Original Research Article

The Minimum Reduced Sombor Index of Unicyclic Graphs in terms of the Girth

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ABSTRACT

Aims: The paper investigates the Reduced Sombor Index (RSO) for unicyclic graphs. Specifically, it aims to determine and characterize the unicyclic graphs that attain the minimum RSO index among all unicyclic graphs of a given order.
Study design: This is a theoretical mathematical study based on graph theory and topological indices. The study involves in defining and analyzing the Reduced Sombor Index by comparing RSO values across different unicyclic graphs. Lemmas and theorems are proved to establish the minimum RSO index graph.
Methodology: Several graph transformation operations are analyzed. The study proves multiple lemmas that compare RSO values before and after transformations by demonstrating whether a specific structural modification increases or decreases the RSO value.
Results: The minimum RSO value in unicyclic graphs is achieved only by cycle graphs. Several lemmas prove that adding pendant vertices or modifying graph structure increases RSO. The final theorem establishes that for any unicyclic graph of order n, $RSO(G) \geq n\sqrt{2}$ with equality if and only if G is a cycle C_n.
Conclusion: The study successfully characterizes unicyclic graphs with the minimum Reduced Sombor Index (RSO). It establishes that cyclic graphs are the unique minimizers of the RSO index among unicyclic graphs. Any structural modification leading to non-cycle unicyclic graphs increases RSO. The findings contribute to chemical graph theory by refining how topological indices behave in molecular graph models.

Keywords: Reduced Sombor index; topological index; graph invariant; trees; extremal problem; characterization.

1. INTRODUCTION

By a graph *G*, in this article, we mean an ordered pair (V_G, E_G), and the members of the sets V_G and E_G respectively are the vertices and edges of the graph. The set of vertices that are adjacent to a vertex *u* in *G* is denoted as $N_G(u)$ and called by "the open neighbourhood" of *u* in *G*. The term "closed neighbourhood" is $N_G[v] = N_G(v) \cup \{v\}$ and by a (v, w)-Path $vv_1v_2 \dots w$ is a sequence of distinct members of the set V_G and the vertices *v*,*w* are usually known as *the origin* and *the terminus* of the path *P* respectively. The concept of distance between any two vertices $x, y \in V_G$ is usually defined as the length of the smallest (x, y)-path that exists in *G*. If $d_G(v) = 1$, then *v* is a pendant vertex and it is adjacent to a unique vertex in *G*, say *u* which is called a support vertex. For more on graphs and related works, the reader is referred to [1, 6, 7].

The topological indices (also known as graph invariants) play a major role in the chemical graph theory because they are used to analyze the behaviour of the molecule structures and their inter-relationships. There are numerous topological indices available in the literature; a few of them are the Sombor index, Zagreb index, and so on. The topological indices were defined with minor and major modifications in the past and several classes of topological indices are available for the Sombor index and Zagreb index. Given a graph *G*, the Sombor (SO) index is defined (by Gutman [8]) as

$SO(G) = \sum_{uv \in E(G)} \sqrt{d_G(u)^2 + d_G(v)^2}$.

The Sombor index, in recent years, received numerous attentions from academics and researchers throughout the globe [4, 9, 12, 13, 16]. For some recent surveys in Sombor index, one can refer to the articles [2, 11]. Chemical applications have been carried out in the articles [5, 14]. For various results and versions of Sombor index, one can refer [10, 15, 18]. The reduced Sombor index is defined as

$RSO(G) = \sum_{uv \in E(G)} \sqrt{(d_G(u) - 1)^2 + (d_G(v) - 1)^2}$.

The reduced Sombor index is a recently introduced term and some of the works can be found in [6, 8].

