A Comprehensive Analysis of Stability and Data Dependency in a Novel Jungck-Type Iteration Algorithm

Abstract

This study introduces a novel Jungck-type iterative algorithm for approximating coincidence points under specific contractive conditions. The research demonstrates the algorithm's strong convergence, stability, and data dependency through rigorous theoretical analysis and numerical experiments. Results indicate that the proposed method achieves a significantly faster convergence rate compared to existing Jungck-type iterations. These findings have practical implications in fields such as optimization, economic modeling, and coupled differential equations, where iterative techniques are vital.

Keywords: Data Dependence, Common Fixed Point, Coincidence Point, and Stability.

MSC: 47H09, 47H10, 54H25

1. Introduction and Preliminaries

The Jungck-type fixed-point iteration extends classical fixed-point theory to address scenarios involving multiple mappings with defined interrelations. This approach demonstrates particular efficacy in addressing problems in which the interaction between the two operators plays a crucial role, ensuring convergence to a common fixed point under specific contractive conditions.

By expanding the scope of fixed-point theory to encompass more complex and hybrid systems, Jungck-type iterations have demonstrated significant applicability in domains such as optimization, economic modeling, and coupled differential equations. The exploration of numerical solutions for differential equations and their equivalent integral equations through the lens of fixed point iterations has become a focal point for many scholars in the field. (Karapinar et al., 2022, Adeyemi et al., 2021). Various problems have utilized fractals, such as in studying the structure of river networks.(Martinez et al., 2022), image encryption (Zhang et al., 2019), and the generation of art (Ouyang et al., 2021; Guran et al., 2023).

Recently, substantial advancements have been achieved in both the theoretical analysis and numerical exploration of various explicit iterative techniques (Berinde, 2004; Olatinwo, 2008; Khan et al., 2014; Olatinwo and Postolache, 2012).

Consider $(W,\|.\|)$ be a Banach space, V be an arbitrary set, $J,H : V \to W$ be arbitrary non-self mapping with $H(V) \subseteq J(V)$, $J(V)$ is a complete subspace of W, and $a_0 \in V$.
For $\alpha_n \in [0,1]$, (Singh et al., 2005) defined the Jungck-Mann iterative scheme as follows:

$$Ja_{n+1} = (1-\alpha_n)Ja_n + \alpha_n Ha_n. \tag{1.1}$$

(Olatinwo, 2008) defined the Jungck-Noor (J-Itr1) iteration scheme as

$$\begin{cases} Jc_n = \left(1-\mu_n\right) Ja_n + \mu_n Ha_n, \\ Jb_n = \left(1-\delta_n\right) Ja_n + \delta_n Hc_n, \\ Ja_{n+1} = \left(1-\lambda_n\right) Ja_n + \lambda_n Hb_n, \end{cases} \tag{1.2}$$

where $\{\mu_n\}_{n=0}^{\infty}, \{\delta_n\}_{n=0}^{\infty}, \{\lambda_n\}_{n=0}^{\infty} \subseteq [0,1]$.
A new three-step iteration process, known as the Jungck-Khan (J-Itr2) method, was proposed by (Khan et al., 2014) as follows

$$\begin{cases} Jc_n = \left(1-\mu_n\right) Ja_n + \mu_n Ha_n, \\ Jb_n = \left(1-\delta_n-\lambda_n\right) Ja_n + \delta_n Hc_n + \lambda_n Ha_n, \\ Ja_{n+1} = \left(1-\theta_n-\gamma_n\right) Ja_n + \theta_n Hb_n + \gamma_n Ha_n, \end{cases} \tag{1.3}$$

where $\{\mu_n\}_{n=0}^{\infty}, \{\delta_n\}_{n=0}^{\infty}, \{\lambda_n\}_{n=0}^{\infty}, \{\theta_n\}_{n=0}^{\infty}$, and $\{\gamma_n\}_{n=0}^{\infty} \subseteq [0,1]$, satisfying $\{\delta_n\}_{n=0}^{\infty} + \{\lambda_n\}_{n=0}^{\infty}$, $\{\theta_n\}_{n=0}^{\infty} + \{\gamma_n\}_{n=0}^{\infty} \subseteq [0,1]$.

To demonstrate the strong convergence of both the Jungck-Mann and Jungck-Ishikawa iterative process, (Olatinwo and Imoru, 2008) introduced the following contractive definition

$$\|Hu - Hv\| \le 2\beta\|Ju - Hu\| + \beta\|Ju - Jv\| \quad \forall u, v \in V, \quad 0 \le \beta < 1. \tag{1.4}$$

(Olatinwo, 2008), building on result (1.4), Olatinwo demonstrated the stability and strong convergence of various iterative techniques. This was achieved by employing a more comprehensive contractive condition, which is represented as

$$\|Hu - Hv\| \le \psi\left(\|Ju - Hu\|\right) + \beta\|Ju - Jv\| \quad \forall u, v \in V, \quad 0 \le \beta < 1, \tag{1.5}$$

where the monotonically increasing function $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ satisfying $\psi(0) = 0$.

Definition 1 (Jungck, 1976). Consider V a non-empty set and $J, H: V \to W$ be two mappings. A coincidence point exists when $J(v) = f = H(v)$ for some v in V, and the associated value f is referred to as the point of coincidence or coincidence value of J and H. If $J(v) = v = H(v)$ for an element v in V then v is called the common fixed point of J and H. The pair (J, H) commutes at the coincidence point, and is said to be weakly compatible.

Definition 2 (Olatinwo and Postolache, 2012). Consider the operators $J, H : V \to W$ such that $H(V) \subseteq J(V)$ and $Jv = f = Hv$, where f is a point of coincidence of J and H. Suppose $a_0 \in V$ is the initial approximation, g is some function, and $\{Ja_n\}_{n=0}^{\infty} \subset W$, be the sequence converges to *f*, generated by an iterative procedure

$$Ja_{n+1} = g(H, a_n) \ , n = 0, 1, 2, \ . \ . \ .$$

Let $\{Jh_n\}_{n=0}^{\infty} \subset W$ be an arbitrary sequence. Set

$$\rho_n = \left\| Jh_{n+1} - g\left(\mathrm{H}, \mathrm{h}_n\right) \right\|, \quad n = 0, 1, 2, \ldots .$$

Then, the iterative procedure $Ja_{n+1} = g(H, a_n)$ is said to be stable if and only if $\lim_{n\to\infty} \rho_n = 0$ implies $\lim_{n\to\infty} Jh_n = f$.

