu_{2j}, 0, \frac{\mu_{2j-1}(\mu_{2j-1} + \mu_{2j})}{2}\Big\}.$$

If $\mu_{2j-1} < \mu_{2j}$, then (2.1.3) reduces to $\mu_{2j}^3 \leq \rho\mu_{2j}^3 - \phi(\mu_{2j}^2)$, which is a contradiction as $0 < \rho < 1$. Hence $\mu_{2j} \leq \mu_{2j-1}$,i.e., $\mu_n \leq \mu_{n-1}$.

Case II. If n is odd, then similar to case I, we can obtain $\mu_{n+1} \leq \mu_n$.

Thus we have the sequence $\{\mu_n\}$ is non-increasing.

Let $\lim_{n\to\infty} \mu_n = \lim_{n\to\infty} \Delta(v_n, v_{n+1}) = \lambda$, for some $\lambda \geq 0$.

Suppose that $\lambda > 0$. On substituting $u = u_{2n}$ and $u = u_{2n+1}$ in (C4), we obtain

$$\Delta^3(fu_{2n}, gu_{2n+1}) \leq \rho\psi\Big\{\Delta^2(hu_{2n}, fu_{2n})\Delta(ku_{2n+1}, gu_{2n+1}),$$
$$\Delta(hu_{2n}, fu_{2n})\Delta^2(ku_{2n+1}, gu_{2n+1})],$$
$$\Delta(hu_{2n}, fu_{2n})\Delta(hu_{2n}, gu_{2n+1})\Delta(ku_{2n+1}, fu_{2n}),$$
$$\Delta(hu_{2n}, gu_{2n+1})\Delta(ku_{2n+1}, fu_{2n})\Delta(ku_{2n+1}, gu_{2n+1})\Big\}$$
$$- \phi(m(hu_{2n}, ku_{2n+1})),$$

where

$$m(hu_{2n}, ku_{2n+1}) = \max\Big\{\Delta^2(hu_{2n}, ku_{2n+1}), \Delta(hu_{2n}, fu_{2n})\Delta(ku_{2n+1}, gu_{2n+1}),$$
$$\Delta(hu_{2n}, gu_{2n+1})\Delta(ku_{2n+1}, fu_{2n}), \frac{1}{2s}[\Delta(hu_{2n}, fu_{2n})\Delta(hu_{2n}, gu_{2n+1})$$
$$+ \Delta(ku_{2n+1}, fu_{2n})\Delta(ku_{2n+1}, gu_{2n+1})]\Big\}.$$

Using (2.1.1), triangular inequality, property of ϕ and approaching as $n \to \infty$, we get

$\lambda^3 \leq \rho\lambda^3 - \phi(\lambda^2) < \rho\lambda^3$, which is a contradiction and hence we have $\lambda = 0$.

Next, we assert that $\{v_n\}$ is a Cauchy sequence. If possible, let $\{v_n\}$ be not a Cauchy sequence. Then there exists $\epsilon > 0$, for which we can find two sequences of positive integers $\{\beta(\tau)\}$ and $\{\gamma(\tau)\}$ such that for all positive integers τ with $\gamma(\tau) > \beta(\tau) \geq \tau$, we have

$$\Delta(v_{\beta(\tau)}, v_{\gamma(\tau)}) \geq \epsilon.$$

Further corresponding to $\beta(\tau)$, we can choose $\gamma(\tau)$ in such a manner that it is the smallest zpositive integer with $\gamma(\tau) > \beta(\tau)$ and satisfying $\Delta(v_{\beta(\tau)}, v_{\gamma(\tau)}) \geq \epsilon$. Then, we get

$$\Delta(v_{\beta(\tau)}, v_{\gamma(\tau)-1}) < \epsilon.$$

Now, $\epsilon \leq \Delta(v_{\beta(\tau)}, v_{\gamma(\tau)}) \leq s\Delta(v_{\beta(\tau)}, v_{\beta(\tau)-1}) + s\Delta(v_{\beta(\tau)-1}, v_{\gamma(\tau)})$.
$\leq s\Delta(v_{\beta(\tau)}, v_{\beta(\tau)-1}) + s\Delta(v_{\beta(\tau)-1}, v_{\gamma(\tau)-1}) + s^2\Delta(v_{\gamma(\tau)-1}, v_{\gamma(\tau)})$
From lemma 1.5, we have

