UNDER PEER REVIEW

Original Research Article

# Some Inequalities via Functional Type Generalization of Cauchy-Bunyakovsky-Schwartz Inequality

## ABSTRACT


In this work, firstly we introduce an inequality of the form

$$[f^{(n)}(x)]^2 \leq k(x) \sum_{k=0}^{m} a_k\, f^{(m-k)} \left(\frac{p}{r}x + q\right) \sum_{k=0}^{l} b_k\, f^{(l-k)} \left(\left(\frac{2}{r} - \frac{p}{r}\right) x - q\right)$$

and by using a functional type generalization of the Cauchy-Bunyakovsky-Schwartz inequality we get some inequalities for derivatives of a one-parameter deformation of the Gamma function to satisfy the introduced inequality. Also, we show that the established results are generalizations of some previous results.



*Keywords: Cauchy-Bunyakovsky-Schwartz inequality; Gamma function; v-Gamma function; inequality*

2020 Mathematics Subject Classification: 26D15; 26D20; 33B15


## 1 INTRODUCTION

The Cauchy-Bunyakovsky-Schwartz inequality is given in [**?** ] as

$$\left(\int_a^b f(t)g(t)dt\right)^2 \leq \int_a^b f^2(t)dt \int_a^b g^2(t)dt \tag{1.1}$$

on the space of continious real valued functions $C[a,b]$. In recent years, many generalizations of the inequality (**??**) have been given, for example, see [**? ? ? ? ?** ]. One of the functional type generalization of the equation (**??**) is given in [**?** ] as

$$\int_a^b F_m(f_1, f_2, \dots, f_m) G_k(g_1, g_2, \dots, g_k)\, dx \leq \left(\int_a^b F_m^2(f_1, f_2, \dots, f_m)\, dx\right)^{\frac{1}{2}} \left(\int_a^b G_k^2(g_1, g_2, \dots, g_k)\, dx\right)^{\frac{1}{2}} \tag{1.2}$$

for $\{f_i\}_{i=1}^m, \{g_j\}_{j=1}^k \in C[a,b]$. A subclass of the inequality (**??**) is when $m = k$ and

$$F_m(f_1, f_2, \dots, f_m) = f_1^{\frac{1+\alpha_1}{2}} f_2^{\frac{1+\alpha_2}{2}} \dots f_m^{\frac{1+\alpha_m}{2}}, \; G_m(g_1, g_2, \dots, g_m) = g_1^{\frac{1-\alpha_1}{2}} g_2^{\frac{1-\alpha_2}{2}} \dots g_m^{\frac{1-\alpha_m}{2}} \tag{1.3}$$

for $\{\alpha_i\}_{i=1}^m \in \mathbb{R}$. In particular, when $m = 2$ and $m = 3$ it gives the following inequalities respectively

$$\left(\int_a^b f(t)g(t)\,dt\right)^2 \leq \int_a^b f^{1+\alpha}(t)g^{1+\beta}(t)\,dt \int_a^b f^{1-\alpha}(t)g^{1-\beta}(t)\,dt, \tag{1.4}$$

$$\left(\int_a^b f(t)g(t)h(t)\,dt\right)^2 \leq \int_a^b f^{1+\alpha}(t)g^{1+\beta}(t)h^{1+\gamma}(t)\,dt \int_a^b f^{1-\alpha}(t)h^{1-\beta}(t)h^{1-\gamma}(t)\,dt \tag{1.5}$$

for $\alpha, \beta, \gamma \in \mathbb{R}$ and $f, g, h$ are real integrable functions such that the integrals in the inequalities (**??**) and (**??**) exist.
In [**?** ], by using the inequality (**??**), the author gives the inequalities for some well-known special functions in order to get new inequalities of the form

$$f^2(x) \leq k(x)\, f(px+q)\, f((2-p)x-q) \quad (p, q \in \mathbb{R},\; k(x) > 0). \tag{1.6}$$

