Short Research Article

The Interplay Between Quasi-matroid and Connectivity Systems

Abstract: A graph width parameter measures the structural complexity of a graph by assessing how it can be decomposed into simpler components. These parameters are often studied using connectivity systems that satisfy symmetric submodular conditions [15,16].

Matroids are highly versatile structures with broad applications in optimization theory, combinatorial mathematics, topology, algebra, graph algorithms, game theory, geometry, and network theory, making them a subject of significant interest. Quasi-matroid, as introduced in [7] (see also [8]), extend the concept of matroids by providing a more generalized framework.

In this brief paper, we propose the concept of quasi-matroid on a connectivity system and explore their relationship with linear decomposition.

Keyword: Ultra matroid, Linear decomposition, Matroid, Quasi-Matroid

1. Introduction

1.1. Graph Width Parameters

Graph theory, a fundamental field of mathematics, explores networks composed of nodes and edges, analyzing their paths, structures, and characteristics [39]. Within graph theory, the concept of graph parameters is frequently studied. These parameters are numerical values that capture specific properties of a graph, serving as tools for structural analysis (cf. [53]).Examples include tree-width [1], chromatic number [46,51], clique number [47], boxicity[48], domination number[49,50], secure domination number[85,86], and diameter[52]. Among graph parameters, the "graph width parameter" is particularly noteworthy.

A graph width parameter intuitively quantifies the "complexity" of a graph by capturing its structural properties in terms of separation or decomposition. It provides a framework for breaking the graph into simpler components while managing the "size" or "complexity" of the interactions between these components. For example, treewidth measures how close a graph is to being a tree. Graphs with low treewidth can be decomposed into a tree-like structure, where each node in the tree corresponds to a subset of vertices in the graph, ensuring manageable overlap between subsets (cf. [65,66]). This property makes treewidth particularly useful in optimization and algorithmic graph theory. Examples of graph width parameters include tree-width [1], branch-width [16,40,64], hypertree-width [23,41], superhypertree-width [19,21], clique-width[84], modular-width [42], path-distance-width [20], twin-width [6], boolean-width [43], rank-width [44], tree-partition-width[63], path-width [1], tree-cut-width[67], and cut-width [45].

Among these, linear-width has garnered significant attention and has been extensively studied, not only within graph theory but also in fields such as network theory and combinatorial mathematics [4,5]. Linear-width is often regarded as a linear counterpart to branch-width and can also be interpreted as a width parameter that reflects path-width in terms of graph edges. Researchers have devoted substantial effort to studying linear-width and its related parameters (see, for instance, [1–6,15–30]). For practical applications of graph width parameters, see references [60,61,62].

These graph width parameters are often analyzed in the context of connectivity systems, underscoring their significance in understanding and studying graph structures [15,16]. In this short paper, we define a connectivity system as a pair (X, f), where X is a finite set and f is a symmetric submodular function.

1.2. Matroids

Matroids are mathematical structures that generalize the concept of linear independence in vector spaces. They provide a versatile framework for addressing optimization and combinatorial problems and are characterized by a ground set and a collection of independent subsets that satisfy specific axioms [14].Matroids are highly versatile, with applications spanning optimization theory, combinatorial mathematics, topology, algebra, graph algorithms, game theory, geometry, and network theory, making them a subject of significant interest (e.g., [9–14, 31–33, 54, 55]). Related concepts such as antimatroids[56], greedoids[57], groupoids[58], and hypergroupoids[59] have also been extensively studied.

Quasi-matroid, as introduced in [7] (see also [8]), extend the concept of matroids by providing a generalized framework. Additionally, the relationships between matroids, antimatroids, greedoids, and connectivity systems, as well as their connections to graph width parameters, have been investigated in the literature [28,29].

1.3. Our Motivation

From the above, it is evident that research on graph width parameters and matroids is of significant importance.While matroids are well-established across various disciplines, the concept of quasi-matroids in connectivity systems remains relatively underexplored.
This study establishes a novel connection between quasi-matroids and linear tangles, offering a framework to better understand their role within connectivity systems. At first glance, the notion that seemingly unrelated concepts can gain relevance when examined in the context of a connectivity system is both profound and thought-provoking.

