Short Research Article

Quasi-Matroid on connectivity system

Abstract: Matroids are versatile concepts with applications across optimization theory, combinatorial mathematics, topology, algebra, graph algorithms, game theory, geometry, and network theory, garnering substantial interest. Quasi-matroids, as defined in reference [7] (cf. [8]), represent a conceptual extension of matroids, known for their generalized nature. In this concise paper, we introduce the concept of quasi-matroids on a connectivity system and discuss the relationship between quasi-matroids and linear decomposition.

Keyword: Ultra matroid, Linear decomposition, Matroid, Quasi-Matroid

1. Introduction

Graph theory is a fundamental branch of mathematics that focuses on the study of networks formed by nodes and edges, examining their paths, structures, and properties [39]. A crucial metric within this field is the "graph width parameter," which measures the width of a graph. This parameter represents the maximum width across all cuts or layers in a hierarchical decomposition of the graph and is essential for analyzing the complexity and properties of a graph's structure.

The concept of the width parameter holds a significant place within the realm of graph theory. Among its various iterations, linear-width has garnered considerable attention, undergoing extensive exploration not only within graph theory but also across disciplines such as network theory, topology, geometry, and combinatorial mathematics. Scholars have dedicated ample research efforts to investigating linear-width and its related parameters (see, for instance, [1-6, 15-30]).

Matroids are mathematical structures that generalize the notion of linear independence in vector spaces. They provide a framework for optimization and combinatorial problems, characterized by a ground set and a collection of independent subsets satisfying specific axioms. Matroids, versatile concepts with applications spanning optimization theory, combinatorial mathematics, topology, algebra, graph algorithms, game theory, geometry, and network theory, have garnered substantial interest. Numerous studies have delved into this subject matter (see, for example, [9-14,31-33]).
Quasi-matroids, as defined in reference [7] (cf. [8]), represent a conceptual extension of matroids, known for their generalized nature.

In this paper, we embark on an exploration of the relationship between quasi-matroids on connectivity systems and linear-width. At first glance, the idea that seemingly unrelated concepts can become relevant when considered within the context of a connectivity system is both profound and surprising.

2. Preparation

This section provides mathematical definitions of each concept.

First, we explain the basic notation used in this paper. In this paper, we use expressions like $A \subseteq X$ to indicate that A is a subset of X, $A \cup B$ to represent the union of two subsets A and B, both of which are subsets of X, or $A = \emptyset$ to signify an empty set. Specifically, $A \cap B$ denotes the intersection of subsets A and B. A similar logic applies to $A \setminus B$. $A \in X$ indicates that A is an element of X. $\forall$ means "for all" or "for every." $\exists$ means "there exists."

2.1 Symmetric submodular function and connectivity system

A symmetric submodular function demonstrates symmetry and adheres to the principle of diminishing returns, often applied in optimization and decision-making contexts, as well as in graph theory and discrete mathematics. The definition of a symmetric submodular function is provided below.

Definition 1: Let X be a finite set. A function $f: X \rightarrow \mathbb{N}$ is called symmetric submodular if it satisfies the following conditions:

· $\forall A \subseteq X,\ f(A) = f(X \backslash A)$.

· $\forall A, B \subseteq X,\ f(A) + f(B) \geq f(A \cap B) + f(A \cup B)$.

A symmetric submodular function satisfies the following lemma.

Lemma 1 [15,16]: A symmetric submodular function f satisfies

1. $\forall A \subseteq X,\ f(A) \geq f(\emptyset) = f(X)$,
2. $\forall A, B \subseteq X,\ f(A) + f(B) \geq f(A \backslash B) + f(B \backslash A)$.

In this short paper, a pair (X, f) of a finite set X and a symmetric submodular function f is called a connectivity system. The concept of a connectivity system is commonly employed when discussing graph width parameters. And we use the notation f for a symmetric submodular function, a finite set X, and a natural number k.