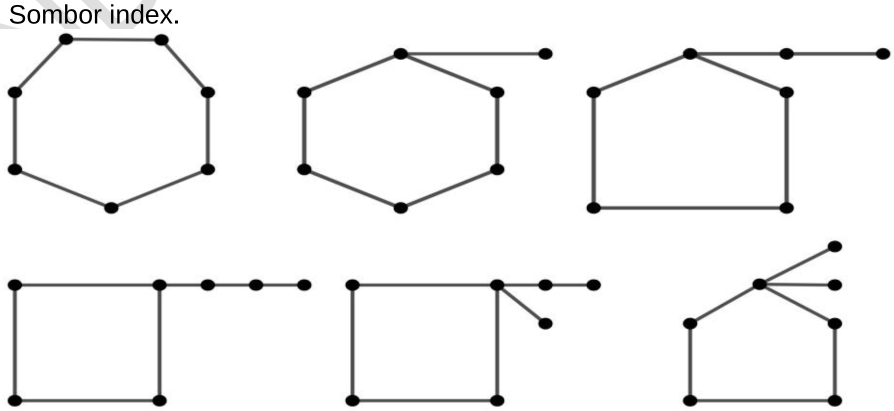
The reduced Sombor index of unicyclic graphs is studied in this article and characterized the graphs with minimum Reduced Sombor index.

2. METHODOLOGY

The main results on the minimum Reduced Sombor index for the class of unicyclic graphs is studied in this section. If the order *n* of *G* is one, there is no edge in the graph; hence by a graph *G*, throughout this article, we mean a graph with minimum two vertices. Also, by a graph *G*, we mean only a connected graph, unless explicitly stated.

2.1 The Graphs with the Minimum Reduced Sombor Index

The minimum and maximum values of topological indices have been an interesting problem in history. In this section, the results on the minimum Reduced Sombor index of unicyclic graphs are provided. Using the behavioural analysis of $RSO(G)$ on this collection of graphs, further, a characterization of minimum $RSO(G)$ of unicyclic graphs is provided. For a given positive integer $n \geq 3$, there exists several unicyclic graphs with order *n*. Here we provide a simple example for the case when $n = 7$. The Figure 1 provides a few unicyclic graphs of oredr 7. Among all the unicyclic graphs with 7 vertices, the cycle C_7 has the minimum Reduced Sombor index.



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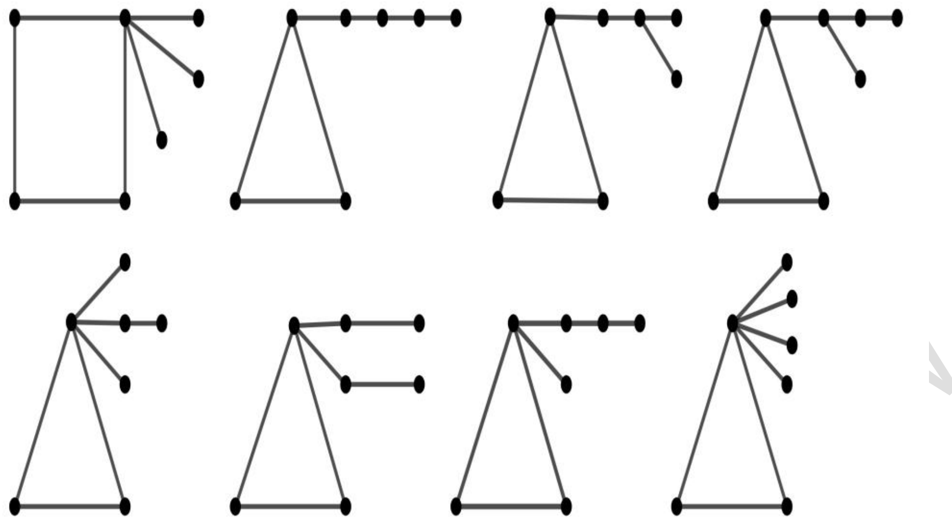

Figure 1: Unicyclic Graphs with order 7

A characterization of graphs with the minimum values of $RSO(G)$ on the class of unicyclic graphs is provided in this section. First, without loss of generality, let the order *n* of *G* be minimum 3 unless the graph contains no cycle. For any integer *n*, assuming it is the order of *G*, we found the unicyclic graphs of order *n* with minimum $RSO(G)$. This characterization provides a unique unicyclic graph for any positive integer $n \geq 3$ with minimum *RSO*(*G*).