Lemma 1 (Berinde, 2007). If $\beta \in [0,1)$ and sequence of positive numbers $\{\omega_n\}_{n=0}^{\infty}$ with $\lim_{n\to\infty} \omega_n = 0$, then for every sequence of positive numbers $\{u_n\}_{n=0}^{\infty}$, which satisfies

$$u_{n+1} \le \beta u_n + \omega_n, \quad n = 0,1,2, \ldots$$

one has $\lim_{n\to\infty} u_n = 0$.

Definition 3 (Khan et al., 2014). Assume an arbitrary set V and the non-self mapping pairs $(J, H), (J_1, H_1): V \to W$ with $H(V) \subseteq J(V)$ and $H_1(V) \subseteq J_1(V)$. If for fixed $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$, and for all $v \in V$, one gets

$$m(Hv, H_1 v) \le \varepsilon_1,$$
$$m(Jv, J_1 v) \le \varepsilon_2,$$

then, (J_1, H_1) is said to be an approximate mapping pair of (J, H).

This paper presents a novel Jungck-type iteration algorithm to determine the coincidence points of contractive-type mappings. The proposed method exhibits an enhanced convergence rate and stability compared to existing Jungck-type iteration approaches. Additionally, we derived the data dependence findings for our newly introduced process.

We now define our novel Jungck-type iteration (New-Itr) scheme, as follows:
For $a_0 \in V$, the sequence $\{Ja_n\}_{n=0}^{\infty}$ in W is given by

$$\begin{cases} Jd_n = \dfrac{s}{s+1} Ja_n + \dfrac{1}{s+1} Ha_n, \\ Jc_n = \dfrac{s+1}{s} Ha_n - \dfrac{1}{s} Hd_n, \\ Jb_n = \dfrac{s'}{s'+1} Jc_n + \dfrac{1}{s'+1} Hc_n, \\ Ja_{n+1} = Hb_n \end{cases} \tag{1.6}$$

where, $s \ge 1$ and $s' > 0$ are real numbers.

Remark. If we consider $V = W$, and J is the identity operator in (1.6), then we get the following iteration outlined in (Sharma et al., 2024, equation(18)):

$$\begin{cases} b_n = \dfrac{1}{s}\left((s+1) Ha_n - H\left(\dfrac{sa_n + Ha_n}{s+1} \right) \right), \\ a_{n+1} = H\left(\dfrac{s' b_n + Hb_n}{s'+1} \right), \end{cases} \tag{1.7}$$

where $s \ge 1$ and $s' > 0$ are real numbers.

2. Main results

Theorem 2.1. Consider $(W,\|.\|)$ a Banach space and V be an arbitrary set. Suppose $J,H:V\to W$ be non-self mappings with $H(V)\subseteq J(V)$, and $J(V)$ is a complete subspace of W. Suppose J and H have a coincidence point v, (that is, $Jv=f=Hv$) and also J,H satisfies the contractive condition (1.5) with $\beta<\frac{1}{\sqrt{3}}$. Let $\{Ja_n\}_{n=0}^{\infty}$ be the new iteration process defined by (1.6). Then, $\{Ja_n\}_{n=0}^{\infty}$ is strongly converges to f. Moreover, (J,H) has a unique common fixed point f provided that $V=W$ and H,J are weakly compatible.

Proof. We shall prove that $\lim_{n\to\infty} Ja_n=f$. Using (1.6), one has

$$\begin{aligned}
\|Jd_n-f\| &= \left\|\frac{s}{s+1}Ja_n+\frac{1}{s+1}Ha_n-f\right\|,\\
&\le \frac{s}{s+1}\|Ja_n-f\|+\frac{1}{s+1}\|Ha_n-Hv\|,\\
&\le \frac{s}{s+1}\|Ja_n-f\|+\frac{1}{s+1}\left[\psi\left(\|Jv-Hv\|\right)+\beta\|Ja_n-Jv\|\right],\\
&= \frac{s+\beta}{s+1}\|Ja_n-f\|. \qquad (2.1)
\end{aligned}$$

Now,

$$\begin{aligned}
\|Jc_n-f\| &= \left\|\frac{s+1}{s}Ha_n-\frac{Hd_n}{s}-f\right\|,\\
&= \left\|\frac{s+1}{s}(Ha_n-f)-\frac{1}{s}(Hd_n-f)\right\|,\\
&\le \frac{s+1}{s}\|Ha_n-Hv\|+\frac{1}{s}\|Hd_n-Hv\|,\\
&\le \frac{s+1}{s}\left[\psi\left(\|Jv-Hv\|\right)+\beta\|Ja_n-Jv\|\right]+\frac{\beta}{s}\|Jd_n-Jv\|,\\
&= \frac{\beta(s+1)}{s}\|Ja_n-f\|+\frac{\beta}{s}\|Jd_n-f\|. \qquad (2.2)
\end{aligned}$$

Substituting (2.1) in (2.2), we obtain

$$\begin{aligned}
\|Jc_n-f\| &\le \frac{\beta(s+1)}{s}\|Ja_n-f\|+\frac{\beta}{s}\left(\frac{s+\beta}{s+1}\right)\|Ja_n-f\|,\\
&= \beta\left(1+\frac{1}{s}+\frac{s+\beta}{s(s+1)}\right)\|Ja_n-f\|. \qquad (2.3)
\end{aligned}$$