2.2 Quasi-Matroids on Boolean algebra $(X, \cup, \cap)$

First, we explain matroids. A matroid is a mathematical structure that generalizes the notion of independence in linear algebra.

The definition of matroid onBoolean algebra $(X, \cup, \cap)$ is shown below. Axiom (MB2) is frequently recognized as the hereditary property, whereas axiom (MB3) is commonly referred to as the augmentation property or independent set exchange property.

Definition 2: In Boolean algebra $(X, \cup, \cap)$, the set family $M \subseteq 2^X$ is called a matroidif the following axioms hold true:

(MB1) $\emptyset \in M$,

(MB2) if $A \in M$ and $B \subseteq A$ then $B \in M$,

(MB3) if $A, B \in M$, $|A| < |B|$ then there exists $e \in B \setminus A$ such that $A \cup \{e\} \in M$.

The conjunction of axiom (MB1) and axiom (MB2) defines a combinatorial notion known as an independence system, which is also referred to as an abstract simplicial complex on Boolean algebra.

Next, let's explain Quasi-Matroids. Quasi-Matroids, as defined in reference [7] (cf. [8]), represent a conceptual extension of Matroids known for their generalized nature. A quasi-matroid Q is characterized as a pair (E, I), where E is a simplicial complex on Boolean algebra and I is a matroid on Boolean algebra such that I is a subset of E. Following condition is essential for defining the relationship between the simplicial complex and the matroid.

(MB4) For each pair $(X, M) \in E \times I$ such that$M \subset X$ and $Y \notin I$ for any $M \subsetneq Y \subseteq X$, if $e \in \cup E$ with $M \cup \{e\} \in E$, then $X \cup \{e\} \in E$.

2.3.Quasi-Matroids on a connectivity system (X,f)

We explain about Quasi-Matroids on a connectivity system (X,f).

First, we explain matroid on a connectivity system (X,f).

Definition3 [29]: Let X be a finite set and f be a symmetric submodular function.In a connectivity system(X,f), the set family $M \subseteq 2^X$ is called a matroid of order $k+1$ on (X,f) if the following axioms hold true:

(M0) For every $A \in M$, $f(A) \leq k$,

(M1)$\emptyset \in M$,

(M2) if $A \in M$, $B \subseteq A$, and $f(B) \leq k$ then $B \in M$,
(M3) if $A, B \in M$, $|A| < |B|$, $e \in X$, $f(\{e\}) \leq k$, and $f(A \cup \{e\}) \leq k$, , then $e \in B \setminus A$ such that $A \cup \{e\} \in M$.

Furthermore, let us define an order $k+1$ matroid $M \subseteq 2^X$ on connectivity system (X,f) as an Ultra Matroid on a connectivity system (X,f) if it satisfies the following Axiom (M4):
(M4): For any subset $A \subseteq X$, if $f(A) \leq k$, then either $A \in M$ or $X \setminus A \in M$.

The combination of axiom (M0), axiom (M1) and axiom (M2) establishes a combinatorial concept called an independence system on a connectivity system, also known as an abstract simplicial complex on a connectivity system. Axiom (M4) mirrors the concept of Ultrafilters, asserting that "either a set A or its complement belongs to the (Ultra)filter."

Next, let's explain quasi-Matroids on a connectivity system. Let X be a finite set and f be a symmetric submodular function. A quasi-matroid Q on a connectivity system (X,f) is characterized as a pair (E, I), where E is a simplicial complex on a connectivity system and I is a matroid on a connectivity system such that I is a subset of E. The following condition is crucial for establishing the connection between the simplicial complex and the matroid within a connectivity system.
The definition of Boolean algebra is extended to encompass conditions for symmetric submodular functions, which then become axioms on the connectivity system (X, f).
(M5) For each pair $(X, M) \in E \times I$ such that $M \subset X$ and $Y \notin I$ for any $M \subsetneq Y \subseteq X$, if $e \in \cup E$ with $M \cup \{e\} \in E$, $f(\{e\}) \leq k$, $f(X \cup \{e\}) \leq k$ then $X \cup \{e\} \in E$.