Numerous extensions and deformations of Euler's classical Gamma function are discussed in the literature; see for example, [**? ? ?** ]. A one-parameter deformation of the classical Gamma function, namely $v$-Gamma function, is defined in [**?** ] as

$$\Gamma_v(x) = \int_0^\infty \left(\frac{t}{v}\right)^{\frac{x}{v}-1} e^{-t}\,dt \quad (x, v > 0). \tag{1.7}$$

Some results and inequalities associated with the $v$-Gamma function are presented in [**? ?** ].
Differentiating the equation (**??**) with respect to $x$ we have

$$\Gamma_v^{(n)}(x) = \frac{1}{v^n} \int_0^\infty \left(\frac{t}{v}\right)^{\frac{x}{v}-1} \ln^n\left(\frac{t}{v}\right) e^{-t}\,dt \quad (x, v > 0). \tag{1.8}$$

Note that when $v = 1$, we have $\Gamma_v^{(n)}(x) = \Gamma^{(n)}(x)$ for $n \in \mathbb{N} = \{0, 1, 2, \dots\}$.

In this presented paper we introduce a generalization form of the inequality (**??**) as

$$[f^{(n)}(x)]^2 \leq k(x) \sum_{k=0}^{m} a_k\, f^{(m-k)}\,(px+q) \sum_{k=0}^{l} b_k\, f^{(l-k)}\,((2-p)\,x-q) \tag{1.9}$$

for $l, m, n \in \mathbb{N}$, $p, q, a_k, b_k \in \mathbb{R}$ and $k(x) > 0$, and show that the inequalities we obtained are satisfied the inequality (**??**).

## 2 MAİN RESULTS

In this section, we prove some inequalities which involve the derivatives of the $v$-Gamma function by using the inequalities (**??**) and (**??**).

**Theorem 2.1.** *Let $x, v > 0$. Then the inequality*

$$[\Gamma_v^{(n)}(x)]^2 \leq \Gamma_v^{(n)}(x + \alpha x - \alpha v)\Gamma_v^{(n)}(x - \alpha x + \alpha v) \tag{2.1}$$

*is valid for* $x + \alpha x - \alpha v > 0$, $x - \alpha x + \alpha v > 0$, $n \in 2\mathbb{N}$,
*and the inequality*

$$[\Gamma_v^{(n)}(x)]^2 \leq \frac{1}{(1+\beta)^{\frac{x}{v}+\frac{\alpha x}{v}-\alpha}(1-\beta)^{\frac{x}{v}-\frac{\alpha x}{v}+\alpha}} \sum_{k=0}^{n(1+\beta)} (-1)^k v^{-k} \binom{n(1+\beta)}{k} \ln^k(1+\beta)$$
$$\times \Gamma_v^{((n(1+\beta)-k)} (x+\alpha x - \alpha v) \sum_{k=0}^{n(1-\beta)} (-1)^k v^{-k} \binom{n(1-\beta)}{k} \ln^k(1-\beta)$$
$$\times \Gamma_v^{(n(1-\beta)-k)} (x - \alpha x + \alpha v) \tag{2.2}$$

*is valid for* $x+\alpha x-\alpha v > 0$, $x-\alpha x+\alpha v > 0$ *and some* $\beta \in (-1,1)/\{0\}$ *such that* $n(1+\beta),\ n(1-\beta) \in 2\mathbb{N}$.

*Proof.* By substituting $[a,b] = [0,\infty)$, $f(t) = \left(\frac{t}{v}\right)^{\frac{x}{v}-1}$, $g(t) = \ln^n\left(\frac{t}{v}\right)e^{-t}$ in the inequality (**??**) we have

$$\left(\int_0^\infty \left(\frac{t}{v}\right)^{\frac{x}{v}-1} \ln^n\left(\frac{t}{v}\right) e^{-t}\,dt\right)^2 \leq \int_0^\infty \left(\frac{t}{v}\right)^{\left(\frac{x}{v}-1\right)(1+\alpha)} \left(\ln^n\left(\frac{t}{v}\right)e^{-t}\right)^{1+\beta} dt$$
$$\times \int_0^\infty \left(\frac{t}{v}\right)^{\left(\frac{x}{v}-1\right)(1-\alpha)} \left(\ln^n\left(\frac{t}{v}\right)e^{-t}\right)^{1-\beta} dt. \tag{2.3}$$

