Theoretical Analysis of Nonlinear partial differential equations with Mixed Boundary Conditions

Abstract

The solution of author's mathematical model of Magnetohydrodynamics (MHD) flow of a Casson fluid is discussed. The given nonlinear partial differential equations are converted into nonlinear ordinary differential equations using similarity transformations. The simple and closed form analytical expression for concentration profiles is provided. The compatibility of analytical results with simulation results can be observed from the graphs and table presented.

Keywords

Mixed Boundary Conditions, Similarity Transformations, Homotopy Analysis Method, SCILAB.

1 Introduction

Magnetohydrodynamics (MHD) is the study of the flow of electrically conducting fluids, which has important applications in polymer industry and engineering fields, observed K.Sharada and B.Shankar [1]. Casson Fluid Model is one of the non-Newtonian models. Casson fluid was originally introduced by Casson. The model finds the rate of shear stress and strain relationships nonlinear. Examples of such are sauce, honey, soup, jelly etc. For the analysis of boundary layer flows, this model is one of the best rheological models.

Casson fluid's Magnetohydrodynamics (MHD) flow over a shrinking sheet and Stretching Sheet has been experimented by S. Nadeem et al. [2] and Krishnendu Bhattacharyya [3]. Ishak [4 – 6] has explained about MHD stagnation point flow on vertical, linearly stretching sheet. T. Hayat et al. [7 – 9] have discussed mixed convection flow of Casson nanofluid with mixed boundary condition.

By applying keller box method, K. Sharada and B. Shankar [1] have obtained the numerical solution of MHD flow of a Casson fluid with mixed boundary condition. In this manuscript, the analytical solution by Homotopy Analysis Method has been derived [10 – 13].

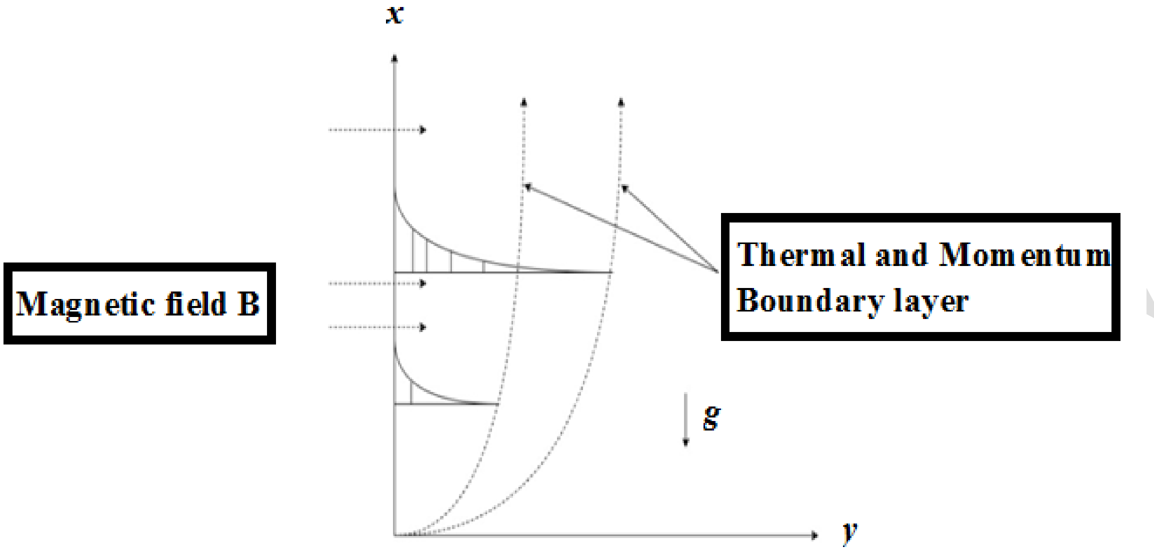

Fig. 1: Physical sketch of the flow [1]

2 Mathematical Formulation

The governing equations of continuity, momentum, energy and concentration are [1]:

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0 \tag{2.1}$$

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=u\left(1+\frac{1}{\beta}\right)\frac{\partial^2 u}{\partial y^2}+g\beta_T(T-T_\infty)-\frac{\sigma B^2}{\rho}\,u \tag{2.2}$$

$$u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}=\alpha\frac{\partial^2 T}{\partial y^2}+\frac{\sigma B^2}{\rho C_p}\,u^2 \tag{2.3}$$

where u and v are velocity components in the directions of x and y respectively.