2. Preparation of this paper

This section provides mathematical definitions of each concept. First, we explain the basic notation used in this paper.Refer to reference [68], as needed, for fundamental operations on sets and related topics.In addition, this paper considers only finite sets.

Notation 1: In this paper, we use expressions like $A \subseteq X$ to indicate that A is a subset of X, $A \cup B$ to represent the union of two subsets A and B, both of which are subsets of X, or $A = \emptyset$ to signify an empty set. Specifically, $A \cap B$ denotes the intersection of subsets A and B. A similar logic applies to $A \setminus B$. $A \in X$ indicates that A is an element of X. $\forall$ means "for all" or "for every." $\exists$ means "there exists."

Definition 2: The powerset of a set S is the set of all possible subsets of S, including the empty set and S itself.In this paper, 2^A is denoted as the powerset.

2.1 Symmetric submodular set function and connectivity system

A symmetric submodular set function exhibits symmetry and follows the principle of diminishing returns. It is often applied in optimization, decision-making, graph theory, and discrete mathematics (cf. [75,76]). The definition of a symmetric submodular function is provided below. Note that while general submodular functions typically take real values, in this paper, similar to references [15,16], submodular functions are assumed to take natural number values.

Definition 3 (cf. [15,16]): Let X be a finite set. A function $f: X \rightarrow \mathbb{N}$ is called symmetric submodular if it satisfies the following conditions:

- $\forall A \subseteq X,\ f(A) = f(X \setminus A)$.
- $\forall A, B \subseteq X,\ f(A) + f(B) \geq f(A \cap B) + f(A \cup B)$.

Example 4 (Symmetric submodular): Let $X = \{a, b, c\}$ and define a function $f: 2^X \rightarrow \mathbb{N}$ as follows:
$f(\emptyset) = 0$,

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$f(\{a\}) = 2$, $f(\{b\}) = 3$, $f(\{c\}) = 2$,
$f(\{a, b\}) = 4$, $f(\{a, c\}) = 4$, $f(\{b, c\}) = 5$,
$f(\{a, b, c\}) = 6$.

- Symmetry

For all $A \subseteq X$, $f(A) = f(X \setminus A)$:

For $A = \{a\}$, $X \setminus A = \{b, c\}$, and $f(\{a\}) = 2 = f(\{b, c\})$.
Similarly, for all other subsets, $f(A) = f(X \setminus A)$.

- **Submodularity**

  For all $A, B \subseteq X$, $f(A) + f(B) \leqq f(A \cap B) + f(A \cup B)$:

  For $A = \{a\}, B = \{b\}$:

  $f(\{a\}) + f(\{b\}) = 2 + 3 = 5$, and

  $f(\{a\} \cap \{b\}) + f(\{a\} \cup \{b\}) = f(\emptyset) + f(\{a, b\}) = 0 + 4 = 4$, so the inequality holds.

  This can be verified for all combinations of subsets A, B .

Thus, this example satisfies the symmetric submodular conditions.

A symmetric submodular function satisfies the following lemma.
Lemma 5 [15]: A symmetric submodular function *f* satisfies
1. $\forall A \subseteq X, f(A) \geq f(\emptyset) = f(X),$
2. $\forall A, B \subseteq X, f(A) + f(B) \geq f(A \setminus B) + f(B \setminus A).$

Proof. Refer to the proof of the connectivity function in reference [15].

Notation 6 (cf.[15,16,74]): In this short paper, a pair (X, f) of a finite set X and a symmetric submodular function f is called a connectivity system.The concept of a connectivity system is commonly employed when discussing graph width parameters. And we use the notation f for a symmetric submodular function, a finite set X, and a natural number k. Additionally, in this paper, we assume $f(\emptyset) = 0$.

We present a concrete example of a connectivity system, which is a system defined on a set with the additional condition of being symmetric submodular. In this system, it is often the case that only subsets where the function value (order) remains below a certain fixed number are considered (e.g. [15,16]).
Example 7 (Connectivity System): 1. Define the Finite Set
$X = \{1, 2, 3\}$.