For simplicity let

$$I_1 = \int_0^\infty \left(\frac{t}{v}\right)^{\frac{x}{v}-1} \ln^n\left(\frac{t}{v}\right) e^{-t}\,dt,$$
$$I_2 = \int_0^\infty \left(\frac{t}{v}\right)^{\left(\frac{x}{v}-1\right)(1+\alpha)} \ln^{n(1+\beta)}\left(\frac{t}{v}\right) e^{-t(1+\beta)}\,dt,$$

and

$$I_3 = \int_0^\infty \left(\frac{t}{v}\right)^{\left(\frac{x}{v}-1\right)(1-\alpha)} \ln^{n(1-\beta)}\left(\frac{t}{v}\right) e^{-t(1-\beta)}\,dt.$$

If $\beta = 0$ we have

$$I_1 = v^n\Gamma_v^{(n)}(x), \quad I_2 = v^n\Gamma_v^{(n)}(x+\alpha x-\alpha v), \quad I_3 = v^n\Gamma_v^{(n)}(x-\alpha x+\alpha v), \tag{2.4}$$

for $x+\alpha x-\alpha v > 0$, $x-\alpha x+\alpha v > 0$, and the inequality (**??**) follows for $n \in 2\mathbb{N}$.
Now, for the inequality (**??**) let $t(1+\beta) = u$ and $\beta \neq 0$ in $I_2$. Then we get

$$I_2 = \int_0^\infty \left(\frac{u}{(1+\beta)v}\right)^{\frac{x}{v}+\frac{\alpha x}{v}-\alpha-1} \ln^{n(1+\beta)}\left(\frac{u}{(1+\beta)v}\right) e^{-u}\,\frac{du}{1+\beta}$$
$$= \left(\frac{1}{1+\beta}\right)^{\frac{x}{v}+\frac{\alpha x}{v}-\alpha} \sum_{k=0}^{n(1+\beta)} (-1)^k \binom{n(1+\beta)}{k} \ln^k(1+\beta)$$
$$\times \int_0^\infty \left(\frac{u}{v}\right)^{\frac{x}{v}+\frac{\alpha x}{v}-\alpha-1} \ln^{n(1+\beta)-k}\left(\frac{u}{v}\right) e^{-u}\,du.$$

By using the equation (**??**) we have

$$I_2 = \left(\frac{1}{1+\beta}\right)^{\frac{x}{v}+\frac{\alpha x}{v}-\alpha} \sum_{k=0}^{n(1+\beta)} (-1)^k \binom{n(1+\beta)}{k} \ln^k(1+\beta)$$
$$\times v^{n(1+\beta)-k}\Gamma_v^{((n(1+\beta)-k)} (x+\alpha x-\alpha v) \tag{2.5}$$

for $x+\alpha x-\alpha v>0$, $\beta>-1$ and $n(1+\beta)\in\mathbb{N}$.
Similarly, let $t(1-\beta)=y$ and $\beta\neq 0$ in $I_3$. Then

$$\begin{aligned} I_3 &= \int_0^\infty \left(\frac{y}{(1-\beta)v}\right)^{\frac{x}{v}-\frac{\alpha x}{v}+\alpha-1} \ln^{n(1-\beta)}\left(\frac{y}{(1-\beta)v}\right) e^{-y}\frac{dy}{1-\beta} \\ &= \left(\frac{1}{1-\beta}\right)^{\frac{x}{v}-\frac{\alpha x}{v}+\alpha} \sum_{k=0}^{n(1-\beta)} (-1)^k \binom{n(1-\beta)}{k} \ln^k(1-\beta) \\ &\qquad \times \int_0^\infty \left(\frac{y}{v}\right)^{\frac{x}{v}-\frac{\alpha x}{v}+\alpha-1} \ln^{n(1-\beta)-k}\left(\frac{y}{v}\right) e^{-y}\,dy. \end{aligned}$$