Also,

$$\begin{aligned}\|Jb_n - f\| &= \left\|\frac{s'}{s'+1}Jc_n + \frac{1}{s'+1}Hc_n - f\right\|,\\ &\le \frac{s'}{s'+1}\|Jc_n - f\| + \frac{1}{s'+1}\|Hc_n - Hv\|,\\ &\le \frac{s'}{s'+1}\|Jc_n - f\| + \frac{1}{s'+1}\left[\psi\left(\|Jv - Hv\|\right) + \beta\|Jc_n - Jv\|\right],\\ &= \frac{s'+\beta}{s'+1}\|Jc_n - f\|. \end{aligned} \tag{2.4}$$

And;

$$\begin{aligned}\|Ja_{n+1} - f\| &= \|Hb_n - Hv\|,\\ &\le \beta\|Jb_n - f\|. \end{aligned} \tag{2.5}$$

By using (2.3), (2.4); (2.5) yields

$$\begin{aligned}\|Ja_{n+1} - f\| &\le \beta^2\left(\frac{s'+\beta}{s'+1}\right)\left(1+\frac{1}{s}+\frac{s+\beta}{s(s+1)}\right)\|Ja_n - f\|,\\ &\dots\\ &\le \left(\beta^2\left(\frac{s'+\beta}{s'+1}\right)\left(1+\frac{1}{s}+\frac{s+\beta}{s(s+1)}\right)\right)^{n+1}\|Ja_0 - f\|. \end{aligned} \tag{2.6}$$

Since, $s \ge 1$, $s' > 0$; and $0 \le \beta < \frac{1}{\sqrt{3}}$, we have

$$0 < \frac{s+\beta}{s+1} < 1; \text{ and } 0 < 1 + \frac{1}{s} + \frac{s+\beta}{s(s+1)} < 3.$$

Hence,

$$\beta^2\left(\frac{s'+\beta}{s'+1}\right)\left(1+\frac{1}{s}+\frac{s+\beta}{s(s+1)}\right) < 1;$$

and therefore

$$\left(\beta^2\left(\frac{s'+\beta}{s'+1}\right)\left(1+\frac{1}{s}+\frac{s+\beta}{s(s+1)}\right)\right)^{n+1} \to 0 \text{ as } n \to \infty.$$

Therefore, (2.6) implies that $\lim_{n\to\infty}\|Ja_{n+1} - f\| = 0.$

Hence, $\{Ja_n\}_{n=0}^{\infty}$ converges to f.

To demonstrate that f is the unique common fixed point of H and J, assume that v and v_1 are coincidence points of H and J such that $Hv = f = Jv$ and $Hv_1 = f_1 = Jv_1$, where f_1 is another point of coincidence of H and J. Applying the contractive condition (1.5), we have:

$$\|f-f_1\|=\|Hv-Hv_1\|\le\beta\|Jv-Jv_1\|<\|f-f_1\|,$$

this leads to a contradiction. Hence, $f_1=f$ means that point of coincidence is unique. Given that H and J are weakly compatible, we have

$$Hf=HHv=HJv=JHv=Jf.$$

which implies $Jf=f=Hf$. This verifies that H and J have f as the point of coincidence. The uniqueness of the point of coincidence indicates that f is a unique common fixed point of H,J.

Theorem 2.2. Let J and H be the same as in Theorem 2.1, and $\{Ja_n\}_{n=0}^{\infty}$ be the iteration scheme generated by (1.6) converging to f. Then $\{Ja_n\}_{n=0}^{\infty}$ is (J,H)- stable.

Proof. Suppose $\{Jp_n\}_{n=0}^{\infty}\subset W$ be an arbitrary sequence, such that

$$\mu_n=\|Jp_{n+1}-Hq_n\|,$$

where $Jq_n=\frac{s'}{s'+1}Jr_n+\frac{1}{s'+1}Hr_n$, $Jr_n=\frac{s+1}{s}Hp_n-\frac{1}{s}Hh_n$, and $Jh_n=\frac{s}{s+1}Jp_n+\frac{1}{s+1}Hp_n$.
To prove that the iterative scheme (1.6) is (J,H)-stable; we have to show that $\lim_{n\to\infty}\mu_n=0$ if and only if $\lim_{n\to\infty}Jp_n=f$.
Let $\lim_{n\to\infty}\mu_n=0$. We have

$$\begin{aligned}\|Jp_{n+1}-f\|&=\|Jp_{n+1}-Hq_n+Hq_n-f\|,\\&\le\|Jp_{n+1}-Hq_n\|+\|Hq_n-f\|,\\&=\mu_n+\|Hq_n-f\|,\\&\le\mu_n+\beta\|Jq_n-f\|.\end{aligned}\tag{2.7}$$

$$\begin{aligned}\|Jq_n-f\|&=\left\|\frac{s'}{s'+1}Jr_n+\frac{1}{s'+1}Hr_n-f\right\|,\\&\le\frac{s'}{s'+1}\|Jr_n-f\|+\frac{1}{s'+1}\|Hr_n-Hv\|,\\&\le\frac{s'}{s'+1}\|Jr_n-f\|+\frac{1}{s'+1}\left[\psi\left(\|Jv-Hv\|\right)+\beta\|Jr_n-Jv\|\right],\\&=\frac{s'+\beta}{s'+1}\|Jr_n-f\|.\end{aligned}\tag{2.8}$$

With ease; similar to estimate (2.3); one can get

$$\|Jr_n-f\|\le\beta\left(1+\frac{1}{s}+\frac{s+\beta}{s(s+1)}\right)\|Jp_n-f\|.\tag{2.9}$$

Substituting (2.8) and (2.9); estimate (2.7) yields

$$\|Jp_{n+1} - f\| \le \mu_n + \beta^2\left(\frac{s'+\beta}{s'+1}\right)\left(1+\frac{1}{s}+\frac{s+\beta}{s(s+1)}\right)\|Jp_n - f\|. \quad (2.10)$$

As $\beta^2\left(\frac{s'+\beta}{s'+1}\right)\left(1+\frac{1}{s}+\frac{s+\beta}{s(s+1)}\right)<1;$ using Lemma 1; inequality (2.10) yields $\lim_{n\to\infty} Jp_n = f$.