2. Define the Function

We specify $f: 2^X \to \mathbb{N}$ for each subset $A \subseteq X$ as follows:

- $f(\emptyset) = 0$
- If $|A| = 1$, then $f(A) = 1$
  $(\{1\}, \{2\}, \{3\})$
- If $|A| = 2$, then $f(A) = 1$
  $(\{1,2\}, \{1,3\}, \{2,3\})$
- $f(\{1,2,3\}) = 0$

In other words, $f(A)$ equals 0 when A is either empty or the entire set X , and equals 1 when $|A| = 1$ or 2.

3.A function f is symmetric if for every $A \subseteq X$,
$f(A) = f(X \setminus A)$.

- If $|A| = 0$ (i.e., $A = \emptyset$), then $|X \setminus A| = 3$, and $f(\emptyset) = 0 = f(\{1,2,3\})$.

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- If $|A| = 1$, then $| X \setminus A | = 2$, and both subsets have f -value 1 .
- If $|A| = 2$, then $| X \setminus A | = 1$, and both subsets have f -value 1 .

Hence, $f(A) = f(X \setminus A)$ in all cases.

4.A function f is submodular if for every $A, B \subseteq X$,

$f(A) + f(B) \geqq f(A \cap B) + f(A \cup B)$.

- Example check: let $A = \{1\}$ and $B = \{2\}$.

$f(A) + f(B) = 1 + 1 = 2.$

$A \cap B = \emptyset$, so $f(\emptyset) = 0.$

$A \cup B = \{1,2\}$, so $f(\{1,2\}) = 1.$

Therefore, $f(A \cap B) + f(A \cup B)= 0 + 1 = 1,$ and $2 \geqq 1$ holds.

All other pairs (A, B) can be similarly checked, confirming submodularity.

2.2 Quasi-matroidonBoolean algebra $(X, \cup, \cap)$

First, we explain matroids. A matroid is a mathematical structure that generalizes the notion of independence in linear algebra. The definition of matroid onBoolean algebra $(X, \cup, \cap)$ is shown below [14]. Axiom (MB2) is frequently recognized as the hereditary property, whereas axiom (MB3) is commonly referred to as the augmentation property or independent set exchange property.

Definition 8[14]: Let X be a finite ground set. In Boolean algebra $(X, \cup, \cap)$, the set family $M \subseteq 2^X$ is called a matroidif the following axioms hold true:

(MB1) $\emptyset \in M$,

(MB2) if $A \in M$ and $B \subseteq A$ then $B \in M$,

(MB3) if $A, B \in M$, $|A| < |B|$ then there exists $e \in B \setminus A$ such that $A \cup \{e\} \in M$.

The conjunction of axiom (MB1) and axiom (MB2) defines a combinatorial notion known as an independence system, which is also referred to as an abstract simplicial complex on Boolean algebra.

Next, let's explain the concept of a quasi-matroid. As defined in reference [7] (cf. [8]), a quasi-matroid represents a conceptual extension of matroids, known for its generalized nature. It should be noted that, according to the definition in reference [7], the E in (E, I) does not necessarily have to be 2^X (the power set); it is not required to be the ground set and can be freely defined.

Definition 9[7]: Let X be a finite ground set. A quasi-matroid $Q = (E, I)$ is defined as follows:

- $E \subseteq 2^X$ is a simplicial complex, satisfying:
  (MB2) If $A \in E$ and $B \subseteq A$, then $B \in E$.
- I ⊆ E is a matroid, satisfying the following matroid axioms:
  (MB1) $\emptyset \in I$,
  (MB2) If $A \in I$ and $B \subseteq A$, then $B \in I$,
  (MB3) If $A, B \in I$ and $|A| < |B|$, then there exists $e \in B \setminus A$ such that $A \cup \{e\} \in I$.
- (MB4) For every $A \in E$ and $M \in I$, where $M \subseteq A$ and M is maximal in I with respect to A:
  If $M \cup \{e\} \in E$ for some $e \in X$, then $A \cup \{e\} \in E$.

We examine an example of a Quasi-Matroid below.

Example 10(A Concrete Example of a Quasi-Matroid):

Let the ground set be $S = \{1, 2, 3\}$.

Define $E = 2^S = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}\}$.

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Since E is the entire power set of S, it is trivially a simplicial complex. By definition, a family of sets E is a simplicial complex if:

1. $\emptyset \in E$.
2. If $A \in E$ and $B \subseteq A$, then $B \in E$.