$$\frac{\epsilon}{s^2} \leq \lim_{\gamma\to+\infty} \Delta(v_{\beta(\tau)-1}, v_{\gamma(\tau)-1}) \leq \lim_{\gamma\to+\infty} sup\Delta(v_{\beta(\tau)-1}, v_{\gamma(\tau)-1})$$

$$\leq s \lim_{\gamma\to+\infty} \Delta(v_{\beta(\tau)-1}, v_{\gamma(\tau)}) + s \lim_{\gamma\to+\infty} \Delta(v_{\beta(\tau)}, v_{\gamma(\tau)-1}) \leq s\epsilon$$

Thus, we have

$$\frac{\epsilon}{s^2} \leq \lim_{\gamma\to+\infty} inf\Delta(v_{\beta(\tau)-1}, v_{\gamma(\tau)-1}) \leq \lim_{\gamma\to+\infty} sup\Delta(v_{\beta(\tau)-1}, v_{\gamma(\tau)-1}) \leq s\epsilon$$

Applying condition (2.1.1) with $u = u_{\beta(\tau)}$ and $v = u_{\gamma(\tau)}$, we have

$$\begin{aligned}\Delta^3(fu_{\beta(\tau)}, gu_{\gamma(\tau)}) \leq & \rho\psi\Big\{\Delta^2(hu_{\beta(\tau)}, fu_{\beta(\tau)})\Delta(ku_{\gamma(\tau)}, gu_{\gamma(\tau)}), \\ & \Delta(hu_{\beta(\tau)}, fu_{\beta(\tau)})\Delta^2(ku_{\gamma(\tau)}, gu_{\gamma(\tau)}), \\ & \Delta(hu_{\beta(\tau)}, fu_{\beta(\tau)})\Delta(hu_{\beta(\tau)}, gu_{\gamma(\tau)})\Delta(ku_{\gamma(\tau)}, fu_{\beta(\tau)}), \\ & \Delta(hu_{\beta(\tau)}, gu_{\gamma(\tau)})\Delta(ku_{\gamma(\tau)}, fu_{\beta(\tau)})\Delta(ku_{\gamma(\tau)}, gu_{\gamma(\tau)})\Big\} \\ & - \phi(m(hu_{\beta(\tau)}, ku_{\gamma(\tau)})),\end{aligned}$$

where

$$\begin{aligned}m(hu_{\beta(\tau)}, ku_{\gamma(\tau)}) = \max\Big\{ & \Delta^2(hu_{\beta(\tau)}, ku_{\gamma(\tau)}), \Delta(hu_{\beta(\tau)}, fu_{\beta(\tau)})\Delta(ku_{\gamma(\tau)}, gu_{\gamma(\tau)}), \\ & \Delta(hu_{\beta(\tau)}, gu_{\gamma(\tau)})\Delta(ku_{\gamma(\tau)}, fu_{\beta(\tau)}), \frac{1}{2}[\Delta(hu_{\beta(\tau)}, fu_{\beta(\tau)}) \\ & .\Delta(hu_{\beta(\tau)}, gu_{\gamma(\tau)}) + \Delta(ku_{\gamma(\tau)}, fu_{\beta(\tau)})\Delta(ku_{\gamma(\tau)}, gu_{\gamma(\tau)})]\Big\}.\end{aligned}$$

