A Comprehensive Study on Stability and Data Dependency for a New Jungck-Type Iteration

ABSTRACT

This study presents a novel Jungck-type iterative algorithm designed to approximate coincidence points under contractive conditions. The research explores the iterative algorithm's strong convergence, stability, and data dependency results. Numerical experiments show that the introduced Jungck-type iterative approach achieves faster convergence than other Jungck-type methods previously documented in the literature.

Keywords: Data Dependence, Common Fixed Point, Coincidence Point, and Stability.

MSC: 47H09, 47H10, 54H25

1. INTRODUCTION AND PRELIMINARIES

The Jungck-type fixed-point iteration extends classical fixed-point theory to address scenarios involving multiple mappings with defined interrelations. This approach demonstrates particular efficacy in addressing problems in which the interaction between two operators plays a crucial role, ensuring convergence to a common fixed point under specific contractive conditions. By expanding the scope of the fixed-point theory to encompass more complex and hybrid systems, Jungck-type iterations have demonstrated significant applicability in domains such as optimization, economic modeling, and coupled differential equations. Recently, substantial advancements have been achieved in both the theoretical analysis and numerical exploration of various explicit iterative techniques [1–9].

Consider $(W,\|.\|)$ be a Banach space, V an arbitrary set, $J,H: V \to W$ be arbitrary non-self mapping with $H(V) \subseteq J(V)$, $J(V)$ is a complete subspace of W, and $a_0 \in V$.
For $\alpha_n \in [0,1]$, Singh et al. [4] defined the Jungck-Mann iterative scheme as follows:

$$Ja_{n+1} = (1-\alpha_n)Ja_n + \alpha_n Ha_n. \tag{1.1}$$

Olantinwo [5] defined the Jungck-Noor (J-Itr1) iteration scheme as

$$\begin{cases} Jc_n = \left(1-\mu_n\right) Ja_n + \mu_n Ha_n, \\ Jb_n = \left(1-\delta_n\right) Ja_n + \delta_n Hc_n, \\ Ja_{n+1} = \left(1-\lambda_n\right) Ja_n + \lambda_n Hb_n, \end{cases} \tag{1.2}$$

where $\{\mu_n\}_{n=0}^{\infty}, \{\delta_n\}_{n=0}^{\infty}, \{\lambda_n\}_{n=0}^{\infty} \subseteq [0,1]$.

A new three-step iteration process, known as the Jungck-Khan (J-Itr2) method, was proposed by Khan et al. [6] as follows

$$\begin{cases} Jc_n = \left(1-\mu_n\right) Ja_n + \mu_n Ha_n, \\ Jb_n = \left(1-\delta_n-\lambda_n\right) Ja_n + \delta_n Hc_n + \lambda_n Ha_n, \\ Ja_{n+1} = \left(1-\theta_n-\gamma_n\right) Ja_n + \theta_n Hb_n + \gamma_n Ha_n, \end{cases} \tag{1.3}$$

where $\{\mu_n\}_{n=0}^{\infty}, \{\delta_n\}_{n=0}^{\infty}, \{\lambda_n\}_{n=0}^{\infty}, \{\theta_n\}_{n=0}^{\infty}$, and $\{\gamma_n\}_{n=0}^{\infty} \subseteq [0,1]$, satisfying $\{\delta_n\}_{n=0}^{\infty} + \{\lambda_n\}_{n=0}^{\infty}$, $\{\theta_n\}_{n=0}^{\infty} + \{\gamma_n\}_{n=0}^{\infty} \subseteq [0,1]$.

To demonstrate the strong convergence of both the Jungck-Mann and Jungck-Ishikawa iterative process, Olatinwo and Imoru [7] introduced the following contractive definition

$$\|Hu - Hv\| \le 2\beta\|Ju - Hu\| + \beta\|Ju - Jv\| \quad \forall u, v \in V, \quad 0 \le \beta < 1. \tag{1.4}$$

In [8, 9], building on result (1.4), Olatinwo showed the stability and strong convergence of various iterative techniques. This was achieved by employing a more comprehensive contractive condition, which is represented as

$$\|Hu - Hv\| \le \psi\left(\|Ju - Hu\|\right) + \beta\|Ju - Jv\| \quad \forall u, v \in V, \quad 0 \le \beta < 1, \tag{1.5}$$

where the monotonically increasing function $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ satisfying $\psi(0) = 0$.

Definition 1 ([10]). Consider V a non-empty set and $J, H: V \to W$ be two mappings. A coincidence point exists when $J(v) = f = H(v)$ for some v in V, and the associated value f is referred to as the point of coincidence or coincidence value of J and H. If $J(v) = v = H(v)$ for an element v in V then v is called the common fixed point of J and H. The pair (J, H) commutes at the coincidence point, and is said to be weakly compatible.

Definition 2 ([11]). Consider the operators $J, H : V \to W$ such that $H(V) \subseteq J(V)$ and $Jv = f = Hv$, where f is a point of coincidence of J and H. Suppose $a_0 \in V$ is the initial approximation, g is some function, and $\{Ja_n\}_{n=0}^{\infty} \subset W$, be the sequence converges to *f*, generated by an iterative procedure

$$Ja_{n+1} = g(H, a_n) \ , n = 0, 1, 2, \ . \ . \ .$$

Let $\{Jh_n\}_{n=0}^{\infty} \subset W$ be an arbitrary sequence. Set

$$\rho_n = \left\|Jh_{n+1} - g\left(\mathrm{H}, \mathrm{h}_n\right)\right\|, \quad n = 0, 1, 2, \ldots.$$

Then, the iterative procedure $Ja_{n+1} = g(H, a_n)$ is said to be stable iff $\lim_{n\to\infty} \rho_n = 0$ implies $\lim_{n\to\infty} Jh_n = f$.