The boundary conditions are defined as [1]:

At $y \to 0$

$u = u_w + L\frac{\partial u}{\partial y}$

$v = 0$

$-K\frac{\partial T}{\partial y}(x,0) = h_f(T_f - T_w)$

At $y \to \infty$

$u \to 0$

$T \to T_\infty$

where T_f, h_f , L, T_w are defined in [1].

Introducing the similarity transformations [1]:

$$\eta = \sqrt{\frac{a}{v}}\, y,\ \ \psi = \sqrt{va}\ \ xf(\eta),\ \ \theta(\eta) = \frac{T - T_\infty}{T_w - T_\infty}\ , u = axf'(\eta),\ \ v = \ -\sqrt{av}\,[f(\eta)]$$

By using the above similarity transformations Eqs. (2.1) to (2.3) are transformed into dimensionless, nonlinear differential equations as follows [1]:

$$\left(1 + \frac{1}{\beta}\right) f''' + \ f\, f'' + \lambda\theta - Mf' = 0 \tag{2.4}$$

$$\frac{1}{P_r}\theta'' + f\theta' + Hf'^2 = 0 \tag{2.5}$$

where $M = \frac{\sigma\, B_0^2}{\rho a}$, $P_r = \frac{v}{\alpha}$, $\lambda = \frac{g\beta_T(T_f - T_\infty)}{a^2 x}$, $Ec = \frac{u_W x}{C_p(T_f - T_\infty)}$, H = M. Ec are the dimensionless parameters [1].

The boundary conditions are

$$\eta = 0\!:\ f(0) = 0, f'(0) = 1 + \lambda_v\, f''(0), \theta'(0) = -\gamma\,(1 - \theta(0)) \tag{2.6}$$

$$\eta = \ \infty\!:\ f'(\infty) = 0, \theta(\infty) = 0 \tag{2.7}$$

where $\lambda_v = L\,\left(\frac{a}{v}\right)$, $\gamma = \left\{\frac{c}{K}\left(\sqrt{\frac{v}{u_\infty}}\right)\right\}$ are in [1].

3 The Approximate Analytical Expression of Fluid and Temperature using Homotopy Analysis Method

The homotopy analysis method is used to solve nonlinear differential equations. This method was introduced by Liao [10]. For few iterations, HAM gives a easy way to converge the solution. The basic idea of Homotopy Analysis Method is furnished in (Appendix A). By solving the nonlinear differential Eqs. (2.4) to (2.7) using the HAM, the approximate analytical expression of the fluid and temperature of casson fluid can be obtained as follows:

$$f(\eta) = \frac{\left(1 - e^{(-\sqrt{k}\,\eta)}\right)}{\lambda_v k + \sqrt{k}} + [\,A(1 + h) - Bh\,]\left(\frac{e^{(-\sqrt{k}\,\eta)}}{\sqrt{k}} - \frac{1}{\sqrt{k}} + \eta\, e^{(-\sqrt{k}\,\eta)}\right) + Ch\left(2e^{(-\sqrt{k}\,\eta)} - 1 - e^{(-2\sqrt{k}\,\eta)}\right) - \ Dh\left(\frac{T}{\sqrt{k}}\, e^{(-\sqrt{k}\,\eta)} - \frac{T}{\sqrt{k}} + 1 - \ e^{(-T\,\eta)}\right) \tag{3.1}$$