Using (2.1.1), we have

$$\begin{aligned}\Delta^3(v_{\beta(\tau)}, v_{\gamma(\tau)}) \leq & \rho \max\Big\{[\Delta^2(v_{\beta(\tau)-1}, v_{\beta(\tau)})\Delta(v_{\gamma(\tau)-1}, v_{\gamma(\tau)}) \\ & + \Delta(v_{\beta(\tau)-1}, v_{\beta(\tau)})\Delta^2(v_{\gamma(\tau)-1}, v_{\gamma(\tau)})], \\ & \Delta(v_{\beta(\tau)-1}, v_{\beta(\tau)})\Delta(v_{\beta(\tau)-1}, v_{\gamma(\tau)})\Delta(v_{\gamma(\tau)-1}, v_{\beta(\tau)}), \\ & \Delta(v_{\beta(\tau)-1}, v_{\gamma(\tau)})\Delta(v_{\gamma(\tau)-1}, v_{\beta(\tau)})\Delta(v_{\gamma(\tau)-1}, v_{\gamma(\tau)})\Big\} \\ & - \phi(m(v_{\beta(\tau)}, v_{\gamma(\tau)})),\end{aligned}$$

where

$$\begin{aligned}m(v_{\beta(\tau)}, v_{\gamma(\tau)}) = & \max\Big\{\Delta^2(v_{\beta(\tau)-1}, v_{\gamma(\tau)-1}), \Delta(v_{\beta(\tau)-1}, v_{\beta(\tau)})\Delta(v_{\gamma(\tau)-1}, v_{\gamma(\tau)}), \\ & \Delta(v_{\beta(\tau)-1}, v_{\gamma(\tau)})\Delta(v_{\gamma(\tau)-1}, v_{\beta(\tau)}), \frac{1}{2s}[\Delta(v_{\beta(\tau)-1}, v_{\beta(\tau)})\Delta(v_{\beta(\tau)-1}, v_{\gamma(\tau)}) \\ & + \Delta(v_{\gamma(\tau)-1}, v_{\beta(\tau)})\Delta(v_{\gamma(\tau)-1}, v_{\gamma(\tau)})]\Big\}.\end{aligned}$$

$0 < s^3 \leq -\phi(s\epsilon^2)$, a contradiction.

Thus $\{v_n\}$ is a Cauchy sequence in $\mathcal{M}$. As $(\mathcal{M}, \Delta)$ is a complete b-metric space, $\{v_n\}$ converges to $w \in \mathcal{M}$ as $n \to \infty$. Consequently, the gsubsequences $\{fu_{2n}\}$, $\{hu_{2n}\}$, $\{gu_{2n+1}\}$ and $\{ku_{2n+1}\}$ of the sequence $\{v_n\}$ also converge to w.

Now, let the map h be continuous. Then $\{hhu_{2n}\}$ and $\{hfu_{2n}\}$ converge to hw as $n \to \infty$. Using the compatibility of the pair (f, h), we have, $\{fhu_{2n}\}$ converges to hw as $n \to \infty$.

First, we assert that $w = hw$. Let $w \neq hw$. On substituting $u = hu_{2n}$ and $v = u_{2n+1}$ in condition (C4), we ghave

$$\begin{aligned}\Delta^3(fhu_{2n}, gu_{2n+1}) \leq & \rho\psi\Big\{[\Delta^2(hhu_{2n}, fhu_{2n})\Delta(ku_{2n+1}, gu_{2n+1}), \\ & \Delta(hhu_{2n}, fhu_{2n})\Delta^2(ku_{2n+1}, gu_{2n+1})], \\ & \Delta(hhu_{2n}, fhu_{2n})\Delta(hhu_{2n}, gu_{2n+1})\Delta(ku_{2n+1}, fhu_{2n}), \\ & \Delta(hhu_{2n}, gu_{2n+1})\Delta(ku_{2n+1}, fhu_{2n})\Delta(ku_{2n+1}, gu_{2n+1})\Big\} \\ & - \phi(m(hhu_{2n}, ku_{2n+1})),\end{aligned}$$

where

$$\begin{aligned}m(hhu_{2n}, ku_{2n+1}) = & \max\Big\{\Delta^2(hhu_{2n}, ku_{2n+1}), \Delta(hhu_{2n}, fhu_{2n})\Delta(ku_{2n+1}, gu_{2n+1}), \\ & \Delta(hhu_{2n}, gu_{2n+1})\Delta(ku_{2n+1}, fhu_{2n}), \frac{1}{2s}[\Delta(hhu_{2n}, fhu_{2n})\Delta(hhu_{2n}, gu_{2n+1}) \\ & + \Delta(ku_{2n+1}, fhu_{2n})\Delta(ku_{2n+1}, gu_{2n+1})]\Big\}.\end{aligned}$$