Lemma 1 Let u, v be vertices of a cycle $\mathrm{G} = \mathrm{C_n}$ such that u and v lies in the graph G. Let H be a graph constructed from G by deleting the vertex u and attaching it to the vertex v as a pendant vertex, and making the two neighbours of u in the cycle to be adjacent in H. Then, $\mathrm{RSO(G)} < \mathrm{RSO(H)}$.

Proof: Let *u* be a vertex of *G* such that *u* lies in the cycle of *G* and u', u'' are the vertices adjacent to u and u', u'' lies in the cycle of *G*. Let u be attached to *v* as a pendant vertex. Note that, we deleted the edges uu', uu'' and added the edges u', u'' and uv. Now,

$$\begin{aligned} SO(H) = RSO(G) &- \sqrt{(d_G(u)-1)^2 + (d_G(u')-1)^2} - \sqrt{(d_G(u)-1)^2 + (d_G(u'')-1)^2} \\ &- \sqrt{(d_G(v)-1)^2 + (d_G(v')-1)^2} - \sqrt{(d_G(v)-1)^2 + (d_G(v'')-1)^2} \\ &+ \sqrt{(d_H(u')-1)^2 + (d_H(u'')-1)^2} + \sqrt{(d_H(u)-1)^2 + (d_H(v)-1)^2} \\ &+ \sqrt{(d_H(v)-1)^2 + (d_H(v')-1)^2} + \sqrt{(d_H(v)-1)^2 + (d_H(v'')-1)^2} \end{aligned}$$

$$= RSO(G) - \sqrt{(2-1)^2 + (2-1)^2} - \sqrt{(2-1)^2 + (2-1)^2} - \sqrt{(2-1)^2 + (2-1)^2}$$

$$- \sqrt{(2-1)^2 + (2-1)^2} + \sqrt{(2-1)^2 + (2-1)^2} + \sqrt{(1-1)^2 + (3-1)^2}$$

$$+ \sqrt{(2-1)^2 + (3-1)^2} + \sqrt{(2-1)^2 + (3-1)^2}$$

$$= RSO(G) - 4\sqrt{2} + \sqrt{2} + \sqrt{2^2} + 2\sqrt{5}$$

$$= RSO(G) - 3\sqrt{2} + 2\sqrt{5} + 2$$

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$= RSO(G) + k,$ say k

$> RSO(G),$ Since $k > 0$

Lemma 2 *Let G be cycle C_n and $u_1, u_2, \dots u_k \in V_G$ be the consecutive vertices lie on the cycle. Let H* be the graph constructed from G by deleting the vertices $u_1, u_2, \dots u_k$ and attaching them to a single vertex v on the cycle as pendant vertices so that the girth of H* is three. Then, $RSO(G) < RSO(H^*)$.*

Proof: Let H^* be the graph constructed as in the hypothesis. Let $u_1, u_2, \dots u_k$ be consecutive vertices on the cycle C_n. The girth of H^* is three. Let *v* be the unique vertex in $V(H)$ with $d_H(v) \geq 3$ and v', v'' be the two neighbours of v in H^*. Now,

$$RSO(H^*) = RSO(G) - \sum_{i=1}^{k-1} \sqrt{(d_G(u_i) - 1)^2 + (d_G(u_{i+1}) - 1)^2}$$
$$+ \sum_{i=1}^{k-1} \sqrt{(d_G(u_i) - 1)^2 + (d_G(v) - 1)^2} - \sqrt{(d_G(v') - 1)^2 + (d_G(v) - 1)^2}$$
$$- \sqrt{(d_G(v'') - 1)^2 + (d_G(v) - 1)^2} + \sqrt{(d_{H^*}(v') - 1)^2 + (d_{H^*}(v) - 1)^2}$$
$$+ \sqrt{(d_{H^*}(v'') - 1)^2 + (d_{H^*}(v) - 1)^2}$$
$$= RSO(G) - \sum_{i=1}^{k-1} \sqrt{(2-1)^2 + (2-1)^2} + \sum_{i=1}^{k-1} \sqrt{(1-1)^2 + (k+2-1)^2}$$
$$- \sqrt{(2-1)^2 + (2-1)^2} - \sqrt{(2-1)^2 + (2-1)^2} + \sqrt{(2-1)^2 + (k+2-1)^2}$$
$$+ \sqrt{(2-1)^2 + (k+2-1)^2}$$
$$= RSO(G) - \sum_{i=1}^{k-1} \sqrt{2} + \sum_{i=1}^{k-1} \sqrt{(1-1)^2 + (k-1)^2} - \sqrt{(2-1)^2 + (2-1)^2}$$
$$- \sqrt{(2-1)^2 + (2-1)^2} + \sqrt{(2-1)^2 + (k+2-1)^2} +$$
$$\sqrt{(2-1)^2 + (k+2-1)^2}$$
$$= RSO(G) - (k-1)\sqrt{2} + (k-1)\sqrt{(k-1)^2} - 2\sqrt{2} + 2\sqrt{(k+1)^2 + 1}$$
$$= RSO(G) - (k-3)\sqrt{2} + (k-1)\sqrt{(k-1)^2} + 2\sqrt{(k+1)^2 + 1}$$
$$= RSO(G) + m, \text{ say m}$$
$$> RSO(G), \text{ since } m > 0$$