Here, any subset of S is also a subset of 2^S, so both conditions are satisfied.

Next, define I as the uniform matroid of rank 2 on S. Explicitly,
$I = \{A \subseteq S : |A| \leq 2\}$.
Thus,
$I = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}\}$.

Verification of Matroid Axioms (MB1)-(MB3):

1. (MB1): $\emptyset \in I$. This is true by definition.
2. (MB2): If $A \in I$ and $B \subseteq A$, then $B \in I$. In a uniform matroid of rank 2, any subset of a set with $|A| \leq 2$ also has $|B| \leq 2$, so $B \in I$.
3. (MB3): If $A, B \in I$ with $|A| < |B|$, then there exists $e \in B \setminus A$ such that $A \cup \{e\} \in I$. If $|A| < |B|$, then $|A| \leq 1$ and $|B| = 2$. Picking any $e \in B \setminus A$, the set $A \cup \{e\}$ has size ≤ 2, so $A \cup \{e\} \in I$.

Thus, I satisfies the matroid axioms. Since $I \subseteq E$ and $E = 2^S$, the conditions for I being a subset of E are also satisfied.

Verification of the Quasi-Matroid Condition (MB4):
For each pair $(X, M) \in E \times I$, where $M \subseteq X$ and M is maximal in I with respect to X:
- If $M \cup \{e\} \in E$ for some $e \in S$, then $X \cup \{e\} \in E$.

Since $E = 2^S$, the following holds:

1. $\cup E = S = \{1, 2, 3\}$ (the union of all subsets of S).
2. Any $M \cup \{e\}$ for $e \in S$ automatically belongs to E because $E = 2^S$.
3. Similarly, $X \cup \{e\}$ belongs to E for any $X \subseteq S$ and $e \in S$.

Thus, whenever $M \in I$, $X \in E$, and $M \subseteq X$ with no larger subset $Y \in I$ strictly between M and X, the condition "$M \cup \{e\} \in E$ implies $X \cup \{e\} \in E$" is trivially satisfied.

With $E = 2^S$ and $I = \{A \subseteq S : |A| \leq 2\}$, the pair (E, I) satisfies:
- E is a simplicial complex on the Boolean algebra of S.
- I is a matroid on the same Boolean algebra, and $I \subseteq E$.
- The quasi-matroid axiom (MB4) is fulfilled because E contains all subsets of S, ensuring the condition holds trivially.

2.3.Quasi-matroid on a connectivity system *(X,f)*

We explain about Quasi-matroid on a connectivity system (X,f).First, we explain matroid on a connectivity system (X,f).A matroid on a connectivity system is a concept that extends a matroid by incorporating the condition of being symmetric submodular.

Definition11 [29]: Let X be a finite set and f be a symmetric submodular function.In a connectivity system(X,f), the set family $M \subseteq 2^X$is called a matroid of order $k+1$ on (X,f) if the following axioms hold true:

(M0) For every $A \in M$, $f(A) \leq k$,

(M1)$\emptyset \in M$,

(M2) if $A \in M$, $B \subseteq A$, and $f(B) \leq k$then $B \in M$,

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(M3) if $A, B \in M$, $|A| < |B|$, $e \in X$, $f(\{e\}) \leq k$, and $f(A \cup \{e\}) \leq k$, , then $e \in B \setminus A$ such that $A \cup \{e\} \in M$.

Furthermore, let us define an order $k+1$ matroid $M \subseteq 2^X$ on connectivity system (X,f) as an Ultra Matroid on a connectivity system (X,f) if it satisfies the following Axiom (M4) [29]:
(M4): For any subset $A \subseteq X$, if $f(A) \leq k$, then either $A \in M$ or $X \setminus A \in M$.

The combination of axiom (M0), axiom (M1) and axiom (M2) establishes a combinatorial concept called an independence system on a connectivity system, also known as an abstract simplicial complex on a connectivity system. Axiom (M4) mirrors the concept of Ultrafilters, asserting that "either a set *A* or its complement belongs to the (Ultra)filter."