Approaching as $n \to \infty$, we have

$$\begin{aligned}\Delta^3(hw, w) \leq & \rho\psi\Big\{\Delta^2(hw, w)\Delta(w, w), \Delta(hw, hw)\Delta^2(w, w),\\ & \Delta(hw, hw)\Delta(hw, w)\Delta(w, hw),\\ & \Delta(hw, w)\Delta(w, hw)\Delta(w, w)\Big\} - \phi(m(hw, w)),\end{aligned}$$

where

$$\begin{aligned}m(hw, w) = \max\Big\{&\Delta^2(hw, w), \Delta(hw, hw)\Delta(w, w), \Delta(hw, w)\Delta(w, hw),\\ &\frac{1}{2s}[\Delta(hw, hw)\Delta(hw, w) + \Delta(w, hw)\Delta(w, w)]\Big\},\end{aligned}$$

which gives that $\Delta^3(hw, w) \leq -\phi(\Delta^2(hw, w))$, a contradiction and hence $hw = w$.

Next, we assert that $fw = w$. On taking $u = w$ and $v = u_{2n+1}$ in (C4),

$$\begin{aligned}\Delta^3(fw, gu_{2n+1}) \leq & \rho\psi\Big\{\Delta^2(hw, fw)\Delta(ku_{2n+1}, gu_{2n+1}), \Delta(hw, fw)\\ & .\Delta^2(ku_{2n+1}, gu_{2n+1}), \tau\Delta(hw, fw)\Delta(hw, gu_{2n+1})\Delta(ku_{2n+1}, fw),\\ & \Delta(hu_{2n}, gu_{2n+1})\Delta(ku_{2n+1}, fw)\Delta(ku_{2n+1}, gu_{2n+1})\Big\}\\ & - \phi(m(hw, ku_{2n+1})),\end{aligned}$$

where

$$\begin{aligned}m(hw, ku_{2n+1}) = \max\Big\{&\Delta^2(hw, ku_{2n+1}), \Delta(hwfw)\Delta(ku_{2n+1}, gu_{2n+1}),\\ &\Delta(hw, gu_{2n+1})\Delta(ku_{2n+1}, fw), \frac{1}{2s}[\Delta(hw, fw)\Delta(hw, gu_{2n+1})\\ &+ \Delta(ku_{2n+1}, fw)\Delta(ku_{2n+1}, gu_{2n+1})]\Big\}.\end{aligned}$$

Approaching as $n \to \infty$ and after simplifying, we get $\Delta^3(fw, w) \leq 0$, i.e., $fw = w$.

Using condition $f(\mathcal{M}) \subset k(\mathcal{M})$, we get $p \in \mathcal{X}$ such that $w = fw = kp$.

Next, we show that $w = gp$. Using condition (C4) for $u = w$ and $v = p$, we obtain

$$\begin{aligned}\Delta^3(fw, gp) \leq & \rho\psi\Big\{\Delta^2(hw, fw)\Delta(kp, gp), \Delta(hw, fw)\Delta^2(kp, gp),\\ & \Delta(hw, fw)\Delta(hw, gp)\Delta(kp, fw), \Delta(hp, gp)\\ & .\Delta(kp, fw)\Delta(kp, gp)\Big\} - \phi(m(hw, kp)),\end{aligned}$$

where

$$\begin{aligned}m(hw, kp) = \max\Big\{&\Delta^2(hw, kp), \Delta(hw, fw)\Delta(kp, gp), \Delta(hw, gp)\Delta(kp, fw),\\ &\frac{1}{2s}[\Delta(hw, fw)\Delta(hw, gp) + \Delta(kp, fw)\Delta(kp, gp)]\Big\}.\end{aligned}$$