$$\theta(\eta) = \ \frac{\gamma}{(T+\gamma)}\ e^{(-T\,\eta)} \ + Eh\left(\frac{-(\sqrt{k}+T)e^{(-T\,\eta)}}{T} + e^{(-(\sqrt{k}+T)\eta)}\right) + Fh\,\left(\frac{(-2\sqrt{k})e^{(-T\,\eta)}}{T} + e^{(-(2\sqrt{k})\eta)}\right) \tag{3.2}$$

where $k = \frac{M}{1+\frac{1}{\beta}}$, $T = \frac{P_r}{\lambda_v k + \sqrt{k}}$, $A = \left[\left(\frac{(1+\beta)\sqrt{k}}{2(\lambda k+\sqrt{k})}\right) - \left(\frac{M}{2\sqrt{k}(\lambda k+\sqrt{k})}\right)\right]$, $B = \frac{1}{2\,(\lambda k+\sqrt{k})^2}$,

$C = \frac{1}{6\sqrt{k}(\lambda k+\sqrt{k})^2}$, $D = \frac{\lambda\gamma}{(T+\gamma)(T)(T^2-k)}$, $E = \frac{T^2}{(\sqrt{k}+T)\sqrt{k}}$, $F = \frac{P_r H k}{(\lambda_v k+\sqrt{k})^2(4k-2\sqrt{k}\,T)}$

4 Result and Discussion

Eqs. (3.1) and (3.2) are the new analytical expression of the fluid and temperature for the different values of $M, \lambda, \beta, \lambda_v, P_r, \gamma, H$. Analytical results are compared with the previous result and numerical result in **Figs.** (2) – (8) for different values of parameters. It gives good agreement with the previous result and numerical result.

The velocity profiles are presented in **Figs. (**2) – (4) for the values of M, λ and β. From this **Fig. (**2), it is inferred that when M increases, the velocity decreases. In **Fig. (**3), it is clear that the velocity profile decreases and reaches the steady state value when $\eta = 2$. As β increases, the boundary layer thickness decreases as shown in **Fig. (**4). **Fig. (**5), indicates that the profile of velocity for various values of λ_v. It is noted velocity decreases when λ_v increases. **Figs.** (6), (7) and (8) demonstrates that profile of temperature for various values of P_r, γ and H. From these **Figs.** (6), (7) and (8), it is understood that P_r, γ and H increases when the temperature profile increases.

The analytical expression of velocity and temperature obtained from Homotopy Analysis Method is compared with numerical results in Table 1 and Table 2 satisfactory agreement is noted. The average error percentage between numerical results and analytical results is less than or equal to 0.13.

5 Conclusion

In this paper, theoretical analysis of nonlinear partial differential equations with mixed boundary condition has been investigated. The dimensionless nonlinear differential equations are solved analytical by Homotopy Analysis Method (HAM). Homotopy analysis method is powerful analytical technique for solving other nonlinear differential equations. It is demonstrated that the obtained results are in good agreement with simulation result and previous result.

References

[1] K. Sharada, B. Shankar, MHD mixed convective flow of a casson fluid with convective

boundary condition and effect of partial slip in the presence of joule heating over a vertically stretching sheet, IJIRSET, vol.6, pp.12852 – 12857, 2017.

[2] S. Nadeem, U. I. Haq Rizwana, C. Lee, MHD flow of a casson fluid over an exponentially shrinking sheet, vol.19, no.6, pp.1550 – 1553, 2012.

[3] Krishnendu Bhattacharya, MHD stagnation point flow of casson fluid and heat transfer over a stretching sheet with thermal radiation, Journal of Thermodynamics, 2013, Article Id 169674, 2013.

[4] A. Ishak, R. Nazar, I. Pop, Mixed convection on the stagnation pit flow toward a vertical, continuously stretching sheet, Journal of Heat Transfer,vol.129, no.8, pp.1087 – 1090, 2007.

[5] A. Ishak, R. Nazar, I. Pop, Post stagnation point boundary layer flow and mixed convection heat transfer over a vertical, linearly stretching sheet. Arch. Mech, vol. 60, pp. 303-322, 2008.