Conversely; let $\lim_{n\to\infty} Jp_n = f$.

$$\begin{aligned}
\mu_n &= \|Jp_{n+1} - Hq_n\|,\\
&= \|Jp_{n+1} - f + f - Hq_n\|,\\
&\le \|Jp_{n+1} - f\| + \|Hq_n - f\|,\\
&\le \|Jp_{n+1} - f\| + \beta^2\left(\frac{s'+\beta}{s+1}\right)\left(1+\frac{1}{s}+\frac{s+\beta}{s(s+1)}\right)\|Jp_n - f\|. \quad (2.11)
\end{aligned}$$

By taking the limit as $n \to \infty$ on both sides of (2.11), we get $\lim_{n\to\infty} \mu_n = 0$.

Theorem 2.3. Let (J, H) be the same as in Theorem 2.1, and (J_1, H_1) be an approximate mapping pair of (J, H) according to Definition 3 such that $J_1(V)$ is complete in W. Suppose $J_1 v = f_1 = H_1 v$. Consider $\{Ja_n\}_{n=0}^{\infty}$ be the iteration scheme generated by (1.6) converging to f and $\{J_1 e_n\}_{n=0}^{\infty}$ be the sequence defined by

$$\begin{cases}
J_1 e_{n+1} = H_1 g_n,\\
J_1 g_n = \frac{s'}{s'+1} J_1 h_n + \frac{1}{s'+1} H_1 h_n,\\
J_1 h_n = \frac{s+1}{s} H_1 e_n - \frac{1}{s} H_1 i_n,\\
J_1 i_n = \frac{s}{s+1} J_1 e_n + \frac{1}{s+1} H_1 e_n.
\end{cases} \quad (2.12)$$

Suppose that $\{J_1 e_n\}_{n=0}^{\infty}$ converges to f_1. Then; we have

$$\|f - f_1\| \le \frac{6(\varepsilon_1 + \beta\varepsilon_2)}{1-3\beta^2}.$$

Proof. Using (1.6) and (2.12), we have

$$\begin{aligned}
\|J_1 e_{n+1} - Ja_{n+1}\| &= \|H_1 g_n - Hb_n\|,\\
&\le \|H_1 g_n - Hg_n\| + \|Hg_n - Hb_n\|,\\
&\le \varepsilon_1 + \psi(\|Jb_n - Hb_n\|) + \beta\|Jg_n - Jb_n\|,\\
&\le \varepsilon_1 + \psi(\|Jb_n - Hb_n\|) + \beta\|Jg_n - J_1 g_n\| + \beta\|J_1 g_n - Jb_n\|,\\
&\le \varepsilon_1 + \psi(\|Jb_n - Hb_n\|) + \beta\varepsilon_2 + \beta\|J_1 g_n - Jb_n\|. \quad (2.13)
\end{aligned}$$

Now,

$$
\begin{aligned}
\|J_1g_n - Jb_n\| &= \left\|\frac{s'}{s'+1}J_1h_n + \frac{1}{s'+1}H_1h_n - \frac{s'}{s'+1}Jc_n - \frac{1}{s'+1}Hc_n\right\|,\\
&\le \frac{s'}{s'+1}\|J_1h_n - Jc_n\| + \frac{1}{s'+1}\|H_1h_n - Hc_n\|,\\
&\le \frac{s'}{s'+1}\|J_1h_n - Jc_n\| + \frac{1}{s'+1}\|H_1h_n - Hh_n\| + \frac{1}{s'+1}\|Hh_n - Hc_n\|,\\
&\le \frac{s'}{s'+1}\|J_1h_n - Jc_n\| + \frac{1}{s'+1}\varepsilon_1 + \frac{1}{s'+1}\psi\left(\|Jc_n - Hc_n\|\right) + \frac{\beta}{s'+1}\|Jh_n - Jc_n\|,\\
&\le \frac{s'}{s'+1}\|J_1h_n - Jc_n\| + \frac{\varepsilon_1}{s'+1} + \frac{1}{s'+1}\psi\left(\|Jc_n - Hc_n\|\right) + \frac{\beta}{s'+1}\left(\|Jh_n - J_1h_n\| + \|J_1h_n - Jc_n\|\right),\\
&\le \frac{\varepsilon_1}{s'+1} + \frac{\beta\varepsilon_2}{s'+1} + \frac{1}{s'+1}\psi\left(\|Jc_n - Hc_n\|\right) + \frac{s'+\beta}{s'+1}\|J_1h_n - Jc_n\|. \qquad (2.14)
\end{aligned}
$$

Also,

$$
\begin{aligned}
\|J_1h_n - Jc_n\| &= \left\|\frac{s+1}{s}H_1e_n - \frac{1}{s}H_1i_n - \frac{s+1}{s}Ha_n + \frac{1}{s}Hd_n\right\|,\\
&\le \frac{s+1}{s}\|H_1e_n - Ha_n\| + \frac{1}{s}\|H_1i_n - Hd_n\|,\\
&\le \frac{s+1}{s}\left(\|H_1e_n - He_n\| + \|He_n - Ha_n\|\right) + \frac{1}{s}\left(\|H_1i_n - Hi_n\| + \|Hi_n - Hd_n\|\right),\\
&\le \frac{\varepsilon_1(s+1)}{s} + \frac{s+1}{s}\psi\left(\|Ja_n - Ha_n\|\right) + \frac{\beta(s+1)}{s}\|Je_n - Ja_n\| + \frac{\varepsilon_1}{s} + \frac{1}{s}\psi\left(\|Jd_n - Hd_n\|\right) + \frac{\beta}{s}\|Ji_n - Jd_n\|,\\
&\le \frac{\varepsilon_1(s+2)}{s} + \frac{s+1}{s}\psi\left(\|Ja_n - Ha_n\|\right) + \frac{1}{s}\psi\left(\|Jd_n - Hd_n\|\right) + \frac{\beta(s+1)}{s}\left(\|Je_n - J_1e_n\| + \|J_1e_n - Ja_n\|\right)\\
&\quad + \frac{\beta}{s}\left(\|Ji_n - J_1i_n\| + \|J_1i_n - Jd_n\|\right),\\
&\le \frac{\varepsilon_1(s+2)}{s} + \frac{s+1}{s}\psi\left(\|Ja_n - Ha_n\|\right) + \frac{1}{s}\psi\left(\|Jd_n - Hd_n\|\right) + \frac{\beta(s+2)}{s}\varepsilon_2 + \frac{\beta(s+1)}{s}\|J_1e_n - Ja_n\| + \frac{\beta}{s}\|J_1i_n - Jd_n\|. \qquad (2.15)
\end{aligned}
$$