Lemma 1 ([12]). If $\beta \in [0,1)$ and sequence of positive numbers $\{\omega_n\}_{n=0}^{\infty}$ with $\lim_{n\to\infty} \omega_n = 0$, then for every sequence of positive numbers $\{u_n\}_{n=0}^{\infty}$, which satisfies

$$u_{n+1} \le \beta u_n + \omega_n, \quad n = 0,1,2, \ldots$$

one has $\lim_{n\to\infty} u_n = 0$.

Definition 3 ([6]). Assume an arbitrary set V and the non-self mapping pairs $(J,H),(J_1,H_1): V \to W$ with $H(V) \subseteq J(V)$ and $H_1(V) \subseteq J_1(V)$. If for fixed $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$, and for all $v \in V$, one gets

$$m(Hv, H_1 v) \le \varepsilon_1,$$
$$m(Jv, J_1 v) \le \varepsilon_2,$$

then, (J_1, H_1) is said to be an approximate mapping pair of (J,H).

This paper presents a novel Jungck-type iteration algorithm for determining the coincidence points of contractive-type mappings. The proposed method exhibits an enhanced convergence rate and stability compared to the already existing Jungck-type iteration approaches. Additionally, we have derived data dependence findings for our newly introduced process.

We now define our novel Jungck-type iteration (New-Itr) scheme, as follows:
For $a_0 \in V$, the sequence $\{Ja_n\}_{n=0}^{\infty}$ in W is given by

$$\begin{cases} Jd_n = \dfrac{s}{s+1} Ja_n + \dfrac{1}{s+1} Ha_n, \\ Jc_n = \dfrac{s+1}{s} Ha_n - \dfrac{1}{s} Hd_n, \\ Jb_n = \dfrac{s'}{s'+1} Jc_n + \dfrac{1}{s'+1} Hc_n, \\ Ja_{n+1} = Hb_n \end{cases} \tag{1.6}$$

where $s \ge 1$ and $s' > 0$ are real numbers.

Remark. If we put $V = W$, and J is identity operator in (1.6), then we get the following iteration outlined in [13, equation(18)]:

$$\begin{cases} b_n = \dfrac{1}{s}\left((s+1)Ha_n - H\left(\dfrac{sa_n + Ha_n}{s+1} \right) \right), \\ a_{n+1} = H\left(\dfrac{s'b_n + Hb_n}{s'+1} \right), \end{cases} \tag{1.7}$$

where $s \ge 1$ and $s' > 0$ are real numbers.

2. MAIN RESULTS

Theorem 2.1. Consider $(W, \|.\|)$ a Banach space and V be an arbitrary set. Suppose $J, H: V \to W$ be non-self mappings with $H(V) \subseteq J(V)$, and $J(V)$ is a complete subspace of W. Suppose J and H have a coincidence point v, (i.e., $Jv = f = Hv$) and also J, H satisfies the contractive condition (1.5) with $\beta < \frac{1}{\sqrt{3}}$. Let $\{Ja_n\}_{n=0}^{\infty}$ be the new iteration process defined by (1.6). Then, $\{Ja_n\}_{n=0}^{\infty}$ is strongly converges to f. Moreover, (J,H) has a unique common fixed point f provided that $V = W$ and H, J are weakly compatible.

Proof. We shall prove that $\lim_{n\to\infty} Ja_n = f$. Using (1.6), one has

$$\begin{aligned}\|Jd_n - f\| &= \left\|\frac{s}{s+1}Ja_n + \frac{1}{s+1}Ha_n - f\right\|,\\ &\le \frac{s}{s+1}\|Ja_n - f\| + \frac{1}{s+1}\|Ha_n - Hv\|,\\ &\le \frac{s}{s+1}\|Ja_n - f\| + \frac{1}{s+1}\left[\psi\left(\|Jv - Hv\|\right) + \beta\|Ja_n - Jv\|\right],\\ &= \frac{s+\beta}{s+1}\|Ja_n - f\|. \end{aligned} \tag{2.1}$$

Now,

$$\begin{aligned}\|Jc_n - f\| &= \left\|\frac{s+1}{s}Ha_n - \frac{Hd_n}{s} - f\right\|,\\ &= \left\|\frac{s+1}{s}(Ha_n - f) - \frac{1}{s}(Hd_n - f)\right\|,\\ &\le \frac{s+1}{s}\|Ha_n - Hv\| + \frac{1}{s}\|Hd_n - Hv\|,\\ &\le \frac{s+1}{s}\left[\psi\left(\|Jv - Hv\|\right) + \beta\|Ja_n - Jv\|\right] + \frac{\beta}{s}\|Jd_n - Jv\|,\\ &= \frac{\beta(s+1)}{s}\|Ja_n - f\| + \frac{\beta}{s}\|Jd_n - f\|. \end{aligned} \tag{2.2}$$

Substituting (2.1) in (2.2), we obtain

$$\begin{aligned}\|Jc_n - f\| &\le \frac{\beta(s+1)}{s}\|Ja_n - f\| + \frac{\beta}{s}\left(\frac{s+\beta}{s+1}\right)\|Ja_n - f\|,\\ &= \beta\left(1 + \frac{1}{s} + \frac{s+\beta}{s(s+1)}\right)\|Ja_n - f\|. \end{aligned} \tag{2.3}$$