[6] A. Ishak, K. Jafar, R. Nazar, Pop, MHD stagnation point flow towards a stretching sheet, Physics A: Statistical Mechanics and its Applications, vol. 388, no. 17, pp. 3377-3383, 2009.

[7] T. Hayat, M. Bilal Ashraf, S. A. Shehzad and A. Alsaedi, Mixed convection flow of Casson nanofluid over a stretching sheet with convectively heated chemical reaction and heat source / sink, Journal of Applied Fluid Mechanics, vol. 8, no. 4, pp. 803 – 813, 2015.

[8] K. Vajravelu, Analysis of MHD boundary layer flow of an upper convected Maxwell fluid with homogeneous heterogeneous chemical ractions, ISPACS, pp.202 – 216, 2017.

[9] V. Ananthaswamy, A. Eswari, L. Rajendran, Non - Linear Reaction - Diffusion Process in a Thin Membrane and Homotopy Analysis Method, International Journal of Automation and Control Engineering Vol.2 Issue 1, pp.10 – 18, 2013

[10] Adebowale M. Obalalu, Heat and mass transfer in an unsteady squeezed Casson fluid flow with novel thermo physical properties: Analytical and numerical solution, Wiley online Library, vol.50, Issue 8, pp. 7988 – 8011, 2021.

[11] Adebowale M. Obalalu, Fatal Adisa Wahaab and Lawal Lanre Adebayo, Heat transfer in an unsteady vertical porous channel with injection / suction in the presence of heat generation, Journal of Taibah University for science, vol. 14, no.1, 541 – 548, 2020.

[12] Adebowale M. Obalalu, Chemical entropy generation and second-order slip condition on hydrodynamic Casson nanofluid flow embedded in a porous medium: a fast convergent method, Journal of the Egyptian Mathematical Society, no1, 2022.

[13] Adebowale M. Obalalu, Lawal Lanre Adebayo, llhami Colak, and Adebayo Olusegun Ajala,

Entropy generation minimization on electromagnetohydrodynamic rediative Casson nanofluid flow over a melting Riga plate, Wiley Online Library, 2022.
[14] S. J. Liao, On the homotopy analysis method for nonlinear problems, Applied Mathematics Computation, vol.147, pp.499 – 513, 2007.

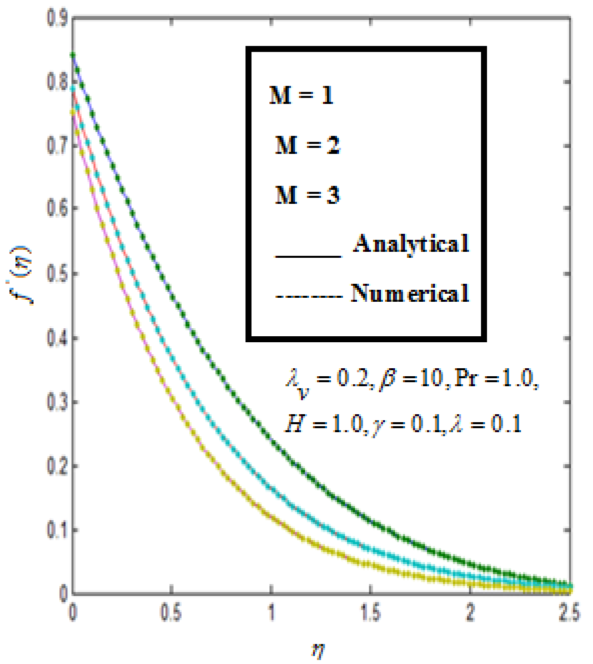

Fig. 2: Velocity profile for various values of M.