Hence, it is proved.

Lemma 3 *Let $G = C_n$ be a cycle on n vertices and let u be a vertex in the cycle of G. Let W be the graph constructed from G by attaching a pendant vertex to u. Then, $RSO(G) > RSO(W)$.*

Proof: Let *u* be a vertex on the cycle of *G* and *W* be a graph constructed from *G* such that a vertex v is attached to u. Let u_1 and u_2 be the neighbours of u in G. Note that now $d_W(u) = 3$. Now,

$$RSO(H) = RSO(G) - \sqrt{(d_G(u) - 1)^2 + (d_G(u_1) - 1)^2} - \sqrt{(d_G(u) - 1)^2 + (d_G(u_2) - 1)^2}$$
$$+ \sqrt{(d_W(u) - 1)^2 + (d_W(u_1) - 1)^2} + \sqrt{(d_W(u) - 1)^2 + (d_W(u_2) - 1)^2}$$
$$+ \sqrt{(d_W(u) - 1)^2 + (d_W(v) - 1)^2}$$
$$= RSO(G) - \sqrt{(2-1)^2 + (2-1)^2} - \sqrt{(2-1)^2 + (2-1)^2} + \sqrt{(3-1)^2 + (2-1)^2}$$
$$+ \sqrt{(3-1)^2 + (2-1)^2} + \sqrt{(3-1)^2 + (1-1)^2}$$
$$= RSO(G) - 2\sqrt{2} + 2\sqrt{5} + 2$$
$$> RSO(G)$$

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Hence, it is proved.

Lemma 4 *Let u be a vertex of a unicyclic graph G such that u lies in the cycle with pendant neighbour $u'$and v be a non-pendant neighbour of u in the cycle of G. Let H_1 be a graph constructed from G by deleting the edge uv' and attaching it to the cycle by introducing the two new edges uv' and $v'v$. Then, $SO^m(G) > SO^m(H)$.*

Proof: Consider the vertex u on the cycle with u' being a pendant neighbour of u and v be a neighbour of *u* in the cycle of *G*. Let w be another neighbour on the cycle of *G*.
Now,

$$RSO(H_1) = RSO(G) - \sqrt{(d_G(u)-1)^2 + (d_G(u')-1)^2} - \sqrt{(d_G(u)-1)^2 + (d_G(w)-1)^2}$$

$$- \sqrt{(d_G(u)-1)^2 + (d_G(v)-1)^2} + \sqrt{\left(d_{H_1}(u)-1\right)^2 + \left(d_{H_1}(u')-1\right)^2}$$

$$+ \sqrt{\left(d_{H_1}(u)-1\right)^2 + \left(d_{H_1}(w)-1\right)^2} + \sqrt{\left(d_{H_1}(u')-1\right)^2 + \left(d_{H_1}(v)-1\right)^2}$$

$$+ \sqrt{\left(d_{H_1}(u)-1\right)^2 + \left(d_{H_1}(v)-1\right)^2}$$

$$= RSO(G) - \sqrt{(3-1)^2 + (1-1)^2} - \sqrt{(3-1)^2 + (2-1)^2} - \sqrt{(3-1)^2 + (2-1)^2}$$