And,

$$
\begin{aligned}
\|J_1i_n - Jd_n\| &= \left\|\frac{s}{s+1}J_1e_n + \frac{1}{s+1}H_1e_n - \frac{s}{s+1}Ja_n - \frac{1}{s+1}Ha_n\right\|,\\
&\le \frac{s}{s+1}\|J_1e_n - Ja_n\| + \frac{1}{s+1}\|H_1e_n - Ha_n\|,\\
&\le \frac{s}{s+1}\|J_1e_n - Ja_n\| + \frac{1}{s+1}\left(\|H_1e_n - He_n\| + \|He_n - Ha_n\|\right),\\
&\le \frac{s}{s+1}\|J_1e_n - Ja_n\| + \frac{\varepsilon_1}{s+1} + \frac{1}{s+1}\psi\left(\|Ja_n - Ha_n\|\right) + \frac{\beta}{s+1}\|Je_n - Ja_n\|,\\
&\le \frac{\varepsilon_1}{s+1} + \frac{1}{s+1}\psi\left(\|Ja_n - Ha_n\|\right) + \frac{s}{s+1}\|J_1e_n - Ja_n\| + \frac{\beta}{s+1}\left(\|Je_n - J_1e_n\| + \|J_1e_n - Ja_n\|\right),\\
&\le \frac{\varepsilon_1}{s+1} + \frac{1}{s+1}\psi\left(\|Ja_n - Ha_n\|\right) + \frac{\beta}{s+1}\varepsilon_2 + \frac{s+\beta}{s+1}\|J_1e_n - Ja_n\|. \qquad (2.16)
\end{aligned}
$$

By combining (2.13), (2.14), (2.15) and (2.16), we get

$$
\begin{aligned}
\|J_1e_{n+1} - Ja_{n+1}\| &\le \frac{\beta^2(s'+\beta)}{s(s'+1)}\left(s+1+\frac{s+\beta}{s+1}\right)\|J_1e_n - Ja_n\| + \varepsilon_1 + \frac{\beta}{s'+1}\varepsilon_1 + \frac{\beta(s'+\beta)(s+2)}{s(s'+1)}\varepsilon_1 + \frac{\beta^2(s'+\beta)}{s(s+1)(s'+1)}\varepsilon_1\\
&\quad + \beta\varepsilon_2 + \frac{\beta^2}{s'+1}\varepsilon_2 + \frac{\beta^2(s'+\beta)(s+2)}{s(s'+1)}\varepsilon_2 + \frac{\beta^3(s'+\beta)}{s(s+1)(s'+1)}\varepsilon_2 + \psi\left(\|Jb_n - Hb_n\|\right) + \frac{\beta}{s'+1}\psi\left(\|Jc_n - Hc_n\|\right)\\
&\quad + \frac{\beta(s'+\beta)}{s(s'+1)}\left(s+1+\frac{\beta}{s+1}\right)\psi\left(\|Ja_n - Ha_n\|\right) + \frac{\beta(s'+\beta)}{s(s'+1)}\psi\left(\|Jd_n - Hd_n\|\right). \qquad (2.17)
\end{aligned}
$$

Since, $s \ge 1$, $s' > 0$, and $\beta < \frac{1}{\sqrt{3}}$, then (2.17) yields

$$\|J_1e_{n+1} - Ja_{n+1}\| \le 3\beta^2 \|J_1e_n - Ja_n\| + 6(\varepsilon_1 + \beta\varepsilon_2) + \psi\left(\|Jb_n - Hb_n\|\right) + \psi\left(\|Jc_n - Hc_n\|\right)$$
$$+\left(s+1+\frac{\beta}{s+1}\right)\psi\left(\|Ja_n - Ha_n\|\right) + \psi\left(\|Jd_n - Hd_n\|\right). \tag{2.18}$$

Now,

$$\begin{aligned}\|Ja_n - Ha_n\| &\le \|Ja_n - f\| + \|f - Ha_n\|,\\ &\le \|Ja_n - f\| + \psi\left(\|Jv - Hv\|\right) + \beta\|Ja_n - Jv\|,\\ &= (1+\beta)\|Ja_n - f\|.\end{aligned} \tag{2.19}$$

Given that, $\lim_{n\to\infty}\|Ja_n - f\| = 0$, equation (2.19) yields $\lim_{n\to\infty}\|Ja_n - Ha_n\| = 0$, which subsequently leads to $\lim_{n\to\infty}\psi\left(\|Ja_n - Ha_n\|\right) = 0.$

Also,

$$\begin{aligned}\|Jd_n - Hd_n\| &\le \|Jd_n - f\| + \|f - Hd_n\|,\\ &\le (1+\beta)\|Jd_n - f\|,\\ &\le (1+\beta)\left(\frac{s}{s+1}\|Ja_n - f\| + \frac{\beta}{s+1}\|Ja_n - f\|\right),\\ &\le (1+\beta)\left(\frac{s+\beta}{s+1}\right)\|Ja_n - f\|.\end{aligned} \tag{2.20}$$

Since, $\lim_{n\to\infty}\|Ja_n - f\| = 0$, equation (2.20) yields $\lim_{n\to\infty}\|Jd_n - Hd_n\| = 0$, which further implies $\lim_{n\to\infty}\psi\left(\|Jd_n - Hd_n\|\right) = 0.$