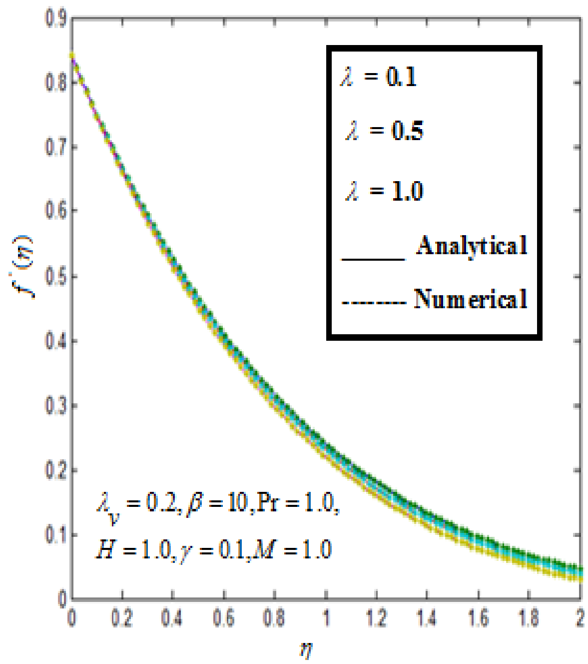

Fig. 3: Velocity profile for various values of λ.

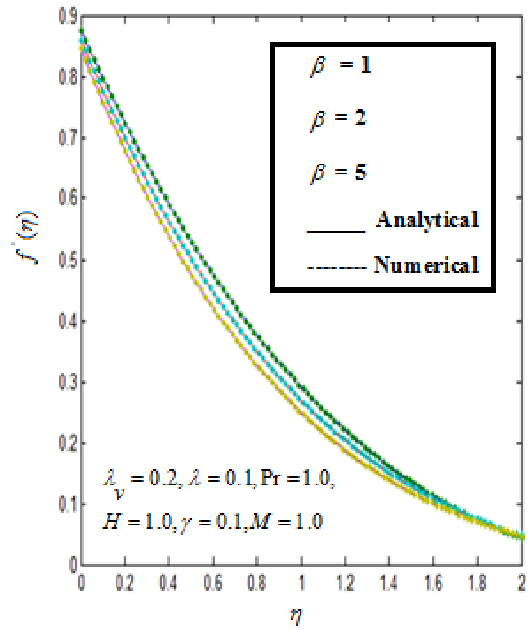

Fig. 4: Velocity profile for various values of β.

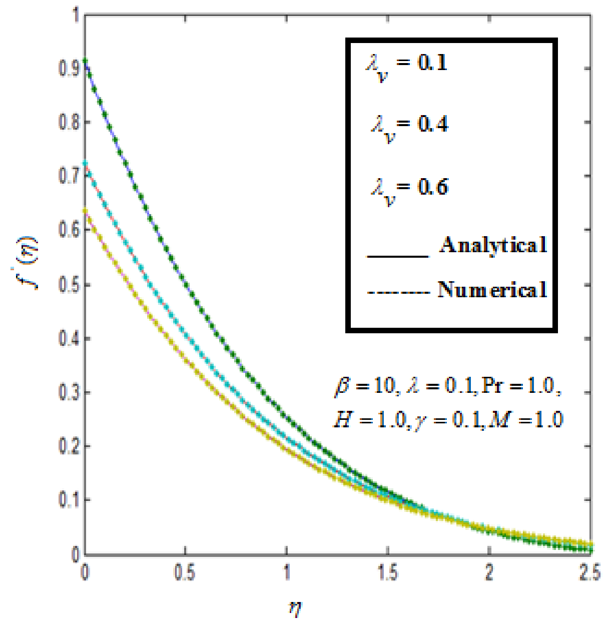

Fig. 5: Velocity profile for various values of λ_v.