Using $w = fw = hw = kp$ and on simplifying, we have

$$\Delta^3(fw, gp) \leq \rho\psi\{0, 0, 0\} - \phi(0),$$

which implies that $\Delta^3(w, gp) = 0$ and hence $w = gp$. Due to the compatibility of the pair (g, k) and $kp = gp = w$, we have, $kgp = gkp$ and hence $kw = kgp = gkp = gw$. Now, suppose that $w \neq kw$. On putting $u = v = w$ in (C4), we have

$$\begin{aligned}\Delta^3(w, kw) = \Delta^3(fw, gw) \leq &\rho\psi\Big\{[\Delta^2(hw, fw)\Delta(kw, gw),\\ &\Delta(hw, fw)\Delta^2(kw, gw)], \Delta(hw, fw)\Delta(hw, gw)\\ &.\Delta(kw, fw), \Delta(hw, gw)\Delta(kw, fw)\Delta(kw, gw)\Big\}\\ &- \phi(m(hw, kw)),\end{aligned}$$

where

$$\begin{aligned}m(hw, kw) = &\max\Big\{\Delta^2(hw, kw), \Delta(hw, fw)\Delta(kw, gw), \Delta(hw, gw)\Delta(kw, fw),\\ &\frac{1}{2s}[\Delta(hw, fw)\Delta(hw, gw) + \Delta(kw, fw)\Delta(kw, gw)]\Big\} = \Delta^2(w, kw).\end{aligned}$$

On simplifying, we get $\Delta^3(w, kw) \leq -\phi(\Delta^2(w, kw))$, a contradiction and hence $w = kw$. Thus $w = kw = hw = fw = gw$. Therefore, w is a common fixed point of f, g, h and k.

In a similar manner, continuity of k can be used to complete the proof.

Next, consider f to be continuous map. Then $\{ffu_{2n}\}$ and $\{fhu_{2n}\}$ converge to fw as $n \to \infty$. Since the pair (f, h) is compatible, therefore,, $\{hfu_{2n}\}$ converges to fw as n tends to ∞.

Now, we show that $w = fw$. Assume that $w \neq fw$. For this, on taking fu_{2n} for u and u_{2n+1} for v in (C4), we get

$$\begin{aligned}\Delta^3(ffu_{2n}, gu_{2n+1}) \leq &\rho\psi\Big\{[\Delta^2(hfu_{2n}, ffu_{2n})\Delta(ku_{2n+1}, gu_{2n+1}),\\ &\Delta(hfu_{2n}, ffu_{2n})\Delta^2(ku_{2n+1}, gu_{2n+1})],\\ &\Delta(hfu_{2n}, ffu_{2n})\Delta(hfu_{2n}, gu_{2n+1})\Delta(ku_{2n+1}, ffu_{2n}),\\ &\Delta(hfu_{2n}, gu_{2n+1})\Delta(ku_{2n+1}, ffu_{2n})\Delta(ku_{2n+1}, gu_{2n+1})\Big\}\\ &- \phi(m(hfu_{2n}, ku_{2n+1})),\end{aligned}$$

where

$$\begin{aligned}m(hfu_{2n}, ku_{2n+1}) = &\max\Big\{\Delta^2(hfu_{2n}, ku_{2n+1}), \Delta(hfu_{2n}, ffu_{2n})\Delta(ku_{2n+1}, gu_{2n+1}),\\ &\Delta(hfu_{2n}, gu_{2n+1})\Delta(ku_{2n+1}, ffu_{2n}), \frac{1}{2s}[\Delta(hfu_{2n}, ffu_{2n})\\ &.\Delta(hfu_{2n}, gu_{2n+1}) + \Delta(ku_{2n+1}, ffu_{2n})\Delta(ku_{2n+1}, gu_{2n+1})]\Big\}.\end{aligned}$$

Letting $n \to \infty$, we have $\Delta^3(fw, w) \leq -\phi(\Delta^2(fw, w))$, a contradiction. Hence $w = fw$.