Similarly,

$$\begin{aligned}\|Jc_n - Hc_n\| &\le \|Jc_n - f\| + \|f - Hc_n\|,\\ &\le (1+\beta)\|Jc_n - f\|,\\ &\le (1+\beta)\left(\frac{s+1}{s}\|Ha_n - f\| + \frac{1}{s}\|f - Hd_n\|\right),\\ &\le (1+\beta)\left(\frac{\beta(s+1)}{s}\|Ja_n - f\| + \frac{\beta}{s}\|Jd_n - f\|\right),\\ &\le (1+\beta)\left(\frac{\beta(s+1)}{s} + \frac{\beta}{s}\left(\frac{s+\beta}{s+1}\right)\right)\|Ja_n - f\|.\end{aligned} \tag{2.21}$$

Given that, $\lim_{n\to\infty}\|Ja_n - f\| = 0$, equation (2.21) yields $\lim_{n\to\infty}\|Jc_n - Hc_n\| = 0$, which further implies $\lim_{n\to\infty}\psi\left(\|Jc_n - Hc_n\|\right) = 0.$

And,

$$\begin{aligned}\|Jb_n - Hb_n\| &\le \|Jb_n - f\| + \|f - Hb_n\|,\\ &\le (1+\beta)\|Jb_n - f\|,\\ &\le (1+\beta)\frac{\beta}{s}\left(\frac{s'+\beta}{s'+1}\right)\left(s+1+\frac{s+\beta}{s+1}\right)\|Ja_n - f\|.\end{aligned} \tag{2.22}$$

Since, $\lim_{n\to\infty}\|Ja_n - f\| = 0$, equation (2.22) yields $\lim_{n\to\infty}\|Jb_n - Hb_n\| = 0$, which subsequently leads to $\lim_{n\to\infty}\psi\left(\|Jb_n - Hb_n\|\right) = 0.$

As $\lim_{n\to\infty} Ja_n = f$ and $\lim_{n\to\infty} J_1 e_n = f_1.$

Taking limit $n \to \infty$ and using above facts, (2.18) yields

$$\| f - f_1\| \le 3\beta^2 \| f - f_1\| + 6(\varepsilon_1 + \beta\varepsilon_2)$$

which further gives

$$\| f - f_1\| \le \frac{6(\varepsilon_1 + \beta\varepsilon_2)}{1 - 3\beta^2}.$$

To verify the numerical feasibility of Theorem 2.3, the following example is provided:

Example 2.4. Consider $V = [0,1]$ and $H, J : V \to W$ is defined by $H(v) = \frac{5+v}{7}$ and $J(v) = \frac{6v}{7}$ satisfy the contractive condition (1.5) when $\psi(v) = v^{\frac{1}{2}}$ and $\beta \geq 0.2$. Define operators H_1 and J_1 as $H_1(v) = \frac{4-v}{5}$ and $J_1(v) = \frac{4v}{5}$. It is clear that $H(v) \subseteq J(v), H_1(v) \subseteq J_1(v), H(1) = J(1) = \frac{6}{7} = f$ and $H_1\left(\frac{4}{5}\right) = J_1\left(\frac{4}{5}\right) = \frac{16}{25} = f_1$. We have $\max_{v\in V}|H - H_1| = 0.257 = \varepsilon_1$ (say) and $\max_{v\in V}|J - J_1| = 0.057 = \varepsilon_2$ (say).Evidently, (H_1, J_1) is an approximate mapping pair of (H, J). With initial approximation $v_0 = 0.5$ and $s = 9, s' = \frac{1}{10}$, the iteratives schemes $\{Ja_{n+1}\}_{n=0}^{\infty}$ and $\{J_1 a_{n+1}\}_{n=0}^{\infty}$ converges to $\frac{6}{7}$ and $\frac{16}{25}$, respectively as shown in Table 1 and the graphical convergence is shown in Fig. 1. The values of the operators $H(v)$, $H_1(v)$, $J(v)$, and $J_1(v)$ displayed in Table 2 corresponding to different values of $v \in V$ and also the graphical representation of the values of Table 2 is provided in Fig. 2(a) – 2(b). Therefore, we obtain the following estimate:

$$0.22 = |f - f_1| \le \frac{6(\varepsilon_1 + \beta\varepsilon_2)}{1 - 3\beta^2} = 1.83.$$

Table 1. Comparison of Ja_{n+1} and J_1a_{n+1} iterations.

Iterations	Ja_{n+1}	J_1a_{n+1}
0	0.42857	0.40000
1	0.85423	0.64207
2	**0.85703**	**0.64007**
3	0.85703	0.64007
4	0.85703	0.64007
5	0.85703	0.64007

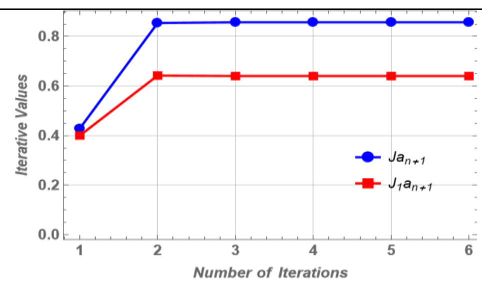

Figure 1. Graph for Table 1.

Table 2. Values of operators over domain $[0,1]$.

v	$H(v)$	$H_1(v)$	$J(v)$	$J_1(v)$
0.	0.71429	0.80000	0.00000	0.00000
0.1	0.72857	0.78000	0.08571	0.08000
0.2	0.74286	0.76000	0.17143	0.16000
0.3	0.75714	0.74000	0.25714	0.24000
0.4	0.77143	0.72000	0.34286	0.32000
0.5	0.78571	0.70000	0.42857	0.40000
0.6	0.80000	0.68000	0.51429	0.48000
0.7	0.81429	0.66000	0.60000	0.56000
0.8	0.82857	0.64000	0.68571	0.64000
0.9	0.84286	0.62000	0.77143	0.72000
1.	0.85714	0.60000	0.85714	0.80000

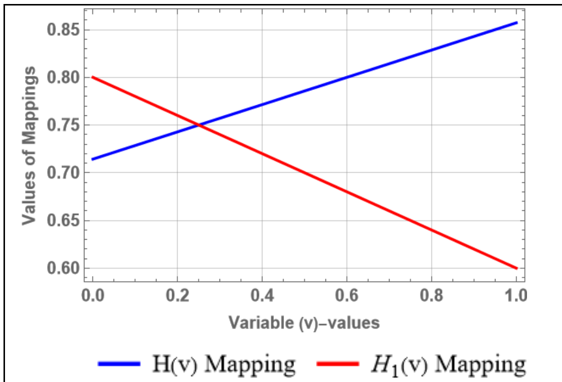

Figure 2(a). Graph for Operators H and H_1.