By using the equation (**??**) we get

$$\begin{aligned} I_3 = \left(\frac{1}{1-\beta}\right)^{\frac{x}{v}-\frac{\alpha x}{v}+\alpha} \sum_{k=0}^{n(1-\beta)} (-1)^k \binom{n(1-\beta)}{k} \ln^k(1-\beta) \\ \times v^{n(1-\beta)-k}\Gamma_v^{(n(1-\beta)-k)}(x-\alpha x+\alpha v) \end{aligned} \tag{2.6}$$

for $x-\alpha x+\alpha v>0$, $\beta<1$ and $n(1-\beta)\in\mathbb{N}$.
Hence by using the equations (**??**), (**??**) and (**??**) and taking $n(1+\beta)\in 2\mathbb{N}$, $n(1-\beta)\in 2\mathbb{N}$ to guarantee the positivity of the right-hand side of the inequality (2.3), we get the desired result (**??**). □

**Remark 2.2.** *The inequality (**??**) satisfy the inequality (**??**) for $p=1+\alpha$, $q=-\alpha v$, $k(x)=1$ and $f=\Gamma_v^{(n)}$.*

**Remark 2.3.** *The inequality (**??**) satisfy the generic form (**??**) for*

$$p=1+\alpha,\ \ q=-\alpha v,\ \ k(x)=\frac{1}{(1+\beta)^{\frac{x}{v}+\frac{\alpha x}{v}-\alpha}(1-\beta)^{\frac{x}{v}-\frac{\alpha x}{v}+\alpha}},$$

$$a_k=(-1)^k v^{-k}\binom{n(1+\beta)}{k}\ln^k(1+\beta),\ \ b_k=(-1)^k v^{-k}\binom{n(1-\beta)}{k}\ln^k(1-\beta)\ \ and\ \ f=\Gamma_v.$$

**Example 2.1.** *Let $n=2$ and $\alpha=\frac{1}{2}$ in the inequality (**??**). Then we get*

$$[\Gamma_v^{''}(x)]^2 \le \Gamma_v^{''}(\frac{3x}{2}-\frac{v}{2})\Gamma_v^{''}(\frac{x}{2}+\frac{v}{2})$$

*for $v>0$ and $x>\frac{v}{3}$.*

**Example 2.2.** *By taking $v=1$, $n=3$, $\alpha=\frac{1}{2}$ and $\beta=\frac{1}{3}$ in the inequaliy (**??**), we get*

$$\begin{aligned} [\Gamma^{'''}(x)]^2 \le 2^{\frac{1}{2}-\frac{7x}{2}}\,3^{2x} \sum_{k=0}^{4}(-1)^k\binom{4}{k}\ln^k(\frac{4}{3})\Gamma^{(4-k)}\left(\frac{3x}{2}-\frac{1}{2}\right) \\ \times \sum_{k=0}^{2}(-1)^k\binom{2}{k}\ln^k(\frac{2}{3})\Gamma^{(2-k)}\left(\frac{x}{2}+\frac{1}{2}\right) \end{aligned} \tag{2.7}$$

*for $x>0$.*

**Corollary 2.4.** *By taking $v = 1$ and $n = 0$ in the inequality (**??**) we get inequality for the Gamma function*

$$[\Gamma(x)]^2 \le \frac{1}{(1+\beta)^{x+\alpha x-\alpha}(1-\beta)^{x-\alpha x+\alpha}}\Gamma\left(x+\alpha x-\alpha\right)\Gamma\left(x-\alpha x+\alpha\right) \tag{2.8}$$

*for $x > 0$, $x+\alpha x-\alpha > 0$, $x-\alpha x+\alpha > 0$ and $\beta \in (-1,1)$ given in [**?** ].*

Now we give the following theorem as an application of the inequality (**??**).