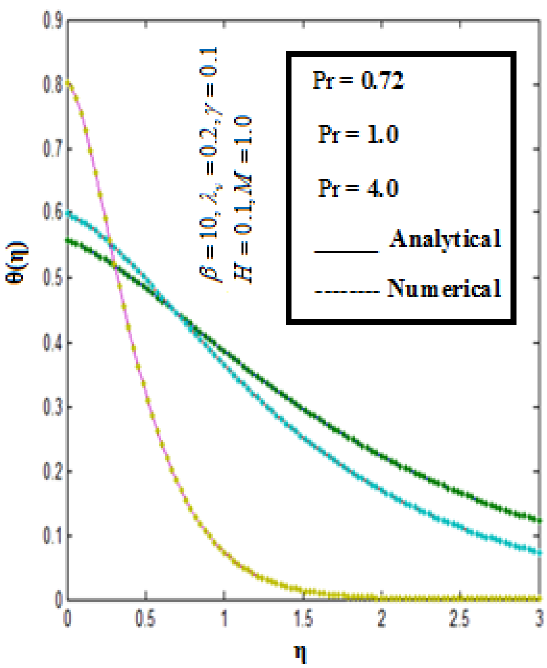

Fig. 6: Temperature profile for various values of Pr.

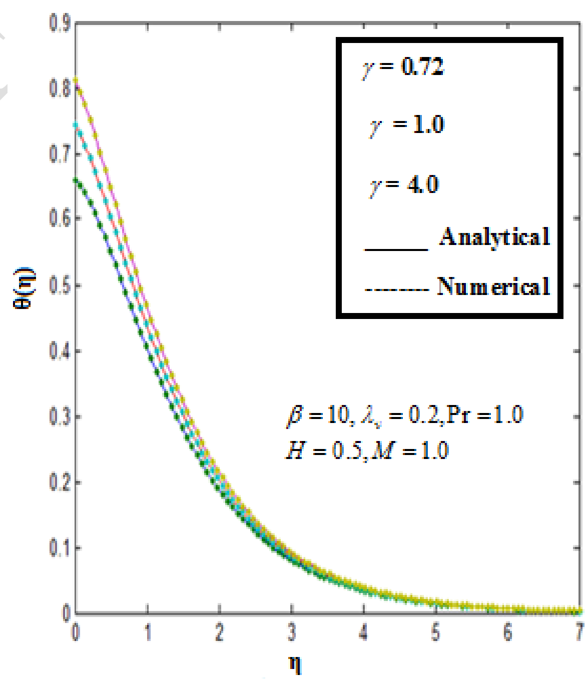

Fig. 7: Temperature profile for various values of γ.

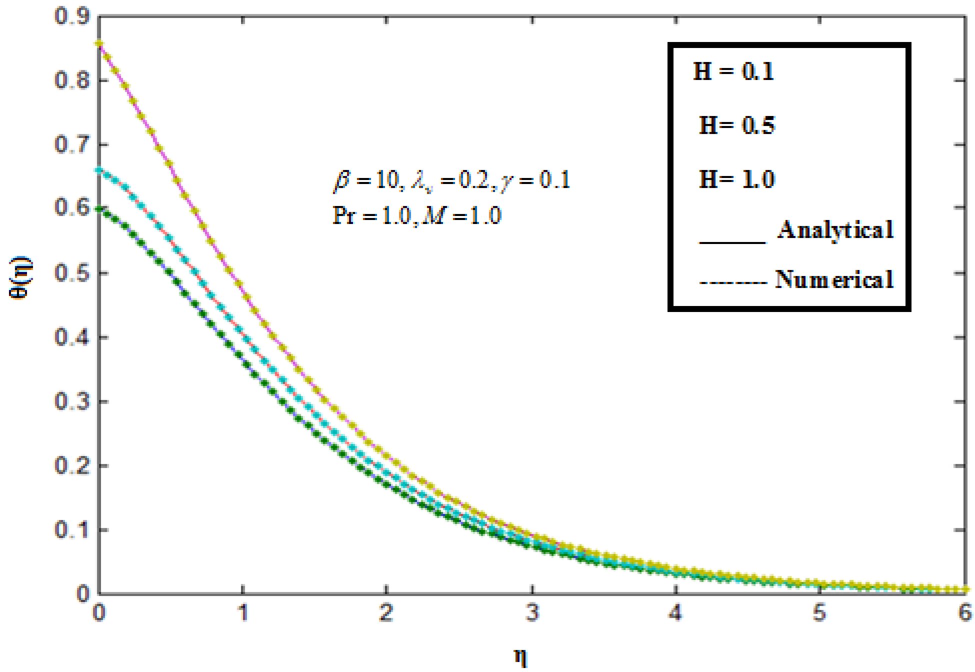

Fig. 8: Temperature profile for various values of H.