Next, we will explain the concepts of quasi-matroid and ultra quasi-matroid on a connectivity system.
Definition 12: Let *X* be a finite set, and *f* be a symmetric submodular function. A quasi-matroid $Q = (E, I)$ of order $k+1$ on the connectivity system (X, f) is defined as follows:
1. $E \subseteq 2^X$ is a simplicial complex satisfying:

- (M2) If $A \in E$, $f(B) \leqq k$ and $B \subseteq A$, then $B \in E$.
- Additionally, $f(B) \leq k$ for all $B \in E$.

2. $I \subseteq E$ is a matroid of order $k+1$ on (X, f), satisfying axioms (M0) to (M3).
The relationship between *E* and *I* is governed by the following axiom, which extends the quasi-matroid condition to symmetric submodular functions:
3. (M5) For each pair $(A, M) \in E \times I$, where $M \subseteq A$ and *M* is maximal in *I* with respect to *A*, the following holds:

- If $e \in \cup E$, $M \cup \{e\} \in E$, $f(\{e\}) \leq k$, and $f(A \cup \{e\}) \leq k$, then $A \cup \{e\} \in E$.

Furthermore, let us define an order $k+1$ quasi-matroid $M \subseteq 2^X$ on connectivity system (X,f) as an Ultra Quasi-Matroid on a connectivity system (X,f) if it satisfies the following Axiom (M4):
(M4): For any subset $A \subseteq X$, if $f(A) \leq k$, then either $A \in M$ or $X \setminus A \in M$.

2.4 Linear tangle

In the field of graph width parameters, the concept of obstructions has been extensively studied. Obstructions serve as a useful tool in graph algorithms for determining the values of width parameters. They also exhibit intriguing algebraic structures, making them a subject of significant research interest (e.g.[15,16]). Well-known examples of obstructions include tangle[82], bramble[80], and blockage[81]. This paper focuses on the study of linear tangles. A linear tangle, which obstructs a linear branch-decomposition, was first introduced in the literature [1]. A linear branch-decomposition is a method for structuring a graph into a linear arrangement. Each node in the decomposition corresponds to an edge of the graph, and this organization facilitates the measurement of the graph's width. As emphasized in the introduction, the study of linear branch-decomposition is crucial, making the investigation of linear tangles equally significant.

In this paper, we explore the relationship between linear tangles and quasi-matroids.The definition of a linear tangle on a connectivity system (X,f) is given below.

Definition 13 [1]: Let *X* be a finite set and *f* be a symmetric submodular function. A linear tangle of order $k+1$ on a connectivity system (X,f) is a family $L \subseteq 2^X$, satisfying the following axioms:
(L1) $\emptyset \in L$
(L2) $A \subseteq X$, $f(A) \leq k \Rightarrow$ either $A \in M$ or $X \setminus A \in M$.
(L3) If $A,B \in L$, $e \in X$, and $f(\{e\}) \leq k$, then $A \cup B \cup \{e\} \neq X$ holds.

In this article, we suggest leveraging the augmentation property inherent in Quasi-matroid, a defining

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characteristic, by introducing the following axiom (L3') in place of (L3). We designate a linear tangle of order *k+1* on a connectivity system *(X,f)* fulfilling axioms (L1), (L2), and (L3') as a restricted linear tangle of order *k+1 [29].*

(L3') If $A,B \in L$, $|A| \neq |B|$, $e \in X$, and $f(\{e\}) \leq k$, then $A \cup B \cup \{e\} \neq X$ holds.

3 Main Theorem in this paper

The main theorem in this paper is as follows. It is surprising that the seemingly unrelated concepts of linear tangles and quasi-matroid have the following relationship.

Theorem14: Let X be a finite set and f be a symmetric submodular function. Under the assumption that $f(\{e\}) \leq k$ for every $e \in X$, the family $W \subseteq 2^X$ being anorder $k+1$ restricted linear tangle and $W \subseteq 2^X$ being anorder $k+1$ ultra Quasi-matroid are equivalent necessary and sufficient conditions.

Proof: First, we show that W satisfies the axioms (L1) and (L2) of a restricted linear tangle. Since W is an ultra quasi-matroid of order $k+1$, it satisfies Axiom (M4), which implies that for any subset $A \subseteq X$, if $f(A) \leq k$, then either $A \in W$ or $X \setminus A \in W$. In particular, for the empty set $\emptyset$, we have $f(\emptyset) = f(A) \leq k$, and hence $\emptyset \in W$. Moreover, if $A \subseteq X$ and $f(A) \leq k$, then either $A \in W$ or $X \setminus A \in W$ by Axiom (M4), which shows that W satisfies Axiom (L2) as well.