Using condition $f(\mathcal{M}) \subset k(\mathcal{M})$, we get $w = fw = kq$, for some $q \in \mathcal{M}$.

Next, we assert that $w = gq$. On replacing u by fu_{2n} and v by q in (C4), we get

$$\begin{aligned}\Delta^3(ffu_{2n}, gq) \leq \rho\psi\Big\{&\Delta^2(hfu_{2n}, ffu_{2n})\Delta(kq, gq), \Delta(hfu_{2n}, ffu_{2n})\Delta^2(kq, gq),\\ &\Delta(hfu_{2n}, ffu_{2n})\Delta(hfu_{2n}, gq)\Delta(kq, ffu_{2n}), \Delta(hfu_{2n}, gq)\Delta(kq, ffu_{2n})\Delta(kq, gq)\Big\}\\ &- \phi(m(hfu_{2n}, kq)),\end{aligned}$$

where

$$\begin{aligned}m(hfu_{2n}, kq) = \max\Big\{&\Delta^2(hfu_{2n}, kq), \Delta(hfu_{2n}, ffu_{2n})\Delta(kq, gq),\\ &\Delta(hfu_{2n}, gq)\Delta(kq, ffu_{2n}), \frac{1}{2s}[\Delta(hfu_{2n}, ffu_{2n})\Delta(hfu_{2n}, gq)\\ &+ \Delta(kq, ffu_{2n})\Delta(kq, gq)]\Big\}.\end{aligned}$$

Approaching as $n \to \infty$, we have

$$\begin{aligned}\Delta^3(w, gq) \leq \rho\psi\Big\{&[\Delta^2(w, w)\Delta(w, gq), \Delta(w, w)\Delta^2(w, gq)],\\ &\Delta(w, w)\Delta(w, gq)\Delta(w, fw),\\ &\Delta(w, gq)\Delta(w, w)\Delta(w, gq)\Big\} - \phi(m(w, kq)),\end{aligned}$$

where

$$\begin{aligned}m(w, kq) = \max\Big\{&\Delta^2(w, kq), \Delta(w, w)\Delta(w, gq), \Delta(w, gq)\Delta(w, fw),\\ &\frac{1}{2s}[\Delta(w, w)\Delta(w, gq) + \Delta(w, fw)\Delta(w, gq)]\Big\} = 0,\end{aligned}$$

which implies that $\Delta^3(w, gq) = 0$ and hence $w = gq$. Using the compatibility of the pair (g, k) and $kq = gq = w$, we have $kgq = gkq$. Thus $kw = kgq = gkq = gw$.

Next, we claim that $w = gw$. Assume that $w \neq gw$. For this, on replacing u by u_{2n} and v by w in (C4), we get

$$\begin{aligned}\Delta^3(fu_{2n}, gw) \leq \rho\psi\Big\{&[\Delta^2(hu_{2n}, fu_{2n})\Delta(kw, gw), \Delta(hu_{2n}, fu_{2n})\Delta^2(kw, gw)],\\ &\Delta(hu_{2n}, fu_{2n})\Delta(hu_{2n}, gw)\Delta(kw, fu_{2n}), \Delta(hu_{2n}, gw)\Delta(kw, fu_{2n})\\ &.\Delta(kw, gw)\Big\} - \phi(m(hu_{2n}, kw)),\end{aligned}$$

where

$$\begin{aligned}m(hu_{2n}, kw) = \max\Big\{&\Delta^2(fu_{2n}, kw), \Delta(hu_{2n}, fu_{2n})\Delta(kw, gw), \Delta(hu_{2n}, gw)\Delta(kw, fu_{2n}),\\ &\frac{1}{2s}[\Delta(hu_{2n}, fu_{2n})\Delta(hu_{2n}, gw) + \Delta(kw, fu_{2n})\Delta(kw, gw)]\Big\}.\end{aligned}$$

Approaching as $n \to \infty$ and on simplifying, we have $\Delta^3(w, gw) \leq -\phi(\Delta^2(w, gw))$, a contradiction and hence $w = gw$. Using condition $g(\mathcal{M}) \subset h(\mathcal{M})$, we get $w = gw = hr$, for some r in $\mathcal{M}$.