Also,

$$\begin{aligned}\|Jb_n - f\| &= \left\|\frac{s'}{s'+1}Jc_n + \frac{1}{s'+1}Hc_n - f\right\|,\\ &\le \frac{s'}{s'+1}\|Jc_n - f\| + \frac{1}{s'+1}\|Hc_n - Hv\|,\\ &\le \frac{s'}{s'+1}\|Jc_n - f\| + \frac{1}{s'+1}\left[\psi\left(\|Jv - Hv\|\right) + \beta\|Jc_n - Jv\|\right],\\ &= \frac{s'+\beta}{s'+1}\|Jc_n - f\|. \end{aligned} \tag{2.4}$$

And;

$$\begin{aligned}\|Ja_{n+1} - f\| &= \|Hb_n - Hv\|,\\ &\le \beta\|Jb_n - f\|. \end{aligned} \tag{2.5}$$

By using (2.3), (2.4); (2.5) yields

$$\|Ja_{n+1} - f\| \le \beta^2 \left(\frac{s'+\beta}{s'+1}\right)\left(1+\frac{1}{s}+\frac{s+\beta}{s(s+1)}\right)\|Ja_n - f\|,$$

$$\dots$$

$$\le \left(\beta^2 \left(\frac{s'+\beta}{s'+1}\right)\left(1+\frac{1}{s}+\frac{s+\beta}{s(s+1)}\right)\right)^{n+1} \|Ja_0 - f\|. \qquad (2.6)$$

Since, $s \ge 1$, $s' > 0$; and $0 \le \beta < \frac{1}{\sqrt{3}}$, we have

$$0 < \frac{s+\beta}{s+1} < 1; \text{ and } 0 < 1+\frac{1}{s}+\frac{s+\beta}{s(s+1)} < 3.$$

Hence,

$$\beta^2 \left(\frac{s'+\beta}{s'+1}\right)\left(1+\frac{1}{s}+\frac{s+\beta}{s(s+1)}\right) < 1;$$

and therefore

$$\left(\beta^2 \left(\frac{s'+\beta}{s'+1}\right)\left(1+\frac{1}{s}+\frac{s+\beta}{s(s+1)}\right)\right)^{n+1} \to 0 \; as \; n \to \infty.$$

Therefore, (2.6) implies that $\lim_{n\to\infty}\|Ja_{n+1} - f\| = 0.$

Hence, $\{Ja_n\}_{n=0}^{\infty}$ converges to f.

To demonstrate that f is the unique common fixed point of H and J, assume that v and v_1 are coincidence points of H and J such that $Hv = f = Jv$ and $Hv_1 = f_1 = Jv_1$, where f_1 is another point of coincidence of H and J. Applying the contractive condition (1.5), we have:

$$\|f - f_1\| = \|Hv - Hv_1\| \le \beta\|Jv - Jv_1\| < \|f - f_1\|,$$

this leads to a contradiction. Hence, $f_1 = f$ means point of coincidence is unique. Given that H and J are weakly compatible, we have

$$Hf = HHv = HJv = JHv = Jf.$$

which implies $Jf = f = Hf$. This verifies that H and J have f as the point of coincidence. The uniqueness of the point of coincidence indicates that f is a unique common fixed point of H, J.

Theorem 2.2. Let J and H be the same as in Theorem 2.1, and $\{Ja_n\}_{n=0}^{\infty}$ be the iteration scheme generated by (1.6) converging to f. Then $\{Ja_n\}_{n=0}^{\infty}$ is (J, H)- stable.

Proof. Suppose $\{Jp_n\}_{n=0}^{\infty} \subset W$ be an arbitrary sequence, such that

$$\mu_n = \left\| Jp_{n+1} - Hq_n \right\|,$$

where $Jq_n = \frac{s'}{s'+1} Jr_n + \frac{1}{s'+1} Hr_n$, $Jr_n = \frac{s+1}{s} Hp_n - \frac{1}{s} Hh_n$, and $Jh_n = \frac{s}{s+1} Jp_n + \frac{1}{s+1} Hp_n$.

To prove that the iterative scheme (1.6) is (J,H)-stable; we have to show that $\lim_{n\to\infty} \mu_n = 0$ if and only if $\lim_{n\to\infty} Jp_n = f$.

Let $\lim_{n\to\infty} \mu_n = 0$. We have

$$\begin{aligned} \left\| Jp_{n+1} - f \right\| &= \left\| Jp_{n+1} - Hq_n + Hq_n - f \right\|, \\ &\le \left\| Jp_{n+1} - Hq_n \right\| + \left\| Hq_n - f \right\|, \\ &= \mu_n + \left\| Hq_n - f \right\|, \\ &\le \mu_n + \beta \left\| Jq_n - f \right\|. \end{aligned} \tag{2.7}$$

$$\begin{aligned} \left\| Jq_n - f \right\| &= \left\| \frac{s'}{s'+1} Jr_n + \frac{1}{s'+1} Hr_n - f \right\|, \\ &\le \frac{s'}{s'+1} \left\| Jr_n - f \right\| + \frac{1}{s'+1} \left\| Hr_n - Hv \right\|, \\ &\le \frac{s'}{s'+1} \left\| Jr_n - f \right\| + \frac{1}{s'+1} \left[\psi\left(\left\| Jv - Hv \right\| \right) + \beta \left\| Jr_n - Jv \right\| \right], \\ &= \frac{s'+\beta}{s'+1} \left\| Jr_n - f \right\|. \end{aligned} \tag{2.8}$$