Furthermore, let us define an order $k+1$ quasi-matroid $M \subseteq 2^X$ on connectivity system (X,f) as an Ultra Quasi-Matroid on a connectivity system (X,f) if it satisfies the following Axiom (M4):
(M4): For any subset $A \subseteq X$, if $f(A) \leq k$, then either $A \in M$ or $X \setminus A \in M$.

2.4 Linear tangle
Linear tangle, which obstructs Linear branch-decomposition, was first introduced in the literature [1]. A linear branch-decomposition is a method for organizing a graph into a linear structure. Each node in the decomposition represents an edge of the graph, and the arrangement helps measure the graph's width. This approach simplifies the analysis of the graph's properties and complexity. As stated in the introduction, the study of linear branch-decomposition is important, and therefore, the study of linear tangles is also considered significant. In this paper, we investigate the relationship between linear tangles and quasi-matroids.

The definition of a linear tangle on a connectivity system (X,f) is given below.

Definition 4 [1]: Let X be a finite set and f be a symmetric submodular function. A linear tangle of order $k+1$ on a connectivity system (X,f) is a family $L \subseteq 2^X$, satisfying the following axioms:
(L1) $\emptyset \in L$
(L2) $A \subseteq X$, $f(A) \leq k \Rightarrow$ *either* $A \in M$ *or* $X \setminus A \in M$.
(L3) If $A,B \in L$, $e \in X$, and $f(\{e\}) \leq k$, then $A \cup B \cup \{e\} \neq X$ holds.

In this article, we suggest leveraging the augmentation property inherent in Quasi-Matroids, a defining characteristic, by introducing the following axiom (L3') in place of (L3). We designate a linear tangle of order $k+1$ on a connectivity system (X,f) fulfilling axioms (L1), (L2), and (L3') as a restricted linear tangle of order $k+1$.
(L3') If $A,B \in L$, $|A| \neq |B|$, $e \in X$, and $f(\{e\}) \leq k$, then $A \cup B \cup \{e\} \neq X$ holds.

3 Main Theorem

The main theorem in this paper is as follows. It is surprising that the seemingly unrelated concepts of linear tangles and quasi-matroids have the following relationship.

Theorem 2: Let X be a finite set and f be a symmetric submodular function. Under the assumption that $f(\{e\}) \leq k$ for every $e \in X$, the family $W \subseteq 2^X$ being anorder $k+1$ restricted linear tangle and $W \subseteq 2^X$ being anorder $k+1$ ultra Quasi-matroid are equivalent necessary and sufficient conditions.

Proof: First, we show that W satisfies the axioms (L1) and (L2) of a restricted linear tangle. Since W is an ultra quasi-matroid of order $k+1$, it satisfies Axiom (M4), which implies that for any subset $A \subseteq X$, if $f(A) \leq k$, then either $A \in W$ or $X \setminus A \in W$. In particular, for the empty set $\emptyset$, we have $f(\emptyset) = f(A) \leq k$, and hence $\emptyset \in W$. Moreover, if $A \subseteq X$ and $f(A) \leq k$, then either $A \in W$ or $X \setminus A \in W$ by Axiom (M4), which shows that W satisfies Axiom (L2) as well.

Next, we prove that W satisfies the axiom (L3') of a restricted linear tangle. Let $A, B \in W$ and $e \in X$ be such that $f(\{e\}) \leq k$. We need to show that $A \cup B \cup \{e\} \neq X$. Suppose, on the contrary, that $A \cup B \cup \{e\} = X$. We choose a triple $(A,B,\{e\})$ that minimizes $|A \cap B|$ among such triples. First, we claim that $A \cap B = \emptyset$. Since $2k \geq f(A) + f(B) \geq f(A \setminus B) + f(B \setminus A)$, at least one of $f(A \setminus B)$ or $f(B \setminus A)$ is at most k. Without loss of generality, assume that $f(A \setminus B)$ is at most k. Hence, by axiom (M2), $A \setminus B \in M$. If $A \cap B \neq \emptyset$, then we have $|A \cap B| > |(A \setminus B) \cap B|$, which contradicts the choice of the triple. Thus, we have shown that $A \cap B = \emptyset$.