**Theorem 2.5.** *Let $x, v > 0$. Then the inequality*

$$[\Gamma_v^{(n)}(x)]^2 \le \Gamma_v^{(n)}(x+\alpha x-\beta v)\Gamma_v^{(n)}(x-\alpha x+\beta v) \tag{2.9}$$

*is valid for $x+\alpha x-\beta v > 0$, $x-\alpha x+\beta v > 0$, $n \in 2\mathbb{N}$,*
*and the inequality*

$$\begin{aligned}[\Gamma_v^{(n)}(x)]^2 \le{}& \frac{1}{(1+\gamma)^{\frac{x}{v}+\frac{\alpha x}{v}-\beta}(1-\gamma)^{\frac{x}{v}-\frac{\alpha x}{v}+\beta}}\sum_{k=0}^{n(1+\gamma)}(-1)^k v^{-k}\binom{n(1+\gamma)}{k}\ln^k(1+\gamma)\\ &\times \Gamma_v^{((n(1+\gamma)-k)}\left(x+\alpha x-\beta v\right)\sum_{k=0}^{n(1-\gamma)}(-1)^k v^{-k}\binom{n(1-\gamma)}{k}\ln^k(1-\gamma)\\ &\times \Gamma_v^{(n(1-\gamma)-k)}\left(x-\alpha x+\beta v\right)\end{aligned} \tag{2.10}$$

*is valid for $x+\alpha x-\beta v > 0$, $x-\alpha x+\beta v > 0$ and some $\gamma \in (-1,1)/\{0\}$ such that $n(1+\gamma)$, $n(1-\gamma) \in 2\mathbb{N}$.*

By substituting $[a,b] = [0,\infty)$, $f(t) = \left(\frac{t}{v}\right)^{\frac{x}{v}}$, $g(t) = \left(\frac{t}{v}\right)^{-1}$ $h(t) = \ln^n\left(\frac{t}{v}\right)e^{-t}$ in the inequality (**??**) we have

$$\left(\int_0^\infty \left(\frac{t}{v}\right)^{\frac{x}{v}-1}\ln^n\left(\frac{t}{v}\right)e^{-t}\,dt\right)^2 \le \int_0^\infty \left(\frac{t}{v}\right)^{\frac{x(1+\alpha)}{v}}\left(\frac{t}{v}\right)^{-1(1+\beta)}\left(\ln^n\left(\frac{t}{v}\right)e^{-t}\right)^{1+\gamma}dt \tag{2.11}$$

$$\times\int_0^\infty \left(\frac{t}{v}\right)^{\frac{x(1-\alpha)}{v}}\left(\frac{t}{v}\right)^{-(1-\beta)}\left(\ln^n\left(\frac{t}{v}\right)e^{-t}\right)^{1-\gamma}dt. \tag{2.12}$$

Again, for simplicity let

$$J_1 = \int_0^\infty \left(\frac{t}{v}\right)^{\frac{x}{v}-1}\ln^n\left(\frac{t}{v}\right)e^{-t}\,dt,$$

$$J_2 = \int_0^\infty \left(\frac{t}{v}\right)^{\frac{x(1+\alpha)}{v}}\left(\frac{t}{v}\right)^{-1(1+\beta)}\ln^{n(1+\gamma)}\left(\frac{t}{v}\right)e^{-t(1+\gamma)}\,dt$$

and

$$J_3 = \int_0^\infty \left(\frac{t}{v}\right)^{\frac{x(1-\alpha)}{v}}\left(\frac{t}{v}\right)^{-(1-\beta)}\ln^{n(1-\gamma)}\left(\frac{t}{v}\right)e^{-t(1-\gamma)}\,dt.$$

If $\gamma = 0$ we have

$$J_1 = v^n\Gamma_v^{(n)}(x), \quad J_2 = v^n\Gamma_v^{(n)}(x+\alpha x-\beta v), \quad J_3 = v^n\Gamma_v^{(n)}(x-\alpha x+\beta v), \tag{2.13}$$

for $x+\alpha x-\beta v>0$, $x-\alpha x+\beta v>0$, and the inequality (**??**) follows for $n\in 2\mathbb{N}$.
Now, for the inequality (**??**) let $t(1+\gamma)=u$ and $\gamma\neq 0$ in $J_2$. Then we get

$$\begin{aligned} J_2 &= \int_0^\infty \left(\frac{u}{(1+\gamma)v}\right)^{\frac{x(1+\alpha)}{v}-\beta-1} \ln^{n(1+\gamma)}\left(\frac{u}{(1+\gamma)v}\right) e^{-u}\,\frac{du}{1+\gamma} \\ &= \left(\frac{1}{1+\gamma}\right)^{\frac{x}{v}+\frac{\alpha x}{v}-\beta} \sum_{k=0}^{n(1+\gamma)} (-1)^k \binom{n(1+\gamma)}{k} \ln^k(1+\gamma) \\ &\quad \times \int_0^\infty \left(\frac{u}{v}\right)^{\frac{x}{v}+\frac{\alpha x}{v}-\beta-1} \ln^{n(1+\gamma)-k}\left(\frac{u}{v}\right) e^{-u}\,du. \end{aligned}$$