Table 1: Comparison of velocity with numerical results for β = 10, Pr = 1.0, H = 1.0, M = 1.0, λ_v = 0.2, γ = 0.1, and λ = 0.5.

η	Analytical (Eqs. 3.1)	Numerical	% of deviation
0	0.8100	0.8112	0.1481
0.2	0.6304	0.6314	0.1586
0.4	0.4699	0.4704	0.1064
0.6	0.3730	0.3132	0.0639
0.8	0.1595	0.1597	0.1254
1.0	0.0000	0.0000	0.0000
Average of % deviation			0.1004

Table 2: Comparison of temperature with numerical results for β = 10, Pr = 1.0, M = 1.0, λ_v = 0.2, γ = 0.1, and H = 1.0.

η	Analytical (Eqs. 3.2)	Numerical	% of deviation
0	0.2115	0.2119	0.1819
0.2	0.1840	0.1843	0.1630
0.4	0.1430	0.1432	0.1399
0.6	0.0937	0.0938	0.1067
0.8	0.0463	0.0464	0.2160
1.0	0.0000	0.0000	0.0000
Average of % deviation			0.1346

List 1 : LIST OF SYMBOLS

Parameters	Description
β	Casson parameter
B	Magnetic
T	Fluid temperature
ν	Kinematic viscosity
α	Thermal diffusivity
ρ	Density of the fluid
σ	Electrical conductivity
μ	Dynamic co-efficient of viscosity
C_p	Heat capacitance

M	Magnetic parameter
λ	Mixed convection parameter
Pr	Prandtl number
Ec	Eckert number
H	Joule heating parameter
L	Proportionality constant
u, v	Velocity components along x and y directions
h_f	Convective heat transfer coefficient
T_f	Temperature of fluid
h_f	Heat transfer co-efficient
λ_v	Slip parameter
γ	Convective parameter
C_{fx} and C_{fy}	Skin friction along the x and y directions
τ_{wx} and τ_{wy}	Wall shear stress along x and y directions
Re_x	Local Reynolds number

Appendix A

Basic concept of the Homotopy Analysis method (HAM) [14]

The general form of linear (or) nonlinear differential equation is of the form

$$N[u\,(x,t)] = 0 \tag{A1}$$

Zero-th order deformation of the above equation is [10]

$$(1-p)L[\psi\,(x,t,p) - u_0\,(x,t)] = ph\,H(x,t)N[\,\psi(x,t,p)] \tag{A2}$$

where $p \in [0,1]$, h ≠ 0.

Substituting $p = 0$ and $p = 1$, in Eq.(A2), we get

$$\psi(x,t,0) = u_0\,,\ \psi(x,t,1) = u \tag{A3}$$

respectively.

By using Taylor's Expansion $\psi(x,t,p)$ can be expanded as

$$\psi(x,t,p) = u_0\,(x,t) + \sum_{m=1}^{\infty} u_m\,(x,t)p^m \tag{A4}$$

where $u_m(x,t) = \frac{1}{m!}\frac{\partial^m \psi(x,t,p)}{\partial p^m}|p=0$ (A5)