$$+\sqrt{(2-1)^2 + (2-1)^2} + \sqrt{(2-1)^2 + (2-1)^2} + \sqrt{(2-1)^2 + (2-1)^2}$$

$$+\sqrt{(2-1)^2 + (2-1)^2}$$

$$= RSO(G) - \sqrt{2^2} - 2\sqrt{5} + 4\sqrt{2}$$

$$= RSO(G) - 2 - 2\sqrt{5} + 4\sqrt{2}$$

$$= RSO(G) + k, \quad \text{say k}$$

$$< RSO(G) \qquad \text{Since } k < 0$$

Hence, it is proved.

3. RESULTS AND DISCUSSION

The following theorem is a consequence of the lemmas proved in the previous section. For any positive integer *n*, this theorem provides a unique graph (up to isomorphism) of order *n* with minimum $RSO(G)$ and hence holds the validity of the existence of the result we provided for unicyclic graphs.

Theorem 5. *Given a positive integer n, the graph G is a unicyclic graph of order n and $RSO(G) \geq n\sqrt{2}$ with equality if and only if the girth $g(G) = n$.*

Proof: If $g(G) = n$, then G is a cycle on n vertices and hence a unicyclic graph and we have $RSO(G) = n\sqrt{2}$. Let us prove the converse of the theorem on induction on the number of vertices n of G. First, if $n = 3$, then $G = C_3$ which is the only unicyclic graph on 3 vertices and $RSO(G) = 3\sqrt{2}$ and hence it is true. If $n = 4$, then $g = 4$ and $RSO(G) = 4\sqrt{2}$. Thus, the induction is true for base cases $n = 3,4$.

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Now, let us assume that the induction is true for any unicyclic graph with its order not more than $n-1$. Now, assume G is a random unicyclic graph on n-vertices. If $g=n$, then we are done. So, let us assume that $g<n$.

Let G be of the form $C_{n,n-g}$. Then, there are g vertices on the cycle of G and the remaining $n-g$ vertices are adjacent to either vertices on the cycle or they can be connected by a path to on the cycle of G. Let v be a random vertex of G.

Case-1: Let v be adjacent to a vertex lying on the cycle of G.
Claim: There exists a unicyclic graph G' of order n such that $RSO(G) < RSO(G')$.

Let u be a vertex on the cycle of G to which v is adjacent in G. Then, by Lemma 1, $RSO(G) > RSO(C_n)$. Next, by Lemma 3, there exists a unicyclic graph G' with $RSO(G) < RSO(G')$. This ensures the existence of a unicyclic graph with minimum Reduced Sombor index than that of G. Again, by Lemma 3, as for as a vertex of degree more than two exists in G, the graph G is not a unicyclic graph with minimum Reduced Sombor index. Now by repeatedly applying Lemma 1 and Lemma 2, G reduces to the form C_n. Thus, the result is proved.

Case-2: Let v be adjacent to a vertex that is not on the cycle of G.
Claim: There exists a unicyclic graph G'' of order n such that $RSO(G) < RSO(G'')$.

Then, there exists a vertex w in the cycle of G through which the vertex w is connected. Thus, there is a vertex w in the cycle of G with $d_G(w) \geq 3$. Then, by Lemma 3, there exists a unicyclic graph G' with $RSO(G) < RSO(G')$. Now, by Lemma 4 we can construct a unicyclic graph H_1 such that $RSO(H_1) < RSO(G)$. Until the vertices of degree 3 exist we can repeatedly apply Lemma 4 and it results in C_n. Now by repeatedly applying Lemma 1 and Lemma 2, G reduces to the form C_n. Hence, the theorem is proved.

4. CONCLUSION

The study successfully characterizes unicyclic graphs with the minimum Reduced Sombor Index (RSO). It establishes that cyclic graphs are the unique minimizers of the RSO index among unicyclic graphs. Any structural modification leading to non-cycle unicyclic graphs increases RSO. The findings contribute to chemical graph theory by refining how topological indices behave in molecular graph models.

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