By using the equation (**??**) we get

$$\begin{aligned} J_2 &= \left(\frac{1}{1+\gamma}\right)^{\frac{x}{v}+\frac{\alpha x}{v}-\beta} \sum_{k=0}^{n(1+\gamma)} (-1)^k \binom{n(1+\gamma)}{k} \ln^k(1+\gamma) \\ &\quad \times v^{n(1+\gamma)-k}\Gamma_v^{((n(1+\gamma)-k)}(x+\alpha x-\beta v) \end{aligned} \tag{2.14}$$

for $x+\alpha x-\beta v>0$, $\gamma>-1$ and $n(1+\gamma)\in\mathbb{N}$.
For the integral $J_3$ let $t(1-\gamma)=y$ and $\gamma\neq 0$. Then

$$\begin{aligned} J_3 &= \int_0^\infty \left(\frac{y}{(1-\gamma)v}\right)^{\frac{x}{v}-\frac{\alpha x}{v}+\beta-1} \ln^{n(1-\gamma)}\left(\frac{y}{(1-\gamma)v}\right) e^{-y}\,\frac{dy}{1-\gamma} \\ &= \left(\frac{1}{1-\gamma}\right)^{\frac{x}{v}-\frac{\alpha x}{v}+\beta} \sum_{k=0}^{n(1-\gamma)} (-1)^k \binom{n(1-\gamma)}{k} \ln^k(1-\gamma) \\ &\quad \times \int_0^\infty \left(\frac{y}{v}\right)^{\frac{x}{v}-\frac{\alpha x}{v}+\beta-1} \ln^{n(1-\gamma)-k}\left(\frac{y}{v}\right) e^{-y}\,dy \\ &= \left(\frac{1}{1-\gamma}\right)^{\frac{x}{v}-\frac{\alpha x}{v}+\beta} \sum_{k=0}^{n(1-\gamma)} (-1)^k \binom{n(1-\gamma)}{k} \ln^k(1-\gamma) \\ &\quad \times v^{n(1-\gamma)-k}\Gamma_v^{(n(1-\gamma)-k)}(x-\alpha x+\beta v) \end{aligned} \tag{2.15}$$

for $x-\alpha x+\beta v>0$, $\gamma<1$ and $n(1-\gamma)\in\mathbb{N}$.
Hence by using the equations (**??**), (**??**) and (**??**) and taking $n(1+\gamma),\, n(1-\gamma)\in 2\mathbb{N}$, the inequality (**??**) follows.

**Remark 2.6.** *The inequality (**??**) satisfy the inequality (**??**) for $p=1+\alpha$, $q=-\beta v$, $k(x)=1$ and $f=\Gamma_v^{(n)}$.*

**Remark 2.7.** *The inequality (**??**) is a special case of the main inequality (**??**) for*

$$p=1+\alpha,\;\; q=-\beta v\;\;,k(x)=\frac{1}{(1+\gamma)^{\frac{x}{v}+\frac{\alpha x}{v}-\beta}(1-\gamma)^{\frac{x}{v}-\frac{\alpha x}{v}+\beta}},$$

$$a_k=(-1)^k v^{-k}\binom{n(1+\gamma)}{k}\ln^k(1+\gamma),\;\; b_k=(-1)^k v^{-k}\binom{n(1-\gamma)}{k}\ln^k(1-\gamma)\;\; and\;\; f=\Gamma_v.$$