Next, we claim that $e \notin A$ and $e \notin B$. Suppose, on the contrary, that $e \in A$ or $e \in B$. If $e \in A$, then $A \cup B = X$, which implies that $X \setminus A = B \in M$, but this contradicts the axiom (M4). Similarly, we know that $e \notin B$ holds. Now, we know that the triple $(A,B,\{e\})$ consists of a partition of X. Hence, we have $f(A \cup \{e\}) = f(X \setminus B) = f(B) \leq k$. If $|A| < |B|$, by axiom (M3) there exists $e' \in B \setminus A$ such that $A \cup \{e'\} \in M$. This contradicts that there not exists $e' \in B \setminus A$ such that $A \cup \{e'\} \in M$ because the triple $(A,B,\{e\})$ consists of a partition of X. If $|B| < |A|$, by axiom (M3) there exists $e'' \in A \setminus B$ such that $B \cup \{e''\} \in M$. This contradicts that the triple $(A,B,\{e\})$ consists of a partition of X. So W satisfies the axiom (L3').

Therefore, we have shown that W satisfies all the axioms (L1), (L2), and (L3') of a restricted linear tangle. Hence, W is a restricted linear tangle of order $k+1$ on the connectivity system (X,f).

Assume that $f(\{e\}) \leq k$ for every $e \in X$ and let W be a restricted linear tangle of order $k+1$ on the connectivity system (X,f). We will show that W satisfies the axioms of an ultra quasi-matroid of order $k+1$ on the connectivity system (X,f).

(M0) For any $A \in W$, we have by definition of a restricted linear tangle that $f(A) \leq k$, hence (M0) is satisfied.

(M1) Since $\emptyset$ is an element of W by definition of a restricted linear tangle, it is also an element of W, hence the axiom (M1) is satisfied.

(M2) Let $A \in W$, $B \subseteq A$, and $f(B) \leq k$. We need to show that $B \in W$. Suppose, to the contrary, that there exist subsets A and B such that $A \subseteq B$, $f(A) \leq k$, $B \in W$, and $A \notin W$. Then, we have $X \setminus A \in W$ by the axiom (L2), and for any $e \in X$, $(X \setminus A) \cup \{e\} \cup B = X$ holds, but this contradicts the axiom (L3') of a restricted linear tangle. So the axiom (M2) is satisfied.

(M4) For any subset $A \subseteq X$, if $f(A) \leq k$, then either $A \in M$ or $X \setminus A \in M$. This directly follows from the definition of a restricted linear tangle, satisfying (L2).

(M5) Let $(X,M) \in E \times I$ such that $M \subset X$ and $Y \notin I$ for any $M \subsetneq Y \subseteq X$. If $e \in \cup E$ with $M \cup \{e\} \in E$, then $X \cup \{e\} \in E$. Suppose, on the contrary, that there exists $(X,M) \in E \times I$ such that $M \subset X$ and $Y \notin I$ for any $M \subsetneq Y \subseteq X$, and $e \in \cup E$ with $M \cup \{e\} \in E$, but $X \cup \{e\} \notin E$. By (M4), we have either $X \cup \{e\} \in E$ or $(X \cup \{e\}) \setminus (X \cup \{e\}) = \emptyset \in W$, contradicting (L2).

Therefore, W satisfies all the axioms of an ultra quasi-matroid of order $k+1$ on the connectivity system (X,f). This proof is completed.

4 Future tasks

This section briefly outlines the future directions of this study.

We plan to explore the concept of feeble matroids [31] within the framework of connectivity systems. Additionally, we aim to investigate their relationships with fuzzy matroids [34–37] and neutrosophic matroids [38].

Data Availability

This research is purely theoretical, involving no data collection or analysis. We encourage future researchers to pursue empirical investigations to further develop and validate the concepts introduced here.