When $p = 1$, the convergent series becomes

$$u(x,t) = u_0(x,t) + \sum_{m=1}^{\infty} u_m(x,t)\,, \quad \text{(A6)}$$

Appendix B

Analytical solution of fluid f and temperature θ by solving the Eqs. (2.4) and (2.5) using HAM

$$\left(1+\frac{1}{\beta}\right) f''' + f f'' + \lambda\theta - Mf' = 0 \quad \text{(B1)}$$

$$\frac{1}{P_r}\theta'' + f\theta' + Hf'^2 = 0 \quad \text{(B2)}$$

The homotopy analysis method for the Eq. (B1) and (B2) can be written as follows:

$$(1-p)\left[\left(1+\frac{1}{\beta}\right) f''' - Mf'\right] = ph\left[\left(1+\frac{1}{\beta}\right) f''' + f f'' + \lambda\theta - Mf' = 0\right] \quad \text{(B3)}$$

$$(1-p)[\theta'' + T\theta'] = ph\left[\theta'' + T\theta' - T\,e^{(-\sqrt{k}\,\eta)}\theta' + P_r\,Hf'^2\right] \quad \text{(B4)}$$

The approximate solution of the Eq. (B1) and (B2) are as follows:

$$f = f_0 + f_1 p + f_2 p^2 + \cdots \quad \text{(B5)}$$

$$\theta = \theta_0 + \theta_1 p + \theta_2 p^2 + \cdots \quad \text{(B6)}$$

where p is the embedding parameters and $p \in [0,1]$. Substituting Eqs.(B5) and (B6) in Eqs.(B3) and (B4) and equating the like coefficients of p on both sides we get,

$$p^0: \left(1+\frac{1}{\beta}\right) f_0''' - Mf_0' = 0 \quad \text{(B7)}$$

$$p^1: \left(1+\frac{1}{\beta}\right) f_1''' - Mf_1' - \left(1+\frac{1}{\beta}\right) f_0''' + Mf_0' = h\left[\left(1+\frac{1}{\beta}\right) f_0''' - Mf_0' + f_0 f_0'' + \lambda\,\theta_0\right] \quad \text{(B8)}$$

$$p^0: \theta_0'' + T\theta_0' = 0 \quad \text{(B9)}$$

$$p^1: \theta_1'' + T\theta_1' - \theta_0'' - T\theta_0' = h\left[\theta_0'' + T\theta_0' - Te^{(-\sqrt{k}\,\eta)}\theta_0' + P_r\,H\,f_0'^2\right] \quad \text{(B10)}$$

With the boundary conditions,

$$f_0(0) = 0, f_0'(0) = 1 + \lambda_v f_0''(0),\ f_0'(\infty) = 0 \quad \text{(B11)}$$

$$f_1(0) = 0,\ f_1'(0) = 0,\ f_1'(\infty) = 0 \quad \text{(B12)}$$

$$\theta(\infty) = 0,\ \theta'(0) = -\gamma\,(1 - \theta(0)) \quad \text{(B13)}$$

$$\theta(\infty) = 0,\ \theta'(0) = 0 \quad \text{(B14)}$$

Solving the Eqs.(B7) to (B10) with the boundary conditions Eqs.(B11) to (B14) we get, approximate solution of the Eqs. (3.1) and (3.2) in the text.

Appendix C

Using MATLAB program simulation of Eqs. (2.4) and (2.5).

```
function sol = ex1
solinit=bvpinit (linspace(0,1,9),[0 -1 1 -2.5 1]);
sol = bvp4c(@ex1ode,@ex1bc,solinit);
xint = linspace( 0,1,9);
yint = deval(sol,xint);
plot(xint,yint(2,:));
end
function dydx = ex1ode(x,y)
N=0.1;
p=1.0;
H=1.0;
M=1.0;
A=1.1;
dydx=[y(2)
    y(3)
    1/A*(y(3)*y(1)-(y(4)*N)+ (M*y(2)))
    y(5)
    -p*(y(1)*y(5))-p*(H*(y(2)*y(2)))
    ];
end
function res = ex1bc(ya,yb)
v=0.2;
r=0.1;
res=[
ya(1)
ya(2)-(1+v*ya(3))
yb(2)
ya(5)+r*(1-ya(4))
yb(4)
];
end
```

To be typed in the command window

```
solution=ex5;
x=solution.x;
y=solution.y;
y2=solution.y(2,:);
y4=solution.y(4,:);
Plot(x,y2,'r',x,y4,'g');
```