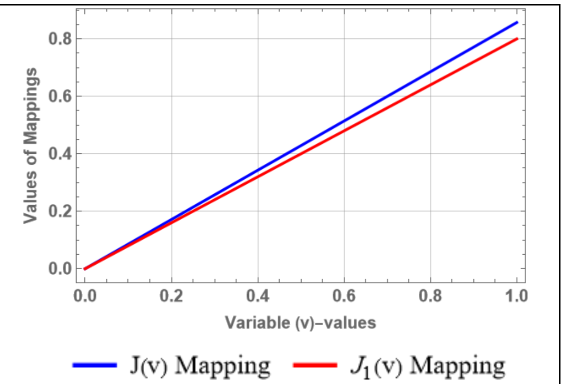

Figure 2(b). Graph for Operators J and J_1.

3. Numerical examples

To thoroughly assess the effectiveness and capabilities of the proposed iterative approach, it is crucial to examine a wide array of mathematical challenges with varying levels of complexity. By implementing the proposed iteration technique across these diverse equation types, we seek to verify its broad applicability and acquire a more comprehensive understanding of its advantages and constraints when addressing intricate, real-world scenarios.

Example 3.1. Consider the equation

$$v^2 - 10 = 3v$$

Let $V = [5,7] \subset \mathbb{R}$ be equipped with a standard metric. Establish $H, J : [5,7] \to [15,40]$ with a coincidence point 5 by $Hv = 3v$ and $Jv = v^2 - 10$. Evidently, $H([5,7]) \subseteq J([5,7])$ and $J([5,7])$ is a complete subset of $[15, 40]$. Assume the initial guess as $v_0 = 7$. Table 3 presents a comparative analysis of the convergence rates for the J-Itr1, J-Itr2, and New-Itr methods towards the point of coincidence, and a graphical representation is shown in Fig. 3.

Table 3. Comparison of iterative algorithms for Example 3.1. with $s = 9, s' = \frac{1}{10}$ and $\alpha_n = \beta_n = \gamma_n = \delta_n = \lambda_n = \frac{1}{\sqrt{(1+n)}}$.

Iterations	*J-Itr1* Ja_{n+1}	*J-Itr2* Ja_{n+1}	*New-Itr* Ja_{n+1}
0	39.00000	39.00000	39.00000
1	15.50300	−9.14960	15.62000
2	15.01300	29.05400	15.02000
3	15.00500	11.43500	15.00100
4	15.00300	14.82800	**15.00000**
5	15.00200	14.96800	15.00000
6	15.00100	14.99100	15.00000
7	15.00100	14.99700	15.00000
8	**15.00000**	14.99900	15.00000
9	15.00000	14.99900	15.00000
10	15.00000	**15.00000**	15.00000
11	15.00000	15.00000	15.00000
12	15.00000	15.00000	15.00000

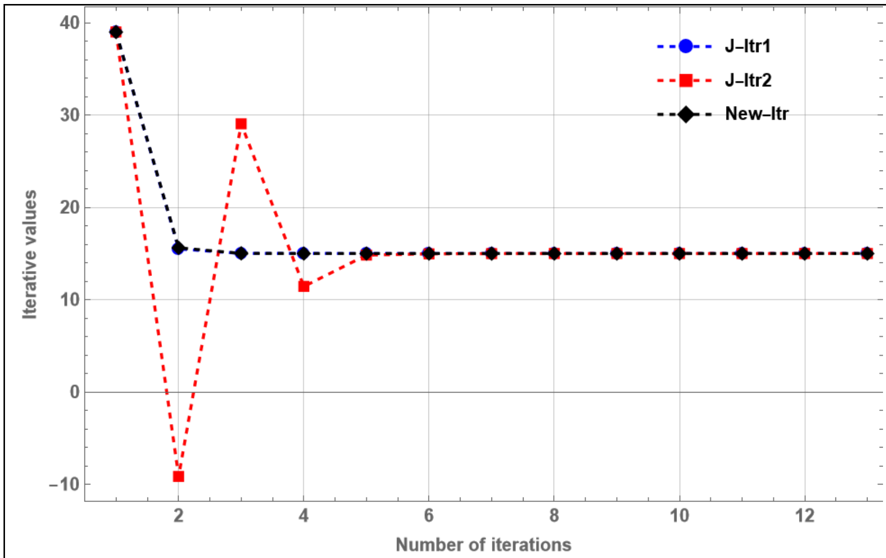

Figure 3. Graph for Table 3.

Example 3.2. Consider the transcendental equation as

$$e^v = Sinv + 2$$

Let $V = [0, 2] \subset \mathbb{R}$ be equipped with a standard metric. We define $H, J : [0, 2] \to [0, 8]$ with a coincidence point 1.054127 by $Hv = 2 + sinv$ and $Jv = e^v$. Evidently, $H([0, 2]) \subseteq J([0, 2])$ and $J([0, 2])$ is a complete subset of $[0,8]$. Suppose the initial guess $v_0 = -1.5$. A comparative study of the convergence between J-Itr1, J-Itr2, and New-Itr to the point of coincidence is presented in Table 4, and a graphical representation is shown in Figure 4.

Table 4. Comparison of iterative algorithms for Example 3.2. with $s = 9, s' = \frac{1}{10}$ and $\alpha_n = \beta_n = \gamma_n = \delta_n = \lambda_n = \frac{1}{\sqrt{(1+n)}}$.