Finally, to show $w = fr$, we replace u by r and v by w in (C4),

$$\begin{aligned}\Delta^3(fr, gw) \leq & \rho\psi\Big\{[\Delta^2(hr, fr)\Delta(kw, gw), \Delta(hr, fr)\Delta^2(kw, gw],\\ & \Delta(hr, fr)\Delta(hr, gw)\Delta(kw, fr), \Delta(hw, gw)\Delta(kw, fr)\Delta(kw, gw)\Big\}\\ & - \phi(m(hr, kw)),\end{aligned}$$

where

$$\begin{aligned}m(hr, kw) = & \max\Big\{\Delta^2(hr, kw), \Delta(hr, fr)\Delta(kw, gw), \Delta(hw, gw)\Delta(kw, fr),\\ & \frac{1}{2s}[\Delta(hr, fr)\Delta(hr, gw) + \Delta(kw, fr)\Delta(kw, gw)]\Big\}.\end{aligned}$$

On simplifying, we conclude $\Delta^3(fr, w) \leq 0$, i.e., $fr = w$. Using the compatibility of the pair (f, h) and $fr = hr = w$, we obtain $fhr = hfr$ and hence $hw = hfr = fhr = fw$. That is, $w = hw = fw = kw = gw$. Therefore, w is a common fixed point of f, g, h and k.

In a similar pattern, proof holds for the continuity of the map g .

To claim uniqueness, assume that w_1 and w_2 ($w_1 \neq w_2$) are two common fixed points of the given mappings. On putting $u = w_1$ and $v = w_2$ in (C4), we obtain

$$\Delta^3(w_1, w_2) = \Delta^3(fw_1, gw_2) \leq -\phi(\Delta^2(w_1, w_2)),$$

which is a contradiction and hence $w_1 = w_2$. Therefore, f, g, h and k possess a unique common fixed point in $\mathcal{X}$. □

Corollary 2.2. Let f and g be two mappings of a complete b-metric space $(\mathcal{M}, \Delta)$ into itself satisfying the following condition:

$$\begin{aligned}\Delta^3(fu, gv) \leq & \rho\psi\Big\{[\Delta^2(u, fu)\Delta(v, gv), \Delta(u, fu)\Delta^2(v, gv)],\\ & \Delta(u, fu)\Delta(u, gv)\Delta(v, fu), \Delta(u, gv)\Delta(v, fu)\Delta(v, gv)\Big\} - \phi(m(u, v)),\end{aligned}$$

for all $u, v \in \mathcal{X}$, where

$$\begin{aligned}m(u, v) = & \max\Big\{\Delta^2(u, v), \Delta(u, fu)\Delta(v, gv), \Delta(u, gv)\Delta(v, fu),\\ & \frac{1}{2s}[\Delta(u, fu)\Delta(u, gv) + \Delta(v, fu)\Delta(v, gv)]\Big\}\end{aligned}$$

and ρ is a real number satisfying $0 < \rho < 1$. Further, $\phi : [0, \infty) \to [0, \infty)$ is a continuous function with $\phi(0) = 0$ and $\phi(t) > 0$ for each $t > 0$. Then f and g have a unique common fixed in $\mathcal{M}$.

Proof. Taking $h = k = I$ (Identity map) in Theorem 2.1, the result holds easily. □

References

[1] Aamri, M. and Moutawakil, D. El, *Some new common fixed point theorems under strict contractive conditions*, J. Math. Anal. Appl., **270** (2002), 181–188.

[2] Agarwal, R.P., Bisht, R.K. and Shahzad, N., *A comparison of various non-commuting conditions in metric fixed point theory and their applications*, Fixed Point Theory and Applications,**1** (2014), 1-33.

[3] Aghajani, A., Abbas, M., Roshan, J.R., *Common fixed point of generalized weak contractive mappings in partially ordered b- metric spaces*, Math. Slovaca, **64(4)**, 941–960 (2014).