With ease; similar to estimate (2.3); one can get

$$\left\| Jr_n - f \right\| \le \beta \left(1 + \frac{1}{s} + \frac{s+\beta}{s(s+1)} \right) \left\| Jp_n - f \right\|. \tag{2.9}$$

Substituting (2.8) and (2.9); estimate (2.7) yields

$$\left\| Jp_{n+1} - f \right\| \le \mu_n + \beta^2 \left(\frac{s'+\beta}{s'+1} \right) \left(1 + \frac{1}{s} + \frac{s+\beta}{s(s+1)} \right) \left\| Jp_n - f \right\|. \tag{2.10}$$

As $\beta^2 \left(\frac{s'+\beta}{s'+1} \right) \left(1 + \frac{1}{s} + \frac{s+\beta}{s(s+1)} \right) < 1$; using Lemma 1; inequality (2.10) yields $\lim_{n\to\infty} Jp_n = f$.

Conversely; let $\lim_{n\to\infty} Jp_n = f$.

$$\begin{aligned}\mu_n &= \|Jp_{n+1} - Hq_n\|,\\ &= \|Jp_{n+1} - f + f - Hq_n\|,\\ &\le \|Jp_{n+1} - f\| + \|Hq_n - f\|,\\ &\le \|Jp_{n+1} - f\| + \beta^2\left(\frac{s'+\beta}{s+1}\right)\left(1+\frac{1}{s}+\frac{s+\beta}{s(s+1)}\right)\|Jp_n - f\|. \end{aligned} \tag{2.11}$$

By taking limit as $n \to \infty$ of both sides of (2.11), we get $\lim_{n\to\infty}\mu_n = 0$.

Theorem 2.3. Let (J,H) be the same as in Theorem 2.1, and (J_1,H_1) be an approximate mapping pair of (J,H) according to Definition 3 such that $J_1(V)$ is complete in W. Suppose $J_1v = f_1 = H_1v$. Consider $\{Ja_n\}_{n=0}^{\infty}$ be the iteration scheme generated by (1.6) converging to f and $\{J_1e_n\}_{n=0}^{\infty}$ be the sequence defined by

$$\begin{cases} J_1e_{n+1} = H_1g_n,\\ J_1g_n = \dfrac{s'}{s'+1}J_1h_n + \dfrac{1}{s'+1}H_1h_n,\\ J_1h_n = \dfrac{s+1}{s}H_1e_n - \dfrac{1}{s}H_1i_n,\\ J_1i_n = \dfrac{s}{s+1}J_1e_n + \dfrac{1}{s+1}H_1e_n. \end{cases} \tag{2.12}$$

Suppose that $\{J_1e_n\}_{n=0}^{\infty}$ converges to f_1. Then; we have

$$\|f - f_1\| \le \frac{6(\varepsilon_1 + \beta\varepsilon_2)}{1-3\beta^2}.$$

Proof. Using (1.6) and (2.12), we have

$$\begin{aligned}\|J_1e_{n+1} - Ja_{n+1}\| &= \|H_1g_n - Hb_n\|,\\ &\le \|H_1g_n - Hg_n\| + \|Hg_n - Hb_n\|,\\ &\le \varepsilon_1 + \psi(\|Jb_n - Hb_n\|) + \beta\|Jg_n - Jb_n\|,\\ &\le \varepsilon_1 + \psi(\|Jb_n - Hb_n\|) + \beta\|Jg_n - J_1g_n\| + \beta\|J_1g_n - Jb_n\|,\\ &\le \varepsilon_1 + \psi(\|Jb_n - Hb_n\|) + \beta\varepsilon_2 + \beta\|J_1g_n - Jb_n\|. \end{aligned} \tag{2.13}$$

Now,

$$\begin{aligned}\|J_1g_n - Jb_n\| &= \left\|\frac{s'}{s'+1}J_1h_n + \frac{1}{s'+1}H_1h_n - \frac{s'}{s'+1}Jc_n - \frac{1}{s'+1}Hc_n\right\|,\\ &\le \frac{s'}{s'+1}\|J_1h_n - Jc_n\| + \frac{1}{s'+1}\|H_1h_n - Hc_n\|,\\ &\le \frac{s'}{s'+1}\|J_1h_n - Jc_n\| + \frac{1}{s'+1}\|H_1h_n - Hh_n\| + \frac{1}{s'+1}\|Hh_n - Hc_n\|,\\ &\le \frac{s'}{s'+1}\|J_1h_n - Jc_n\| + \frac{1}{s'+1}\varepsilon_1 + \frac{1}{s'+1}\psi(\|Jc_n - Hc_n\|) + \frac{\beta}{s'+1}\|Jh_n - Jc_n\|,\\ &\le \frac{s'}{s'+1}\|J_1h_n - Jc_n\| + \frac{\varepsilon_1}{s'+1} + \frac{1}{s'+1}\psi(\|Jc_n - Hc_n\|) + \frac{\beta}{s'+1}(\|Jh_n - J_1h_n\| + \|J_1h_n - Jc_n\|),\\ &\le \frac{\varepsilon_1}{s'+1} + \frac{\beta\varepsilon_2}{s'+1} + \frac{1}{s'+1}\psi(\|Jc_n - Hc_n\|) + \frac{s'+\beta}{s'+1}\|J_1h_n - Jc_n\|. \end{aligned} \tag{2.14}$$