Ethical Approval

As this research is entirely theoretical in nature and does not involve human participants or animal subjects, no ethical approval is required.

Disclaimer

This work presents theoretical concepts that have not yet undergone practical testing or validation. Future researchers are encouraged to apply and assess these ideas in empirical contexts. While every effort has been made to ensure accuracy and appropriate referencing, unintentional errors or omissions may still exist. Readers are advised to verify referenced materials on their own. The views and conclusions expressed here are the authors' own and do not necessarily reflect those of their affiliated organizations.

Reference

[1] Bienstock, D. (1989). Graph searching, path-width, tree-width and related problems (a survey). In Reliability of Computer and Communication Networks (Vol. 5 of DIMACS. Series in Discrete Mathematics and Theoretical Computer Science, pp. 33–50).
[2] Takaaki Fujita, "Quasi-Ultrafilter on the Connectivity System: Its Relationship to Branch-Decomposition," International Journal of Mathematics Trends and Technology (IJMTT), vol. 70, no. 3, pp. 13-16, 2024. Crossref, https://doi.org/10.14445/22315373/IJMTT-V70I3P102.
[3] Beisegel, Jesse, et al. "The Simultaneous Interval Number: A New Width Parameter that Measures the Similarity to Interval Graphs." arXiv preprint arXiv:2404.10670 (2024).
[4] Thilikos, D. M. (2000). Algorithms and obstructions for linear-width and related search parameters. Discrete Applied Mathematics, 105(1-3), 239–271.
[5] Gurski, F., & Rehs, C. (2019). Comparing linear width parameters for directed graphs. Theory of Computing Systems, 63, 1358-1387.
[6] Bonnet, Édouard, et al. "Twin-width II: small classes." Proceedings of the 2021 ACM-SIAM Symposium on Discrete Algorithms (SODA). Society for Industrial and Applied Mathematics, 2021.
[7] Kawahara, Yukihito. "On matroids and Orlik-Solomon algebras." Annals of Combinatorics 8 (2004): 63-80.
[8] Yukito Kawahara. "Combinatorics coming from hyperplane arrangements and the Orlik-Solomon algebra (Algebraic Combinatorics)." RIMS Kokyuroku 1394 (2004): 97-108.
[9] Ardila, Federico. "The geometry of geometries: matroid theory, old and new." arXiv preprint arXiv:2111.08726 (2021).
[10]Im, Sungjin, Benjamin Moseley, and Kirk Pruhs. "The matroid intersection cover problem." Operations Research Letters 49.1 (2021): 17-22.
[11] Fujita, Takaaki. "A review of fuzzy and neutrosophic offsets: Connections to some set concepts and normalization function." Advancing Uncertain Combinatorics through Graphization, Hyperization, and Uncertainization: Fuzzy, Neutrosophic, Soft, Rough, and Beyond (2024): 74.
[12] Bérczi, K., et al. (2023). Hypergraph characterization of split matroids. Journal of Combinatorial Theory, Series A, 194, 105697.
[13] Cameron, A., et al. (2022). Flag matroids: algebra and geometry. In Interactions with Lattice Polytopes: Magdeburg, Germany, September 2017 (pp. 73-114).
[14] Oxley, J. G. (2022). Matroid theory. In Handbook of the Tutte Polynomial and Related Topics (pp. 44-85).