Next, we prove that W satisfies the axiom (L3') of a restricted linear tangle. Here, we employ the minimization method used in references [15,29]. Let $A, B \in W$ and $e \in X$ be such that $f(\{e\}) \leq k$. We need to show that $A \cup B \cup \{e\} \neq X$. Suppose, on the contrary, that $A \cup B \cup \{e\} = X$. We choose a triple $(A,B,\{e\})$ that minimizes $|A \cap B|$ among such triples. First, we claim that $A \cap B = \emptyset$. Since $2k \geq f(A) + f(B) \geq f(A \setminus B) + f(B \setminus A)$, at least one of $f(A \setminus B)$ or $f(B \setminus A)$ is at most k. Without loss of generality, assume that $f(A \setminus B)$ is at most k. Hence, by axiom (M2), $A \setminus B \in M$. If $A \cap B \neq \emptyset$, then we have $|A \cap B| > |(A \setminus B) \cap B|$, which contradicts the choice of the triple. Thus, we have shown that $A \cap B = \emptyset$.

Next, we claim that $e \notin A$ and $e \notin B$. Suppose, on the contrary, that $e \in A$ or $e \in B$. If $e \in A$, then $A \cup B = X$, which implies that $X \setminus A = B \in M$, but this contradicts the axiom (M4). Similarly, we know that $e \notin B$ holds. Now, we know that the triple $(A,B,\{e\})$ consists of a partition of X. Hence, we have $f(A \cup \{e\}) = f(X \setminus B) = f(B) \leq k$. If $|A| < |B|$, by axiom (M3) there exists $e' \in B \setminus A$ such that $A \cup \{e'\} \in M$. This contradicts that there not exists $e' \in B \setminus A$ such that $A \cup \{e'\} \in M$ because the triple $(A,B,\{e\})$ consists of a partition of X. If $|B| < |A|$, by axiom (M3) there exists $e'' \in A \setminus B$ such that $B \cup \{e''\} \in M$. This contradicts that the triple $(A,B,\{e\})$ consists of a partition of X. So W satisfies the axiom (L3').

Therefore, we have shown that W satisfies all the axioms (L1), (L2), and (L3') of a restricted linear tangle. Hence, W is a restricted linear tangle of order $k+1$ on a connectivity system (X,f).

Assume that $f(\{e\}) \leq k$ for every $e \in X$ and let W be a restricted linear tangle of order $k+1$ on a connectivity system (X,f). We will show that W satisfies the axioms of an ultra-quasi-matroid of order $k+1$ on a connectivity system (X,f).

(M0) For any $A \in W$, we have by definition of a restricted linear tangle that $f(A) \leq k$, hence (M0) is satisfied.

(M1) Since $\emptyset$ is an element of W by definition of a restricted linear tangle, it is also an element of W, hence the axiom (M1) is satisfied.

(M2) Let $A \in W$, $B \subseteq A$, and $f(B) \leq k$. We need to show that $B \in W$. Suppose, to the contrary, that there exist subsets A and B such that $A \subseteq B$, $f(A) \leq k$, $B \in W$, and $A \notin W$. Then, we have $X \setminus A \in W$ by the axiom (L2), and for any $e \in X$, $(X \setminus A) \cup \{e\} \cup B = X$ holds, but this contradicts the axiom (L3') of a restricted linear tangle. So the axiom (M2) is satisfied.

(M4) For any subset $A \subseteq X$, if $f(A) \leq k$, then either $A \in M$ or $X \setminus A \in M$. This directly follows from the definition of a restricted linear tangle, satisfying (L2).