Iterations	*J-Itr1* Ja_{n+1}	*J-Itr2* Ja_{n+1}	*New-Itr* Ja_{n+1}
0	0.22313	0.22313	0.22313
1	2.63990	3.63310	2.60050
2	2.86810	2.04090	2.86710
3	2.86900	3.07580	**2.86950**
4	2.86930	2.85580	2.86950
5	2.86940	2.86810	2.86950
6	2.86940	2.86920	2.86950
7	2.86940	2.86940	2.86950
8	2.86940	2.86940	2.86950
9	2.86940	**2.86950**	2.86950
10	**2.86950**	2.86950	2.86950
11	2.86950	2.86950	2.86950
12	2.86950	2.86950	2.86950

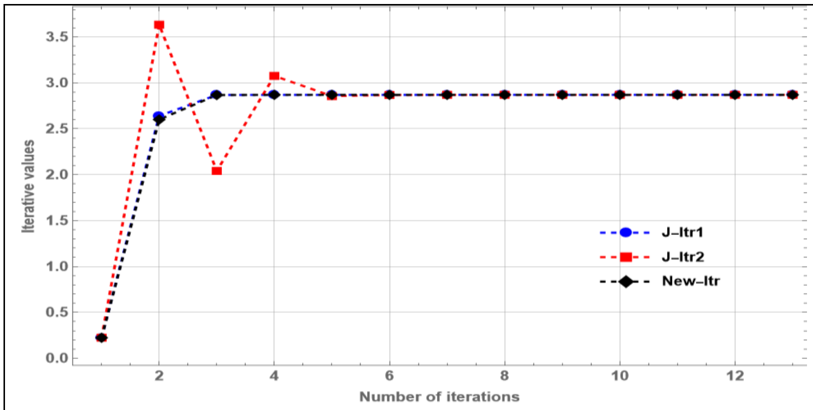

Figure 4. Graph for Table 4.

4. Conclusion

We introduced an innovative Jungck-type iterative method, and examined its strong convergence, stability, and data dependence characteristics followed by non-trivial examples. Furthermore, computational experiments indicated that the proposed method demonstrated superior convergence speed compared to several established iteration techniques.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Author's Contributions

All authors contributed equally to this study.

References

Karapinar, E., Fulga, A., Shahzad, N., & Roldan hopez de hierro, A. F. (2022). Solving integral equations by means of fixed point theory. Journal of Function Spaces, 1-16. Article Id 7667499. https://doi.org/10.1155/2022/7667499.

Adeyemi, T. A., Akutsah, F., Mebawondu, A. A., Adewole, M. O., & Narain, O. K. (2021). The existence of a solution of the non-linear integral equation via the fixed point approach. Advanced in Mathematics Scientific Journal, 10, 2977-2998. https://doi.org/10.37418/amsj.10.7.5.

Martinez, F., Manriquez, H., Ojeda, A., & Olea, G. (2022). Organization patterns of complex river networks in Chile: a fractal morphology. Mathematics, 10(11), 1806. https://doi.org/10.3390/math10111806.

Zhang, X., Wang, L., Zhou, Z., & Niu, Y. (2019). A chaos-based image encryption technique utilizing Hilbert curves and H-fractals. IEEE Access, 7, 74734–74746. https://doi.org/10.1109/ACCESS.2019. 2921309.

Ouyang, P., Chung, K., Nicolas, A., Gdawiec, K. (2021). Self-similar fractal drawings inspired by M.C. Escher's print Square Limit. ACM Trans. Graph, 40(3), 31.

Guran, L.; Shabbir, K.; Ahmad, K.; Bota, M.-F. (2023). Stability, Data Dependence, and Convergence Results with Computational Engendering of Fractals via Jungck–DK Iterative Scheme. Fractal Fractional 7, 418. https://doi.org/10.3390/fractalfract7060418.

Berinde, V. (2004). On the convergence of the Ishikawa iteration in the class of quasi-contractive operators. Acta Mathematica Universitatis Comenianae, 73, 119-126.

Jungck, G., & Hussain, N. (2017). Compatible maps and invariant approximations. Journal of Mathematical Analysis and Applications, 32, 1003-1012.

Khan, A. R., & Fukhar-ud-din, H. (2014). Iterative construction of common fixed points in convex metric spaces. In Saleh, A. R., Al-Mezel, F. R. M., Al-Solamy, F. R. M., & Ansari, Q. H. (Eds.), Fixed Points Theory, Variational Analysis, and Optimization (pp. 3-43). Champan and Hall/CRC.

Singh, S. L., Bhatnagar, C., & Mishra, S. N. (2005). Stability of Jungck-type iterative procedures. International Journal of Mathematics and Mathematical Sciences, 19, 3035-3043.

Olatinwo, M. O. (2008). A generalization of some convergence results using the Jungck-Noor three-step iteration process in arbitrary Banach space. Fasciculi Mathematici, 40, 37-47.

Khan, A. R., Kumar, V., & Hussain, N. (2014). Analytical and numerical treatment of Jungck-type iterative schemes. Applied Mathematics and Computation, 231, 521-535.

Olatinwo, M. O., & Imoru, C. O. (2008). Some convergence results for the Jungck-Mann and the Jungck-Ishikawa iteration processes in the class of generalized Zamfirescu operators. Acta Mathematica Universitatis Comenianae, 77, 299-304.

Olatinwo, M. O. (2008). Some stability and strong convergence results for the Jungck-Ishikawa iteration process. Creative Mathematics and Informatics, 17, 33-42.

Jungck, G. (1976). Commuting mappings and fixed points. American Mathematical Monthly, 83, 261-263.

Olatinwo, M. O., & Postolache, M. (2012). Stability results for Jungck-type iterative processes in convex metric spaces. Computers & Applied Mathematics, 218, 6727-6732.

Berinde, V. (2007). Iterative Approximation of Fixed Points. Springer.

Sharma, P., Ramos, H., Kanwar, V., et al. (2024). Efficient iterative procedures for approximating fixed points of contractive-type mappings with applications. Numerical Algorithms. https://doi.org/10.1007/s11075-024-01869-x.