**Corollary 2.8.** *By taking $v=1$ in the inequality (**??**) we get the following inequality*

$$[\Gamma^{(n)}(x)]^2 \le \Gamma^{(n)}(x+\alpha x-\beta)\Gamma^{(n)}(x-\alpha x+\beta)$$

*for $x>0$, $x+\alpha x-\beta>0$, $x-\alpha x+\beta>0$, $n\in 2\mathbb{N}$.*

**Corollary 2.9.** *By taking $v = 1$ in the inequality (**??**) we get*

$$[\Gamma^{(n)}(x)]^2 \leq \frac{1}{(1+\gamma)^{x+\alpha x-\beta}(1-\gamma)^{x-\alpha x+\beta}} \sum_{k=0}^{n(1+\gamma)} (-1)^k \binom{n(1+\gamma)}{k} \ln^k(1+\gamma) \times \Gamma^{((n(1+\gamma)-k)}(x+\alpha x-\beta) \times \sum_{k=0}^{n(1-\gamma)} (-1)^k \binom{n(1-\gamma)}{k} \ln^k(1-\gamma)\Gamma^{(n(1-\gamma)-k)}(x-\alpha x+\beta) \quad (2.16)$$

*for $x > 0$, $x + \alpha x - \beta > 0$, $x - \alpha x + \beta > 0$, $\gamma \in (-1, 1)/\{0\}$ and $n(1+\gamma)$, $n(1-\gamma) \in 2\mathbb{N}$.*

# 3 CONCLUSIONS

In this work, based on the Cauchy-Bunyakovsky-Schwartz inequality, we introduced an inequality. By getting some new inequalities, we showed that a one-parameter deformation of the Gamma function satisfies this type of inequality. We also show that the established results are generalizations of some previous ones.

# REFERENCES


[1] Alzer, H. (1992). A refinement of the Cauchy-Schwarz inequality, Journal of Mathematical Analysis and Applications, 168(2):596-604.

[2] Alzer, H. (1999). On the Cauchy-Schwarz Inequality, Journal of Mathematical Analysis and Applications, 234(1):6-14.

[3] Díaz, R., Teruel, C. (2005). q, k-Generalized gamma and beta functions, Journal of Nonlinear Mathematical Physics, 12(1), 118-134.

[4] Ege, I. (2022). Some Results on the v-Analogue of Gamma Function, Earthline Journal of Mathematical Sciences, 10(1):109-123.

[5] Ege, İ. (2023). On Inequalities for the Ratio of v-Gamma and v-Polygamma Functions, Earthline Journal of Mathematical Sciences, 13(1):121-131.

[6] Djabang, E., Nantomah, K., Iddrisu, M. M. (2020). On a v-Analogue of the Gamma Function and Some Associated Inequalities, Journal of Mathematical and Computational Science, 11(1):74-86.

[7] Dragomir, S. S. (2003). A survey on Cauchy-Bunyakovsky-Schwarz Type Discrete Inequalities, J. Inequal. Pure Appl. Math., 4(3):1-142.

[8] Kokologiannaki, C. G., Krasniqi, V. (2013). Some properties of the k-gamma function, Le Matematiche, 68(1):13-22.

[9] Masjed-Jamei, M. (2009). A Functional Generalization of the Cauchy–Schwarz Inequality and Some Subclasses, Applied Mathematics Letters, 22(9):1335-1339.

[10] Masjed-Jamei, M. (2010). A Main Inequality for Several Special Functions, Computers and Mathematics with Applications, 60(5):1280-1289.

[11] Nantomah, K., Ege, I. (2022). A Lambda Analogue of the Gamma Function and its Properties, Researches in Mathematics, 30(2), 18-29.

[12] Mitrinovic D. S., Pecaric J. E., Fink A. M. (1993). Classical and New Inequalities in Analysis, Kluwer Academic Publishers, Dordrecht, Boston, London.

[13] Steiger, W. L. (1969). On a Generalization of the Cauchy-Schwarz Inequality, The American Mathematical Monthly, 76(7):815-816.

[14] Zheng, L. (1998). Remark on a Refinement of the Cauchy-Schwarz Inequality, Journal of Mathematical Analysis and Applications, 218(1):13-21.