[4] Alber, Y.I. and Guerre-Delabriere, S., *Principle of weakly contractive maps in Hilbert spaces, New results*, Oper. Theory Adv. Appl., **98** (1997), 7-22.

[5] Ali, J., Imdad, M., *An implicit function implies several contraction conditions*, Sarajevo J. Math., **4(17)** (2008), 269–285.

[6] Bakhtin, I.A., *The contraction mapping principle in almost metric spaces*, J. Funct. Anal., **30**, 26–37 (1989).

[7] Banach, S., *Surles operations dans les ensembles abstraites et leursapplications*, Fundam. Math., **3**(1922), 133-181.

[8] Bellman, R., Methods of Nonlinear Analysis, Vol II, Academic Press, New York, 1973.

[9] Bellman, R. and Lee, E.S., *Functional equations arising in dynamic programming*, Aequationes Math., **17** (1978), 1-18.

[10] Boyd, D.W. and Wong, J.S.W., *On nonlinear contractions*, Proc. Amer. Math. Soc., **20(2)** (1969), 458-464.

[11] Ciric, B. L., *Generalized contractions and fixed point theorems*, Publ. Inst. Math. (Beograd) (N.S.), **12(26)**, (1971), 19–26.

[12] Gopal, D. and Ranadive, A.S., *Common fixed points of absorbing maps*, Bull. Marathwada Math. Soc. **9**(2008), 43-48.

[13] Imdad, M., Pant, B.D., Chauhan, S., *Fixed point theorems in Menger spaces using the* (CLR_{ST}) *property and applications*, J. Nonlinear Anal. Optim., **3(2)** (2012), 225–237.

[14] Jungck, G., *Commuting mappings and fixed points*, Amer. Math. Monthly, **83** (1976), 261-263.

[15] Jungck, G. and Pathak, H.K., *Fixed points via biased Maps*, Proc. Amer. Math. Soc., **123 (7)**, 1995.

[16] Jungck, G., *Compatible mappings and common fixed points*, Int. J. Math. Math. Sci., **9** (1986), 771-779.

[17] Jungck, G., Rhoades, B.E., *Fixed points for set valued functions without continuity*, Indian J. Pure Appl. Math., **29** (1998), 227-238.

[18] Jungck, B.E., Murthy, P. P. and Cho, Y. J., *Compatible mappings of type (A) and common fixed points*, Math. Jpn., **38(2)** (1993), 381-390.

[19] Khan, M.S., Swalek, M. and Sessa, S.,*Fixed point theorems by altering distances between two points*, Bull. Austra. Math. Soc., **30** (1984), 1-9.

[20] Kumar, R. and Kumar, S., *Fixed points for weak contraction involving cubic terms of distance function*, J. Math. Comput. Sci., **11** (2021), 1922-1954.

[21] Murthy, P.P., *Important tools and possible applications of metric fixed point theory*, Nonlinear Anal., **47(5)** (2001), 3479–3490.

[22] Pant, R.P., *Noncompatible mappings and common fixed points*, Soochow J. Math., **26(1)** (2000), 29–35.

[23] Pant, R.P., *Discontinuity and fixed points*, J. Math. Anal. Appl., **240(1)** (1999), 280–283.

[24] Pant, R.P., *Common fixed points of non commuting mappings*. J. Math. Anal. Appl. **188** (1994), 436-440.

[25] Pant, R.P., *A common fixed point theorem under a new condition*. Indian J. Pure Appl. Math. **30(2)**(1999), 147-152.

[26] Rhoades, B.E., *Some theorems on weakly contractive maps*, Nonlinear Anal., **47** (2001),2683-2693.

[27] Sessa, S., *On a weak commutativity conditions of mappings in fixed point consideration*, Publ. Inst. Math., **32(46)** (1982), 146-153.

[28] Sintunavarat, W. and Kumam, P., *Common fixed point theorems for a pair of weakly compatible mappings in fuzzy metric spaces*, Hindawi Publishing Corporation, Journal of applied Mathematics, Volume (2011), Article ID 637958 (2011).