Also,

$$\|J_1h_n - Jc_n\| = \left\|\frac{s+1}{s}H_1e_n - \frac{1}{s}H_1i_n - \frac{s+1}{s}Ha_n + \frac{1}{s}Hd_n\right\|,$$
$$\leq \frac{s+1}{s}\|H_1e_n - Ha_n\| + \frac{1}{s}\|H_1i_n - Hd_n\|,$$
$$\leq \frac{s+1}{s}\left(\|H_1e_n - He_n\| + \|He_n - Ha_n\|\right) + \frac{1}{s}\left(\|H_1i_n - Hi_n\| + \|Hi_n - Hd_n\|\right),$$
$$\leq \frac{\varepsilon_1(s+1)}{s} + \frac{s+1}{s}\psi\left(\|Ja_n - Ha_n\|\right) + \frac{\beta(s+1)}{s}\|Je_n - Ja_n\| + \frac{\varepsilon_1}{s} + \frac{1}{s}\psi\left(\|Jd_n - Hd_n\|\right) + \frac{\beta}{s}\|Ji_n - Jd_n\|,$$
$$\leq \frac{\varepsilon_1(s+2)}{s} + \frac{s+1}{s}\psi\left(\|Ja_n - Ha_n\|\right) + \frac{1}{s}\psi\left(\|Jd_n - Hd_n\|\right) + \frac{\beta(s+1)}{s}\left(\|Je_n - J_1e_n\| + \|J_1e_n - Ja_n\|\right)$$
$$+ \frac{\beta}{s}\left(\|Ji_n - J_1i_n\| + \|J_1i_n - Jd_n\|\right),$$
$$\leq \frac{\varepsilon_1(s+2)}{s} + \frac{s+1}{s}\psi\left(\|Ja_n - Ha_n\|\right) + \frac{1}{s}\psi\left(\|Jd_n - Hd_n\|\right) + \frac{\beta(s+2)}{s}\varepsilon_2 + \frac{\beta(s+1)}{s}\|J_1e_n - Ja_n\| + \frac{\beta}{s}\|J_1i_n - Jd_n\|. \tag{2.15}$$

And,

$$\|J_1i_n - Jd_n\| = \left\|\frac{s}{s+1}J_1e_n + \frac{1}{s+1}H_1e_n - \frac{s}{s+1}Ja_n - \frac{1}{s+1}Ha_n\right\|,$$
$$\leq \frac{s}{s+1}\|J_1e_n - Ja_n\| + \frac{1}{s+1}\|H_1e_n - Ha_n\|,$$
$$\leq \frac{s}{s+1}\|J_1e_n - Ja_n\| + \frac{1}{s+1}\left(\|H_1e_n - He_n\| + \|He_n - Ha_n\|\right),$$
$$\leq \frac{s}{s+1}\|J_1e_n - Ja_n\| + \frac{\varepsilon_1}{s+1} + \frac{1}{s+1}\psi\left(\|Ja_n - Ha_n\|\right) + \frac{\beta}{s+1}\|Je_n - Ja_n\|,$$
$$\leq \frac{\varepsilon_1}{s+1} + \frac{1}{s+1}\psi\left(\|Ja_n - Ha_n\|\right) + \frac{s}{s+1}\|J_1e_n - Ja_n\| + \frac{\beta}{s+1}\left(\|Je_n - J_1e_n\| + \|J_1e_n - Ja_n\|\right),$$
$$\leq \frac{\varepsilon_1}{s+1} + \frac{1}{s+1}\psi\left(\|Ja_n - Ha_n\|\right) + \frac{\beta}{s+1}\varepsilon_2 + \frac{s+\beta}{s+1}\|J_1e_n - Ja_n\|. \tag{2.16}$$

By combining (2.13), (2.14), (2.15) and (2.16), we get

$$\|J_1e_{n+1} - Ja_{n+1}\| \leq \frac{\beta^2(s'+\beta)}{s(s'+1)}\left(s+1+\frac{s+\beta}{s+1}\right)\|J_1e_n - Ja_n\| + \varepsilon_1 + \frac{\beta}{s'+1}\varepsilon_1 + \frac{\beta(s'+\beta)(s+2)}{s(s'+1)}\varepsilon_1 + \frac{\beta^2(s'+\beta)}{s(s+1)(s'+1)}\varepsilon_1$$
$$+ \beta\varepsilon_2 + \frac{\beta^2}{s'+1}\varepsilon_2 + \frac{\beta^2(s'+\beta)(s+2)}{s(s'+1)}\varepsilon_2 + \frac{\beta^3(s'+\beta)}{s(s+1)(s'+1)}\varepsilon_2 + \psi\left(\|Jb_n - Hb_n\|\right) + \frac{\beta}{s'+1}\psi\left(\|Jc_n - Hc_n\|\right)$$
$$+ \frac{\beta(s'+\beta)}{s(s'+1)}\left(s+1+\frac{\beta}{s+1}\right)\psi\left(\|Ja_n - Ha_n\|\right) + \frac{\beta(s'+\beta)}{s(s'+1)}\psi\left(\|Jd_n - Hd_n\|\right). \tag{2.17}$$