[15] Geelen, J., et al. (2006). Obstructions to branch-decomposition of matroids. Journal of Combinatorial Theory, Series B, 96(4), 560-570.
[16] Oum, S.-I., & Seymour, P. (2007). Testing branch-width. Journal of Combinatorial Theory, Series B, 97(3), 385-393.
[17] Diestel, Reinhard, Fabian Hundertmark, and Sahar Lemanczyk. "Profiles of separations: in graphs, matroids, and beyond." Combinatorica 39 (2019): 37-75.
[18] Lozin, Vadim, and Igor Razgon. "Tree-width dichotomy." European Journal of Combinatorics 103 (2022): 103517.
[19] Fujita, Takaaki, and Florentin Smarandache. "Some graph parameters for superhypertree-width and neutrosophictree-width." Neutrosophic Systems with Applications 25 (2025): 53-77.
[20] Fujita, Takaaki. "Bounded Tree-depth, Path-distance-width, and Linear-width of Graphs." Journal of Fundamental Mathematics and Applications (JFMA) 7.2 (2024).
[21] Fujita, Takaaki. "Short note of supertree-width and n-superhypertree-width." Neutrosophic Sets and Systems 77 (2025): 54-78.
[22] Fujita, Takaaki. "Novel Idea on Edge-Ultrafilter and Edge-Tangle." Asian Research Journal of Mathematics 20.4 (2024): 18-22.
[23] Fujita, Takaaki. 2024. "Reevaluating the Complementary Relationship Between Single Ultrafilters and Linear Obstacles in Connectivity Systems". Asian Research Journal of Mathematics 20 (8):69-75. https://doi.org/10.9734/arjom/2024/v20i8817.
[24] Robertson, Neil, and Paul D. Seymour. "Graph minors. II. Algorithmic aspects of tree-width." Journal of algorithms 7.3 (1986): 309-322.
[25] Chekuri, Chandra, and Julia Chuzhoy. "Large-treewidth graph decompositions and applications." Proceedings of the forty-fifth annual ACM symposium on Theory of computing. 2013.
[26] Gurski, Frank, and Egon Wanke. "On the relationship between NLC-width and linear NLC-width." *Theoretical Computer Science* 347.1-2 (2005): 76-89.
[27] Gurski, Frank, and Carolin Rehs. "Comparing linear width parameters for directed graphs." *Theory of Computing Systems* 63 (2019): 1358-1387.
[28] Takaaki Fujita, "Matroid, Ideal, Ultrafilter, Tangle, and so on :Reconsideration of Obstruction to Linear Decomposition," International Journal of Mathematics Trends and Technology (IJMTT), vol. 70, no. 7, pp. 18-29, 2024. Crossref, https://doi.org/10.14445/22315373/IJMTT-V70I7P104.
[29] Fujita, Takaaki. Relation between ultra matroid and Linear decomposition. Italian Journal of Pure and Applied Mathematics. 52–2024 (18–24).
[30] Demaine, Erik D., and Mohammad Taghi Hajiaghayi. "Equivalence of local treewidth and linear local treewidth and its algorithmic applications." (2003).
[31] Al-Hawary, T. (2003) Feeble-Matroids. Italian Journal of Pure and Applied Mathematics, 14, 87-94.
[32] Al-Hawary, T. (2017) On Functions of K-Balanced Matroids. Open Journal of Discrete Mathematics, 7, 103-107. doi: 10.4236/ojdm.2017.73011.
[33] Al-Hawary, T. (2001) Characterization of Matroids via OFR-Sets. Turkish Journal of Math, 24, 1-11.
[34] Sarwar, Musavarah, and Muhammad Akram. "New applications of m-polar fuzzy matroids." Symmetry 9.12 (2017): 319.
[35] Novak, Ladislav. "On Goetschel and Voxman fuzzy matroids." Fuzzy sets and systems 117.3 (2001): 407-412.
[36] Li, Xiaonan, and Huangjian Yi. "Intuitionistic fuzzy matroids." Journal of Intelligent & Fuzzy Systems 33.6 (2017): 3653-3663.
[37] Lu, Ling-Xia, and Wei-Wei Zheng. "Categorical relations among matroids, fuzzy matroids and fuzzifying matroids." (2010): 81-89.
[38] Fujita, Takaaki. "Short Note of Neutrosophic Closure matroids." Advancing Uncertain Combinatorics through Graphization, Hyperization, and Uncertainization: Fuzzy, Neutrosophic, Soft, Rough, and Beyond: 146.
[39] Diestel, Reinhard. Graph theory. Springer (print edition); Reinhard Diestel (eBooks), 2024.