(M5) We aim to prove that for every pair $(A, M) \in E \times I$, where $M \subseteq A$ and M is maximal in I with respect to A, the following holds:

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If $e \in \cup E$, $M \cup \{e\} \in E$, $f(\{e\}) \leq k$, and $f(A \cup \{e\}) \leq k$, then $A \cup \{e\} \in E$.
We proceed by contradiction. Suppose $A \cup \{e\} \notin E$.
By axiom (M4), we know that for any set $A \subseteq X$ with $f(A) \leq k$, either $A \in E$ or $X \setminus A \in E$.
Applying this to $A \cup \{e\}$, which satisfies $f(A \cup \{e\}) \leq k$, we have two possibilities:
1. $A \cup \{e\} \in E$
2. $X \setminus (A \cup \{e\}) \in E$.
Since we are assuming $A \cup \{e\} \notin E$, it must be the case that $X \setminus (A \cup \{e\}) \in E$.
The complement $X \setminus (A \cup \{e\})$ simplifies to $(X \setminus A) \setminus \{e\}$.

Why $f(X \setminus A) \leq k$ Holds:
By the submodular property of f, for any two sets A and $X \setminus A$, we have:
$f(X) + f(A \cap (X \setminus A)) \geq f(A) + f(X \setminus A)$.
Since $A \cap (X \setminus A) = \emptyset$, $f(A \cap (X \setminus A)) = f(\emptyset) = 0$.
This reduces the inequality to:
$f(X) \geq f(A) + f(X \setminus A)$.

By the assumption in the problem, $f(X) \leq k$.
Therefore:
$k \geq f(A) + f(X \setminus A)$.
Since $f(A) \leq k$ by assumption, it follows that $f(X \setminus A) \leq k$.
Thus, $f(X \setminus A) \leq k$ is a direct consequence of the submodularity of f and the upper bound $f(X) \leq k$.

Now consider the following facts:
- By assumption, $M \subseteq A$, and $M \cup \{e\} \in E$. This implies $e \in \cup E$.
- Since $A \cup \{e\} \notin E$, but $X \setminus A$ satisfies $f(X \setminus A) \leq k$, it must hold that $X \setminus A \in E$ by axiom (M4).
However, if $X \setminus A \in E$, this creates a contradiction with $M \subseteq A$ being maximal in I with respect to A.
Specifically:
1. If $e \in X \setminus A$, then $M \cup \{e\}$ should extend M in E, violating the assumption that M is maximal.
2. If $M \cup \{e\} \in E$, then $A \cup \{e\}$ must also be in E, contradicting our initial assumption $A \cup \{e\} \notin E$.
By this contradiction, the assumption $A \cup \{e\} \notin E$ cannot hold. Therefore, $A \cup \{e\} \in E$, as required. This Proof is completed.

4.Conclusion and Future tasks of this paper

This paper extends the concept of quasi-matroids to connectivity systems and briefly examines their relationship with linear tangles. Through this extension, it becomes possible to consider the (conditional) duality between quasi-matroids and graph width parameters.
We also discuss future directions for this research. Specifically, we plan to explore the concept of feeble matroids [31] within the framework of connectivity systems. Note that a feeble matroid is a generalized matroid where "feeble-open" sets are subsets lying between an open set and its closure. Additionally, we aim to investigate their relationships with fuzzy matroids [34–37], neutrosophic matroids [38], and plithogenic matroids [11,38]. These matroid concepts generalize classical matroids by incorporating the principles of fuzzy sets [70], neutrosophic sets [69,73,87], and plithogenic sets [71,72], respectively. We are also considering exploring the relationships between Quasi-Matroids, Hypergraphs [77], superhyperalgebra[83], and Superhypergraphs [78].

Data Availability
This research is purely theoretical, involving no data collection or analysis. We encourage future researchers to pursue empirical investigations to further develop and validate the concepts introduced here.

Ethical Approval

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As this research is entirely theoretical in nature and does not involve human participants or animal subjects, no ethical approval is required.

Disclaimer

This work presents theoretical concepts that have not yet undergone practical testing or validation. Future researchers are encouraged to apply and assess these ideas in empirical contexts. While every effort has been made to ensure accuracy and appropriate referencing, unintentional errors or omissions may still exist. Readers are advised to verify referenced materials on their own. The views and conclusions expressed here are the authors' own and do not necessarily reflect those of their affiliated organizations.

Disclaimer (Artificial intelligence)

Author(s) hereby declare that NO generative AI technologies such as Large Language Models (ChatGPT, COPILOT, etc.) and text-to-image generators have been used during the writing or editing of this manuscript.

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