Since, $s \geq 1$, $s' > 0$, and $\beta < \frac{1}{\sqrt{3}}$, then (2.17) yields

$$\|J_1e_{n+1} - Ja_{n+1}\| \leq 3\beta^2\|J_1e_n - Ja_n\| + 6(\varepsilon_1 + \beta\varepsilon_2) + \psi\left(\|Jb_n - Hb_n\|\right) + \psi\left(\|Jc_n - Hc_n\|\right)$$
$$+\left(s+1+\frac{\beta}{s+1}\right)\psi\left(\|Ja_n - Ha_n\|\right) + \psi\left(\|Jd_n - Hd_n\|\right). \tag{2.18}$$

Now,

$$\|Ja_n - Ha_n\| \leq \|Ja_n - f\| + \|f - Ha_n\|,$$
$$\leq \|Ja_n - f\| + \psi\left(\|Jv - Hv\|\right) + \beta\|Ja_n - Jv\|,$$
$$= (1+\beta)\|Ja_n - f\|. \tag{2.19}$$

Given that, $\lim_{n\to\infty}\|Ja_n - f\| = 0$, equation (2.19) yields $\lim_{n\to\infty}\|Ja_n - Ha_n\| = 0$, which subsequently leads to $\lim_{n\to\infty}\psi\left(\|Ja_n - Ha_n\|\right) = 0.$

Also,

$$
\begin{aligned}
\|Jd_n - Hd_n\| &\le \|Jd_n - f\| + \|f - Hd_n\|,\\
&\le (1+\beta)\|Jd_n - f\|,\\
&\le (1+\beta)\left(\frac{s}{s+1}\|Ja_n - f\| + \frac{\beta}{s+1}\|Ja_n - f\|\right),\\
&\le (1+\beta)\left(\frac{s+\beta}{s+1}\right)\|Ja_n - f\|.
\end{aligned} \tag{2.20}
$$

Since, $\lim_{n\to\infty}\|Ja_n - f\| = 0$, equation (2.20) yields $\lim_{n\to\infty}\|Jd_n - Hd_n\| = 0$, which further implies $\lim_{n\to\infty}\psi\left(\|Jd_n - Hd_n\|\right) = 0.$

Similarly,

$$
\begin{aligned}
\|Jc_n - Hc_n\| &\le \|Jc_n - f\| + \|f - Hc_n\|,\\
&\le (1+\beta)\|Jc_n - f\|,\\
&\le (1+\beta)\left(\frac{s+1}{s}\|Ha_n - f\| + \frac{1}{s}\|f - Hd_n\|\right),\\
&\le (1+\beta)\left(\frac{\beta(\mathrm{s}+1)}{s}\|Ja_n - f\| + \frac{\beta}{s}\|Jd_n - f\|\right),\\
&\le (1+\beta)\left(\frac{\beta(\mathrm{s}+1)}{s} + \frac{\beta}{s}\left(\frac{s+\beta}{s+1}\right)\right)\|Ja_n - f\|.
\end{aligned} \tag{2.21}
$$

Given that, $\lim_{n\to\infty}\|Ja_n - f\| = 0$, equation (2.21) yields $\lim_{n\to\infty}\|Jc_n - Hc_n\| = 0$, which further implies $\lim_{n\to\infty}\psi\left(\|Jc_n - Hc_n\|\right) = 0.$

And,

$$
\begin{aligned}
\|Jb_n - Hb_n\| &\le \|Jb_n - f\| + \|f - Hb_n\|,\\
&\le (1+\beta)\|Jb_n - f\|,\\
&\le (1+\beta)\frac{\beta}{s}\left(\frac{\mathrm{s}'+\beta}{\mathrm{s}'+1}\right)\left(s+1+\frac{s+\beta}{s+1}\right)\|Ja_n - f\|.
\end{aligned} \tag{2.22}
$$

Since, $\lim_{n\to\infty}\|Ja_n - f\| = 0$, equation (2.22) yields $\lim_{n\to\infty}\|Jb_n - Hb_n\| = 0$, which subsequently leads to $\lim_{n\to\infty}\psi\left(\|Jb_n - Hb_n\|\right) = 0.$

As $\lim_{n\to\infty} Ja_n = f$ and $\lim_{n\to\infty} J_1 e_n = f_1$.

Taking limit $n \to \infty$ and using above facts, (2.18) yields

$$\|f - f_1\| \le 3\beta^2\|f - f_1\| + 6(\varepsilon_1 + \beta\varepsilon_2)$$

which further gives

$$\|f - f_1\| \le \frac{6(\varepsilon_1 + \beta\varepsilon_2)}{1 - 3\beta^2}.$$

To check out the numerical feasibility of Theorem 2.3, the following example has been provided.

Example 2.4. Consider $V = [0,1]$ and $H, J : V \to W$ is defined by $H(v) = \frac{5+v}{7}$ and $T(v) = \frac{6v}{7}$ satisfy the contractive condition (5) when $\psi(v) = v^{\frac{1}{2}}$ and $\beta \geq 0.2$. Define operators H_1 and J_1 as $H_1(v) = \frac{4-v}{5}$ and $J_1(v) = \frac{4v}{5}$. It is clear that $H(v) \subseteq J(v), H_1(v) \subseteq J_1(v), H(1) = J(1) = \frac{6}{7} = f$ and $H_1\left(\frac{4}{5}\right) = J_1\left(\frac{4}{5}\right) = \frac{16}{25} = f_1$. We have $\max_{v \in V}|H - H_1| = 0.257 = \varepsilon_1$ (say) and $\max_{v \in V}|J - J_1| = 0.857 = \varepsilon_2$ (say). Obviously, (H_1, J_1) is an approximate mapping pair of (H, J). With initial approximation $v_0 = 0.5$ and $s = 9, s' = \frac{1}{10}$, the iteratives schemes $\{Ja_{n+1}\}_{n=0}^{\infty}$ and $\{J_1 a_{n+1}\}_{n=0}^{\infty}$ converges to $\frac{6}{7}$ and $\frac{16}{25}$, respectively as shown in Table 1 and the graphical convergence is shown in Fig. 1. The values of the operators $H(v)$, $H_1(v)$, $J(v)$, and $J_1(v)$ displayed in Table 2 corresponding to different values of $v \in V$ and also the graphical representation of the values of Table 2 is provided in Fig. 2(a) – 2(b). Therefore, we have the estimate:

$$0.22 = |f - f_1| \leq \frac{6(\varepsilon_1 + \beta\varepsilon_2)}{1 - 3\beta^2} = 2.92.$$

Table 1. Comparison of Ja_{n+1} and $J_1 a_{n+1}$ iterations.

Iterations	Ja_{n+1}	$J_1 a_{n+1}$
0	0.42857	0.40000
1	0.85423	0.64207
2	**0.85703**	**0.64007**
3	0.85703	0.64007
4	0.85703	0.64007
5	0.85703	0.64007

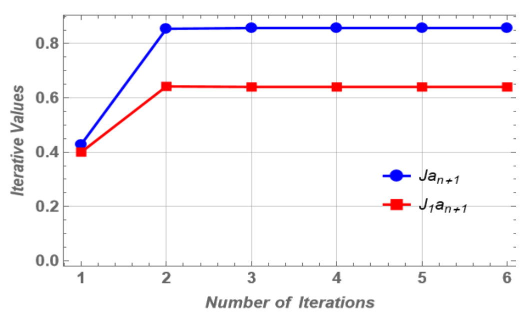

Figure 1. Graph for Table 1.

Table 2. Values of Operators on $[0,1]$.

v	$H(v)$	$H_1(v)$	$J(v)$	$J_1(v)$
0.	0.71429	0.80000	0.00000	0.00000
0.1	0.72857	0.78000	0.08571	0.08000
0.2	0.74286	0.76000	0.17143	0.16000
0.3	0.75714	0.74000	0.25714	0.24000
0.4	0.77143	0.72000	0.34286	0.32000
0.5	0.78571	0.70000	0.42857	0.40000
0.6	0.80000	0.68000	0.51429	0.48000
0.7	0.81429	0.66000	0.60000	0.56000
0.8	0.82857	0.64000	0.68571	0.64000
0.9	0.84286	0.62000	0.77143	0.72000
1.	0.85714	0.60000	0.85714	0.80000

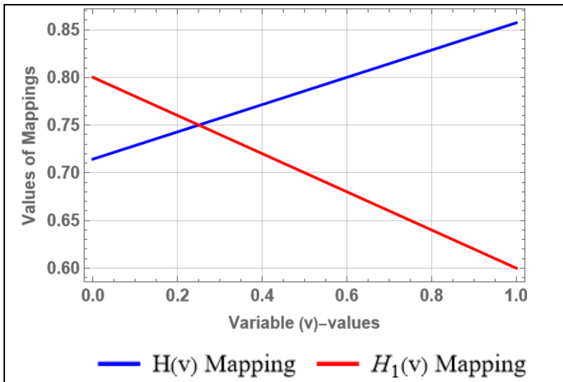

Figure 2(a). Graph for Operators H and H_1.

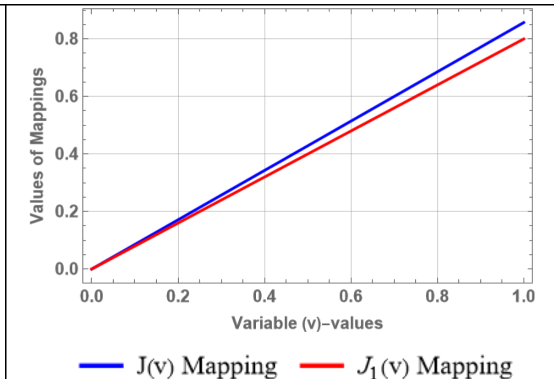

Figure 2(b). Graph for Operators J and J_1.

3. NUMERICAL EXAMPLES

To assess the effectiveness and capabilities of the proposed iterative approach thoroughly, it is crucial to examine a wide array of mathematical challenges with varying levels of complexity. By implementing the proposed iteration technique across these diverse equation types, we seek to not only verify its broad applicability but also acquire a more comprehensive understanding of its advantages and constraints when tackling intricate, real-world scenarios.

Example 3.1. Consider the equation

$$v^2 - 10 = 3v$$

Let $V = [5,7] \subset \mathbb{R}$ be equipped with a standard metric. Establish $H, J : [5,7] \to [15,40]$ with a coincidence point 5 by $Hv = 3v$ and $Jv = v^2 - 10$. Evidently, $H([5,7]) \subseteq J([5,7])$ and $J([5,7])$ is a complete subset of $[15, 40]$. Assume the initial guess as $v_0 = 7$. Table 3 presents a comparative analysis of the convergence rates for the J-Itr1, J-Itr2, and New-Itr methods towards the point of coincidence, and a graphical representation is shown in Fig. 3.

Table 3. Comparison of iterative algorithms for Example 3.1. with $s = 9, s' = \frac{1}{10}$ and $\alpha_n = \beta_n = \gamma_n = \delta_n = \lambda_n = \frac{1}{\sqrt{(1+n)}}$.

Iterations	*J-Itr1* Ja_{n+1}	*J-Itr2* Ja_{n+1}	*New-Itr* Ja_{n+1}
0	39.00000	39.00000	39.00000
1	15.50300	−9.14960	15.62000
2	15.01300	29.05400	15.02000
3	15.00500	11.43500	15.00100
4	15.00300	14.82800	**15.00000**
5	15.00200	14.96800	15.00000
6	15.00100	14.99100	15.00000
7	15.00100	14.99700	15.00000
8	**15.00000**	14.99900	15.00000
9	15.00000	14.99900	15.00000
10	15.00000	**15.00000**	15.00000
11	15.00000	15.00000	15.00000
12	15.00000	15.00000	15.00000

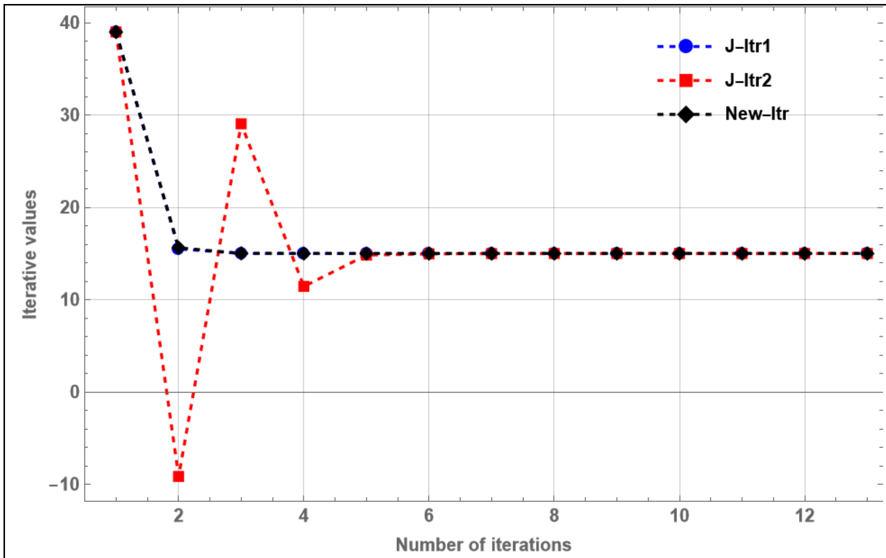

Figure 3. Graph for Table 3.

Example 3.2. Consider the transcendental equation as

$$e^v = Sinv + 2$$

Let $V = [0,2] \subset \mathbb{R}$ be equipped with a standard metric. We define $H, J : [0,2] \to [0,8]$ with a coincidence point 1.054127 by $Hv = 2 + sinv$ and $Jv = e^v$. Evidently, $H([0,2]) \subseteq J([0,2])$ and $J([0,2])$ is a complete subset of $[0,8]$. Suppose the initial guess $v_0 = -1.5$. A comparative study of the convergence between J-Itr1, J-Itr2, and New-Itr to the point of coincidence is shown in Table 4, and a graphical representation is shown in Figure 4.

Table 4. Comparison of iterative algorithms for Example 3.2. with $s = 9, s' = \frac{1}{10}$ and $\alpha_n = \beta_n = \gamma_n = \delta_n = \lambda_n = \frac{1}{\sqrt{(1+n)}}$.

Iterations	*J-Itr1* Ja_{n+1}	*J-Itr2* Ja_{n+1}	*New-Itr* Ja_{n+1}
0	0.22313	0.22313	0.22313
1	2.63990	3.63310	2.60050
2	2.86810	2.04090	2.86710
3	2.86900	3.07580	**2.86950**
4	2.86930	2.85580	2.86950
5	2.86940	2.86810	2.86950
6	2.86940	2.86920	2.86950
7	2.86940	2.86940	2.86950
8	2.86940	2.86940	2.86950
9	2.86940	**2.86950**	2.86950
10	**2.86950**	2.86950	2.86950
11	2.86950	2.86950	2.86950
12	2.86950	2.86950	2.86950

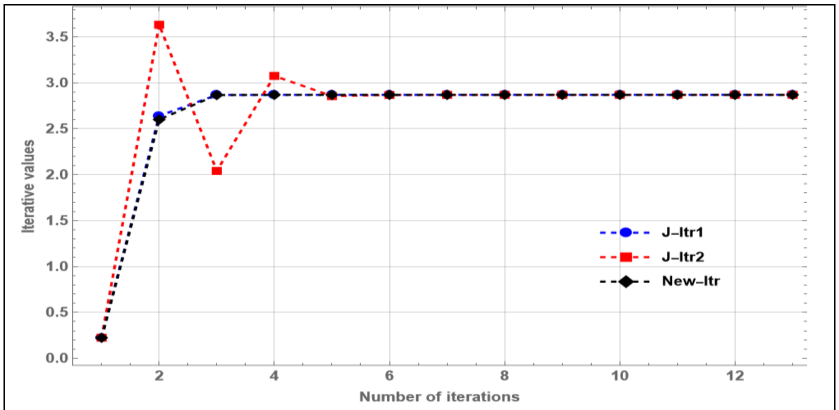

Figure 4. Graph for Table 4.

4. CONCLUSION

We introduced an innovative Jungck-type iterative method, and examined its strong convergence, stability, and data dependence characteristics followed by non-trivial examples. Furthermore, computational experiments indicated that the proposed method demonstrates a superior convergence speed compared to several established iteration techniques.

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